

# DIAGRAM UNIQUENESS FOR HIGHLY TWISTED PLATS

YOAV MORIAH AND JESSICA S. PURCELL

ABSTRACT. Frequently, knots are enumerated by their crossing number. However, the number of knots with crossing number  $c$  grows exponentially with  $c$ , and to date computer-assisted proofs can only classify diagrams up to around twenty crossings. Instead, we consider diagrams enumerated by bridge number, following the lead of Schubert who classified 2-bridge knots in the 1950s. We prove a uniqueness result for this enumeration. Using recent developments in geometric topology, including distances in the curve complex and techniques with incompressible surfaces, we show that infinitely many knot and link diagrams have a unique simple  $m$ -bridge diagram. Precisely, if  $m$  is at least three, if each twist region of the diagram has at least three crossings, and if the length  $n$  of the diagram is sufficiently long, i.e.,  $n > 4m(m - 2)$ , then such a diagram is unique up to obvious rotations. This projection gives a canonical form for such knots and links, and thus provides a classification of these knots or links.

## 1. INTRODUCTION

One way of studying knots in  $S^3$  is via their regular projections on 2-spheres in  $S^3$ . Such projections are called *knot diagrams*. Deciding when two diagrams correspond to the same knot is a difficult problem, going all the way back to work of Tait in the 1870s. In 1926, K. Reidemeister proved that any two regular projections are equivalent by a sequence of *Reidemeister moves*; see [22]. However, determining when two diagrams are equivalent by Reidemeister moves is also a very difficult problem, and remains an area of active research, for example see [17, 5].

The earliest attempts to classify knots began in the 1870's by Tait, using the *crossing number* of a knot as the classifying parameter. Today, some one hundred and thirty years later, only knots with less than 17 crossings have been completely classified.<sup>1</sup> For prime knots with 16 crossings, there are 1,701,936 different knots (where knots and their mirror image are not distinguished); see [14]. Thus distinguishing diagrams using crossing number is currently possible only for knots with a small number of crossings.

Another way around the problem of deciding when diagrams are equivalent would be to obtain “canonical” projections for knots. One attempt to do so was to use the

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<sup>1</sup>That is, the classification for knots with up to 16 crossings has been published. Others, including Thistlethwaite, have worked on knots with up to 21 crossings.

fact that all knots have  $2m$ -plat projections for some  $m \in \mathbb{N}$ , which are very closely related to  $m$ -bridge diagrams. This was used successfully on the class of 4-plats, more commonly known as 2-bridge knots and links. In 1956, H. Schubert gave a complete classification of 2-bridge knots and links. He showed in [23] that these knots are classified by a rational number, and that any two continued fraction expansions of this number correspond to a 4-plat projection. For a discussion of this see [3].

In a somewhat different flavor W. Menasco and M. Thistlethwaite proved in 1993 that any two alternating projections of a knot  $\mathcal{K} \subset S^3$  are equivalent by flype moves [19]. Flypes involve the existence of essential 4-punctured spheres in the diagram, and they can be detected in a finite number of steps. However, the sequence of such steps can be arbitrarily long.

In this paper, we prove a *uniqueness of diagrams* statement for an infinite class of plats of arbitrary large width and length. In a sense that can be made precise, these plats are a generic collection in the set of all plats. We show that such plats have a *canonical* form that is unique, and can be read off the diagram immediately, without any need to consider equivalence relations such as Reidemeister moves or flypes.

For a precise definition of a *plat* and some facts about them see Section 2 and references [4] and [15]. The main theorem of this paper is:

**Theorem 1.1** (Uniqueness of Diagrams). *Let  $\mathcal{K} \subset S^3$  be a knot or link with a  $2m'$ -plat projection  $K'$ , where  $m' \geq 3$  and  $n'$ , the length of  $K'$ , satisfies  $n' > 4m'(m' - 2)$ . Suppose also that each twist region of  $K'$  contains at least three crossings. Finally, suppose that  $K$  is another plat or even plat projection of  $\mathcal{K}$  of width  $m$ . Then  $m \geq m'$ , and if  $m = m'$  and each twist region of  $K$  has at least one crossing, then  $K = K'$  up to rotation in a horizontal or vertical axis.*

**1.1. The preferred notion of generic.** Making mathematical sense of the term “generic” is particularly problematic when considering subsets of countable sets  $X$ . In the classic setting, a set  $A \subset X$  would be called *generic* if and only if the complement  $X \setminus A$  is finite. This definition is very restrictive and, in practice, there are many cases where intuitively one would like to call a set generic, yet it does not satisfy this criterion. A common method to deal with this situation is the following: One considers a family of finite subsets  $X_n \subset X$  that exhaust  $X$  (typically given by enumerating  $X = \{x_1, x_2, \dots\}$ , and by defining  $X_n = \{x_1, x_2, \dots, x_n\}$ ). One then computes the ratio  $\rho_n = \#(X_n \cap A) / \#X_n$  and defines  $A$  to be generic if  $\rho_n$  tends to 1. Of course, the obtained limit ratio depends heavily on the particular chosen enumeration of  $X$ .

Another difficulty with statements about “genericity” is that of *double counting*. A typical example here is to count group presentations instead of groups, so that the same group may well be counted several times, or even infinitely often. The problem arises when in the given mathematical context it is impossible to count the objects in question directly, so that double counting becomes unavoidable.

With these difficulties in mind we would like to advocate for the idea of enumerating knot by plats.

Assume Theorem 1.1. Choose  $X = \mathbb{Z}$  with the obvious enumeration. Further choose  $X_M = [-M, M] \cap \mathbb{Z}$  for some large interval  $[-M, M] \subset \mathbb{R}$  and  $M \in \mathbb{N}^+$ . Choose

at random a set  $A \subset X_M$  consisting of one integer for each twist region in a  $2m$ -plat diagram of length  $n > 4m(m-2)$  (approximately  $m \cdot n$  integers). Compute the probability  $\rho_M$  that  $A$  does not contain elements from  $\{-2, -1, 0, 1, 2\} \subset X_M$ . It is clear that  $\rho_M \rightarrow 1$  as  $M \rightarrow \infty$ . In fact, even better, any such chosen set  $A$  determines a *very large* number of knots as the integers in  $A$  can be distributed in the twist regions, of the  $2m$ -plat, in approximately  $(n \cdot m)! > (4m^2(m-2))!$  different ways.

As the diagram is unique (and the knots are enumerated by their diagrams) this enumeration also takes care of the issue of double counting discussed above. Any diagram will be counted exactly once.

Hence the question of which notions of “genericness” one should adopt here is not merely a set theoretic issue, but it should depend on the topological consequence of the enumeration. Furthermore, in the case of the  $2m$ -plat, the canonical form has a large number of other nice features:

The canonical form gives information about incompressible surfaces in  $S^3 \setminus \mathcal{N}(K)$ ; see [7], [6] and [26]. In many cases, it gives information about  $\pi_1(S^3 \setminus \mathcal{N}(K))$  and its rank, i.e. minimal number of generators [18]. It also gives information about the bridge number for  $K$  and Heegaard splittings of  $S^3 \setminus \mathcal{N}(K)$  and about manifolds obtained by Dehn surgery on  $K$ ; see [18] and [26]. By combining work of Johnson and Moriah [15] and Bachman and Schleimer [1, Corollary 6.2], it follows that such plats are hyperbolic. A subclass was shown to have a property called *semi-adequacy*, with no 2-edge loops in an associated graph [13]. For this subclass, we obtain another proof of hyperbolicity [11], a bound on the volume [9], and the existence of essential spanning surfaces that do not have accidental parabolics [10].

In addition, a knot or link  $\mathcal{K} \subset S^3$  with at least  $C$  crossings in each twist region, for appropriate  $C$ , is known to satisfy several nice geometric properties, whether or not it has the plat projection required by Theorem 1.1. For example, if  $C = 6$ , then  $\mathcal{K}$  and all Dehn fillings of  $\mathcal{K}$  are hyperbolic [12]. Additionally, closed embedded essential surfaces are known to be high genus [2]. If  $C = 7$ , there are known explicit upper and lower bounds on the hyperbolic volume of  $\mathcal{K}$  [8]. And if  $C = 116$ , then the shape of the cusp of  $\mathcal{K}$  is bounded [20]. It is conjectured that knots and links with at least 3 crossings per twist region are always hyperbolic.

From a hyperbolic geometry point of view, we argue further that the knots of Theorem 1.1 are generic in the same sense that hyperbolic Dehn filling is generic, following Thurston [24]. In his celebrated hyperbolic Dehn surgery theorem, Thurston showed that Dehn fillings of hyperbolic 3-manifolds yield hyperbolic 3-manifolds “generically”, in the sense that only finitely many Dehn filling slopes need to be excluded per boundary component, and the result is hyperbolic [24]. The links of Theorem 1.1 can also be seen as obtained by Dehn filling. Encircle each twist region by a simple unknot, called a crossing circle. The result is called a fully augmented plat. When one performs  $(1, t_{i,j})$  Dehn filling on the  $\{i, j\}$ -th crossing circle of a fully augmented plat diagram for each pair  $\{i, j\}$ , one obtains a regular plat diagram. Details of fully augmented links and their Dehn fillings can be found in [21]. Links of Theorem 1.1 are obtained when we exclude Dehn fillings with  $|t_{i,j}| < 2$ .

Thus the links of Theorem 1.1 all lie in a Dehn filling neighborhood of a fully augmented plat. Note this is a very large, explicitly defined neighborhood: The bound of three crossings per twist region excludes far fewer knots than the best current bounds for hyperbolic Dehn filling (e.g. [12] requires six). Moreover, each such knot or link lies in a Dehn filling neighborhood of one of all possible Dehn filling parents of sufficiently long plat diagrams.

For the reasons above, we argue that enumeration of knots by bridge number makes more sense topologically and geometrically than enumeration by crossing number, and that the knots satisfying the conditions of Theorem 1.1 can be viewed as “generic”.

**1.2. Even plats.** Note that the statement of Theorem 1.1 includes knots and links with an “even plat” projection, which we now describe. Consider a braid with an even number of rows of twist regions on an even number of strings. At the top of the braid, we connect pairs of adjacent strings by bridges, just as in the usual case of a plat. At the bottom of the braid, we connect the second string to the third, the fourth to the fifth, and so on. At the end we connect the  $2m$  string to the first one. We obtain a knot or link  $L$  depending on the parity of  $a_{i,j}$ . We will call a projection as above an *even plat*. Technically,  $L$  is not a plat in this case but Theorem 1.2 of [15] applies, so that we can compute the bridge distance. As in the case of a plat we can require that the number of crossings in each twist region be  $|a_{i,l}| \geq 3$  and that the number of rows of twist regions plus one satisfies  $n > 4m(m - 2)$ , so we have a unique bridge sphere by [25]. We conjecture:

**Conjecture 1.2.** *Theorem 1.1 holds when  $K'$  is an even plat.*

Going back to the question of crossing number, we conjecture that these plat projections also have minimal crossing number, as follows.

**Conjecture 1.3.** *Knot and link diagrams satisfying the conditions of Theorem 1.1 realize the crossing number of the associated links.*

Theorem 1.1 raises two questions:

**Question 1.4.** Suppose  $\mathcal{K} \subset S^3$  is a knot that has a  $2m$ -plat projection satisfying the conditions of Theorem 1.1. Suppose that  $\mathcal{K} \subset S^3$  is given by some knot diagram  $D(\mathcal{K})$ . Is there an “efficient” algorithm to isotope  $D(\mathcal{K})$  to its canonical  $2m$ -plat projection?

**Question 1.5.** Given a knot  $\mathcal{K} \subset S^3$  can one decide if  $\mathcal{K}$  has a  $2m$ -plat projection satisfying the conditions of Theorem 1.1?

**Remark 1.6.** Highly twisted plats are defined carefully in Section 2 in terms of a specific family of words in the braid group. It follows from Theorem 1.1 that the corresponding braids are uniquely determined in the braid group by such a braid word.

**1.3. Idea of the proof.** Let  $\mathcal{K}$ ,  $K'$ , and  $K$  be as in the statement of the theorem, as well as other notation. Recall that plat diagrams determine collections of bridge-spheres (see Definition 2.1) and vertical 2-spheres (see Definition 2.9). The main idea is to see how these two collections behave with respect to each other under an ambient isotopy of the knot from  $K'$  to  $K$ .

The condition that  $n' > 4m'(m' - 2)$  implies that the distance of the bridge sphere is greater than  $2m'$ . Results of Tomova [25], Johnson-Tomova [16], and Johnson-Moriah [15] imply  $m$  has to be minimal; thus  $m \geq m'$ .

Suppose  $m = m'$ . Consider the ambient isotopy  $\varphi : (S^3, K') \rightarrow (S^3, K)$ . We show first in Lemma 3.2 that we may take the ambient isotopy to preserve the foliation of  $(S^3, K')$  by bridge spheres. Using this, we show in Proposition 3.4 that the image of a vertical 2-sphere (see Definition 2.9) is mapped to a sphere which does not meet any twist region. As a consequence we show in Lemma 3.8 that the image of a vertical 2-sphere can intersect  $K$  at most once in each level, and that allows us to obtain Proposition 4.4, which states that the length of the two plats must be the same.

It remains to show that the coefficients in each of the twist regions are the same and that they occur in the same order up to reflection in horizontal and vertical axes.

For each twist region  $t'_{i,j}$  we construct an isolating sphere  $\Omega'$ ; see Definition 4.6. Each isolating sphere is composed of two disks, each on a vertical or almost vertical 2-sphere (see Definition 2.12). We show in Theorem 4.1 that all vertical and almost vertical 2-spheres are carried to vertical and almost vertical 2-spheres simultaneously. It follows that isolating spheres are mapped by  $\varphi$  to isolating spheres.

The bounded component of  $S^3 \setminus \Omega'$  is a 2-tangle of type  $1/a'_{i,j}$  in  $K'$ . We then show that such tangles must be mapped by  $\varphi$  to tangles of type  $1/a_{i,j}$  in  $K$ , where  $a_{i,j} = a'_{i,j}$ . Since tangles  $t_{i,j}$  are connected to adjacent tangles, the order of the twist regions is preserved up to reflection in horizontal and vertical axes.

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## 2. PRELIMINARIES

For any knot or link  $\mathcal{K} \subset S^3$  there is an  $m \in \mathbb{N}$  so that  $\mathcal{K}$  has a  $2m$ -plat projection, as indicated in Figure 1. This follows from the classical fact that any knot or link can be presented as a closed braid, and then strands of the braid closure can be pulled across the braid diagram; for example see [4, p. 24].

In Figure 1 a box labeled  $a_{i,j}$  indicates a *twist region* with  $a_{i,j}$  crossings, where  $a_{i,j}$  can be positive, negative, or zero. Note that crossings in a twist region must be alternating. That is, some number of negative crossings cannot be followed by some number of positive crossings, else the diagram could be reduced within the twist regions by cancelling crossings. Thus  $a_{i,j}$  indicates the number of crossings of the same sign in the *reduced* twist region. For the example in the figure,  $a_{1,1} = a_{2,2} = -3$ , and all other  $a_{i,j} = -4$ . The twist region associated with the box labeled  $a_{i,j}$  will be denoted by  $t_{i,j}$ . We can think of the  $t_{i,j}$  twist region as a projection to  $P$ , the plane of projection, of a  $1/a_{i,j}$ -tangle considering the segments of  $K$  as strands contained in the 3-ball (box  $\times I$ ).

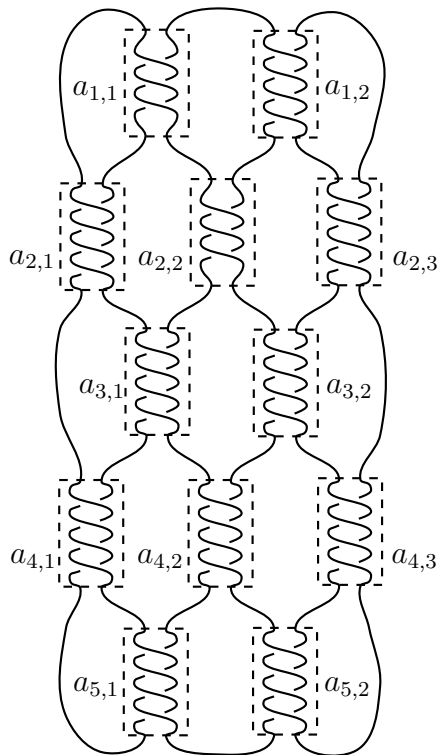


FIGURE 1. A 6-plat projection of a 3-bridge knot.

Notice that a  $2m$ -plat projection  $K$  of  $\mathcal{K}$  corresponds to an element  $\mathcal{B}_{2m}$  in the braid group on  $2m - 1$  generators  $\{\sigma_1, \dots, \sigma_{2m-1}\}$ . This implies that a plat can be given a “standard” form by requiring that the element corresponding to  $\mathcal{B}_{2m}$  be written as a concatenation of sub-words  $\mathcal{B}_{2m} = b_1 \cdot b_2 \cdot \dots \cdot b_{n-1}$ , where  $b_i$  has the following properties:

- (1) When  $i$  is odd,  $b_i$  is a product of all  $\sigma_j$  with  $j$  even. Namely:

$$b_i = \sigma_2^{a_{i,1}} \cdot \sigma_4^{a_{i,2}} \cdot \dots \cdot \sigma_{2m-2}^{a_{i,m-1}}$$

- (2) When  $i$  is even,  $b_i$  is a product of all  $\sigma_j$  with  $j$  odd. Namely:

$$b_i = \sigma_1^{a_{i,1}} \cdot \sigma_3^{a_{i,2}} \cdot \dots \cdot \sigma_{2m-1}^{a_{i,m}}$$

We say  $m$  is the *width* of the plat and  $n$ , which is *even*, is the *length* of the plat.

In Figure 1, there are five rows, so the length is  $n = 6$ . A  $2m$ -plat projection with properties (1) and (2) that also minimizes the width  $m$  will be called *standard*.

A  $2m$ -plat will be called *c-highly twisted* if  $|a_{i,j}| \geq c$  for some constant  $c$ , for all  $i, j$ . Similarly, a knot or link that admits a *c-highly twisted* plat projection will be called a *c-highly twisted knot or link*.

**Definition 2.1.** An  $m$ -bridge sphere of a knot or link  $\mathcal{K} \subset S^3$  is a 2-sphere which meets  $\mathcal{K}$  in  $2m$  points and cuts  $(S^3, \mathcal{K})$  into two  $m$ -string trivial tangles  $(B_1, \mathcal{T}_1)$  and  $(B_2, \mathcal{T}_2)$ . An  $m$ -string *trivial tangle* is a pair  $(B^3, \mathcal{T})$  of 3-ball  $B^3$  and a collection  $\mathcal{T}$  of  $m$  arcs properly embedded in  $B^3$  that are simultaneously isotopic into  $\partial B^3$  fixing  $\partial B^3$ .

**Definition 2.2.** Two bridge spheres  $\Sigma_1$  and  $\Sigma_2$  for a knot or link  $\mathcal{K} \subset S^3$  are said to be *isotopic* if there is a homeomorphism of pairs  $\psi : (S^3, \mathcal{K}) \rightarrow (S^3, \mathcal{K})$  so that  $\psi(\Sigma_1) = \psi(\Sigma_2)$  and  $\psi$  is isotopic to the identity on  $S^3$ .

**Definition 2.3** (Schubert [23] 1956). The *bridge number*  $b(\mathcal{K})$  of a knot or link  $\mathcal{K} \subset S^3$  is the minimal number of over-crossing arcs over all regular projections of  $\mathcal{K}$ . Equivalently, the bridge number is the minimal value of  $m$  such that  $\mathcal{K}$  has an  $m$ -bridge sphere; see [4, p. 23].

Note that every  $2m$ -plat projection defines an  $m$ -bridge knot or link with  $m \geq b(\mathcal{K})$  and every  $m$ -bridge knot or link has a  $2m$ -plat projection (see [4, p. 24]). One can also determine a bridge sphere  $\Sigma$  from a  $2m$ -plat projection as follows. Arrange the maximum and minimum points of the plat projection to lie on horizontal lines at the top and bottom of the diagram, as in Figure 1. Consider a horizontal line in the projection plane  $P$  to lie just below the maximum points, meeting  $K$  in exactly  $2m$  points. Connect endpoints of the line by a simple arc on the projection plane that does not meet  $K$ , forming a closed curve. Then cap this curve by two disks, one in front of the plane  $P$  and one behind  $P$ , to obtain the bridge sphere  $\Sigma$ .

With the diagram arranged in this manner, there is an “obvious” height function on a  $2m$ -plat  $K$ , where the maximum point on the top bridges corresponds to height  $t = 1$  and the minimum point on the bottom bridges corresponds to height  $t = 0$ . Note the bridge sphere  $\Sigma$  defined above corresponds to some height,  $t = 1 - \varepsilon$ . More generally, a *horizontal* bridge sphere  $\Sigma_t$  can be constructed for any height  $t \in (0, 1)$  in the same manner. Note that all bridge spheres  $\Sigma_t$  for  $0 < t < 1$  are isotopic. We can further define horizontal spheres at height  $t = 0$  and  $t = 1$ , but these do not meet  $K$  transversely, and meet it in only  $m$  points.

**Definition 2.4.** Let  $\Sigma_{g,p}$  be a surface of genus  $g$  with  $p$  punctures. A simple closed curve  $\gamma \subset \Sigma_{g,p}$  is *inessential* in  $\Sigma_{g,p}$  if it bounds either a disk or a once punctured disk in  $\Sigma_{g,p}$ . A simple closed curve in  $\Sigma_{g,p}$  is *essential* if it is not inessential. The curve complex  $\mathcal{C}(\Sigma)$  is a simplicial complex defined as follows.

Let  $[\gamma]$  denote the isotopy class of an essential simple closed curve  $\gamma \subset \Sigma$ .

- (1) The set of vertices of  $\mathcal{C}(\Sigma)$  is  $V(\Sigma) = \{[\gamma] \mid \gamma \subset \Sigma \text{ is essential}\}$ .
- (2) An  $n$ -simplex is an  $(n + 1)$ -tuple  $\{[\gamma_0], \dots, [\gamma_n]\}$  of vertices that have pairwise disjoint curve representatives.

Define a metric  $d$  on the 1-skeleton  $\mathcal{C}^1(\Sigma)$  of  $\mathcal{C}(\Sigma)$  by setting the length of each 1-simplex to be 1, and taking minimal path length.

**Definition 2.5.** Given a bridge surface  $\Sigma$  for  $K$ , define  $\Sigma_K = \Sigma \setminus K$ ,  $V_K = B_1 \setminus \mathcal{T}_1$  and  $W_K = B_2 \setminus \mathcal{T}_2$ . Let  $\mathcal{D}(V_K)$  (resp.  $\mathcal{D}(W_K)$ ) be the set of all essential simple closed curves in  $\Sigma_K$  that bound disks in  $V_K$  (resp.  $W_K$ ). Define the (*bridge*) *distance* of  $\Sigma$  to be  $d(\Sigma_K) = d(\mathcal{D}(V_K), \mathcal{D}(W_K))$  measured in  $\mathcal{C}(\Sigma_K)$ .

We now refer to two theorems that provide the basis for this paper. The first is a special case of the main theorem of Tomova for knots [25], extended to links by Johnson and Tomova [16, Theorem 4.4].

**Theorem 2.6** (Tomova [25] and Johnson-Tomova [16]). *Let  $\mathcal{K} \subset S^3$  be a knot or link with two bridge spheres  $\Sigma_1$  and  $\Sigma_2$  meeting  $\mathcal{K}$  in  $2m_1$  and  $2m_2$  points, respectively, where  $m_1$  and  $m_2$  are at least 3. Then either  $\Sigma_2$  is isotopic rel  $\mathcal{K}$  to  $\Sigma_1$  or  $d(\Sigma_1) \leq 2m_2$  or  $d(\Sigma_2) \leq 2m_1$ .*

**Theorem 2.7** (Johnson-Moriah [15]). *Let  $\mathcal{K} \subset S^3$  be a 3-highly twisted link with a  $2m$ -plat projection  $K$ , with  $m \geq 3$  and  $n - 1$  rows of twist boxes. Then the distance  $d(\Sigma)$  of the induced bridge surface  $\Sigma$  is exactly*

$$d(\Sigma) = \lceil n/(2(m-2)) \rceil,$$

where  $\lceil x \rceil$  is the smallest integer greater than or equal to  $x$ .

Combining these two, we obtain:

**Corollary 2.8.** *If the 3-highly twisted  $2m$ -plat  $K$  corresponding to the knot or link  $\mathcal{K} \subset S^3$  has length  $n$ , where  $m \geq 3$  and  $n > 4m(m-2)$ , then the knot  $\mathcal{K}$  is an  $m$ -bridge knot or link, and it has a unique  $m$ -bridge sphere, up to isotopy.*

*Proof.* As  $n > 4m(m-2)$ , Theorem 2.7 implies  $d(\Sigma) > 2m$ . Assume that  $\mathcal{K}$  is an  $m_1$ -bridge knot or link with  $m_1 < m$ . Then Theorem 2.6 implies  $d(\Sigma) < 2m_1 < 2m$ , a contradiction. Further, any other bridge sphere  $\Sigma_1$  meeting  $\mathcal{K}$  in  $2m$  points must be isotopic to  $\Sigma$ .  $\square$

Finally we define the following surfaces first defined in [6]; see also [26].

**Definition 2.9.** Let  $K$  be a  $2m$ -plat projection of a knot or link  $\mathcal{K} \subset S^3$ , with projection plane  $P$ . Assume  $K$  has length  $n$ . Let  $\alpha = \alpha(c_1, \dots, c_{n-1})$  be an arc running monotonically from the top of the plat to the bottom so that  $\alpha$  is disjoint from all twist regions and there are  $c_i$  twist regions to the left of the arc at the  $i$ -th row, and  $\alpha$  intersects  $K$  in precisely  $n$  points. Now connect the endpoints of  $\alpha$  by a simple arc  $\beta \subset P$  that is disjoint from  $K$  so that the simple closed curve  $\gamma = \alpha \cup \beta$  has one maximum and one minimum with respect to the height function. Cap the curve  $\gamma = \alpha \cup \beta$  by two disks, one in front of  $P$  and the other behind  $P$ . Thus, we have obtained a 2-sphere  $S = S(c_1, \dots, c_{n-1})$ .

If there is at least one twist region on each side of  $\alpha$  at each level, the 2-sphere  $S(c_1, \dots, c_{n-1})$  is called a *vertical 2-sphere*; see Figure 2.

We say that a 2-sphere  $S$  embedded in  $(S^3, \mathcal{K})$  is *compressible* or *boundary compressible* if  $S \setminus \mathcal{N}(\mathcal{K})$  is.

**Theorem 2.10** (Finkelstein-Moriah [7] and Wu [26]). *Let  $\mathcal{K} \subset S^3$  be a 3-highly twisted link with a  $2m$ -plat projection  $K$ , with  $m \geq 3$  and length  $n \geq 5$ , and let  $S$  be a vertical 2-sphere for  $K$ . Then  $S \setminus \mathcal{N}(\mathcal{K})$  is incompressible and boundary incompressible.*



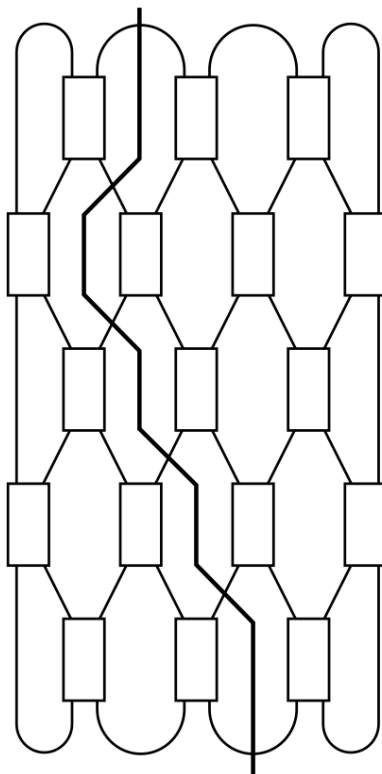


FIGURE 2. A template for a 4-plat with  $n = 6$  and an  $\alpha$  arc.

**Remark 2.11.** A vertical 2-sphere separates a knot or link  $\mathcal{K}$  in a plat projection into two  $n/2$  string tangles  $(B_1, \mathcal{T}_1)$ ,  $(B_2, \mathcal{T}_2)$ . These tangles are non-trivial by Theorem 2.10. We always assume that the projection plane and vertical spheres intersect transversally.

Vertical 2-spheres are required to have a twist region on either side on each row of twist regions. We will also need to work with similar 2-spheres that relax this restriction in the following way:

**Definition 2.12.** Let  $S = S(c_1, \dots, c_{n-1})$  and  $\alpha$  be as in Definition 2.9. If there is at least one twist region on each side of  $\alpha$  at each level except possibly level  $i$  for  $i$  odd, or levels  $i$  and  $i + 2$  for  $i$  odd, or levels  $i - 1$ ,  $i$ , and  $i + 1$  for  $i$  even, we say that  $S$  is an *almost vertical 2-sphere*. Note that the set of almost vertical 2-spheres contains the set of vertical 2-spheres.

### 3. ISOTOPIES

If a knot or link  $\mathcal{K} \subset S^3$  has two plat projections  $K$  and  $K'$  with respect to projection planes  $P$  and  $P'$  respectively, we can think of  $K$  and  $K'$  as embedded in an  $\varepsilon$  neighborhood of  $P$  and  $P'$  respectively. Because they are projections of the same link,

there is an ambient isotopy  $\varphi : (S^3, K') \rightarrow (S^3, K)$  taking  $K'$  to  $K$ , and taking vertical 2-spheres  $S' \subset (S^3, K')$  to spheres  $S = \varphi(S') \subset (S^3, K)$ .

**Remark 3.1.** Note that throughout the paper, objects and parameters, such as surfaces and twist coefficients relevant to the pair  $(S^3, K')$  of the  $K'$  plat will be denoted by a ‘ $'$  e.g.  $S' \subset (S^3, K')$ . Those relevant to  $(S^3, K)$  will have no extra mark, e.g.  $S \subset (S^3, K)$ .

**Lemma 3.2.** *Let  $K', K$  be two  $2m$ -plat projections of the same knot or link  $\mathcal{K} \subset S^3$ , where  $m \geq 3$  and  $n'$ , the length of  $K'$ , is greater than  $4m(m - 2)$ . Assume that  $K'$  is 3-highly twisted. Let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy. Then there exists an isotopy of pairs  $\psi : (S^3, K) \rightarrow (S^3, K)$  with the following properties.*

- (a) *Let  $S'$  be any almost vertical 2-sphere for  $K'$ , and let  $S = \varphi(S')$  be its image. The image  $\psi(S)$  is a 2-sphere meeting each horizontal bridge sphere of  $K$  in a single essential curve, so that any such curve meets  $K$  at most once.*
- (b) *For each horizontal bridge sphere  $\Sigma'_t$  for  $K'$ , the image  $(\psi \circ \varphi)(\Sigma'_t)$  is a horizontal bridge sphere for  $K$ .*

*Proof.* The lemma follows from Corollary 2.8, which states that the bridge sphere is unique up to isotopy, as follows:

The link space  $(S^3, K')$  is foliated by horizontal bridge spheres. That is, the link  $K'$  is contained in a region  $S^2 \times [0, 1] \subset S^3$  in which each  $S^2 \times \{t\}$  is a bridge sphere  $\Sigma'_t$ , with the exception of  $t = 0$  and  $t = 1$ , where the spheres meet the diagram in  $m$  points. By the definition of an almost vertical 2-sphere, each horizontal sphere  $\Sigma'_t$  intersects  $S'$  in a single essential curve, and no such curve meets  $K'$  more than once.

Now the isotopy  $\varphi : (S^3, K') \rightarrow (S^3, K)$  takes this foliation of  $K'$  to a foliation of  $K$  by bridge spheres. Note that images  $\varphi(S')$  intersect images  $\varphi(\Sigma'_t)$  in a single essential curve. Because the bridge sphere is unique up to isotopy, each level  $\varphi(\Sigma'_t)$  is isotopic to a horizontal bridge sphere for  $K$ , under an isotopy fixing  $K$ . The images of  $\varphi(S')$  and  $\varphi(\Sigma'_t)$  will still meet in single curves under this isotopy. Thus we need to ensure that the bridge spheres  $\varphi(\Sigma'_t)$  can be simultaneously isotoped to the horizontal foliation of  $K$ . We do this by an ‘‘open-closed’’ argument:

Let  $I \subset (0, 1)$  be the sub-interval containing  $1/2$  for which we may simultaneously isotope  $\Sigma'_t$  to  $\Sigma_t$  for any  $t \in I$ . Then  $I$  is nonempty by Corollary 2.8: We may isotope  $\Sigma'_{1/2}$  to  $\Sigma_{1/2}$ .

The interval  $I$  is open, because if  $t \in I$ , then for small enough  $\varepsilon > 0$ , the image  $\varphi(\Sigma' \times (t - \varepsilon, t + \varepsilon))$  is contained in some  $\Sigma \times (t - \delta, t + \delta)$ , taking  $\Sigma'_t$  to  $\Sigma_t$ . There is an isotopy of the ball with boundary  $\Sigma_t$  keeping  $\Sigma_t$  fixed and taking the small neighborhood  $\varphi(\Sigma' \times [t, t + \varepsilon])$  of  $\Sigma_t$  to a small fibered neighborhood. By stretching or contracting, we may assume  $\varphi(\Sigma' \times \{s\})$  goes to  $\varphi(\Sigma \times \{s\})$ . Similarly for  $\Sigma' \times (t - \varepsilon, t]$ . So  $I$  is open.

Moreover,  $I$  is closed, for if  $t_n$  is a sequence in  $I$  with limit  $t$ , then  $\varphi(\Sigma'_{t_n})$  is contained in  $\Sigma_{t_n}$ ; by continuity  $\varphi(\Sigma'_t)$  lies on  $\Sigma_t$ . So  $I = (0, 1)$ .

Thus,  $\varphi(S^2 \times (0, 1))$  can be isotoped to be horizontal under an isotopy of pairs  $\psi : (S^3, K) \rightarrow (S^3, K)$ . Now consider the images of  $\varphi(S^2 \times \{0\})$  and  $\varphi(S^2 \times \{1\})$  under this isotopy. They each meet  $K$  in  $m$  points. Moreover,  $K$  is contained in the image  $(\psi \circ \varphi)(S^2 \times [0, 1])$ . Thus  $(\psi \circ \varphi)(S^2 \times \{0\})$  and  $(\psi \circ \varphi)(S^2 \times \{1\})$  must also be horizontal. Under this isotopy, the image of  $S'$  is taken to  $S$ , satisfying the conclusions of the lemma.  $\square$

Let  $S_{i,j}$  denote a 4-punctured sphere embedded in  $S^3 \setminus K$  that encloses the twist region  $t_{i,j}$ . Each  $S_{i,j}$  is chosen so that the collection of all such 4-punctured spheres over all pairs  $(i, j)$  is disjoint. Let  $B_{i,j}$  denote the bounded closed ball in  $S^3$  whose boundary contains  $S_{i,j}$ . Thus,  $B_{i,j} \subset S^3$  contains the  $1/a_{i,j}$  tangle determined by the  $t_{i,j}$  twist region of  $K$ .

**Lemma 3.3.** *With the hypotheses on  $K'$  and  $K$  from Lemma 3.2, let  $S'$  be a fixed vertical 2-sphere. Then there exists an isotopy  $\psi_2 : (S^3, K) \rightarrow (S^3, K)$  that preserves properties (a) and (b) of that lemma, and in addition does the following.*

- (c) *For all  $i, j$ , it takes  $S$  to a surface either disjoint from  $B_{i,j}$  or meeting  $B_{i,j}$  in disjoint disks, each separating the two strands of  $K$  within  $B_{i,j}$ , and each disjoint from  $K$ . Note this implies all points of  $S \cap K$  lie outside of twist regions.*

The proof of the lemma uses incompressibility and boundary incompressibility to slide surfaces past each other. However, the argument is not quite standard, because we need to take care to ensure that isotopies preserve horizontal bridge spheres at every level.

*Proof of Lemma 3.3.* Assume we have already isotoped  $(S^3, K)$  as in Lemma 3.2, and denote the composition of isotopies from  $(S^3, K')$  to  $(S^3, K)$  to  $(S^3, K)$  by  $\varphi$ .

Recall that  $S_{i,j} = \partial B_{i,j}$ . Consider the intersection of spheres  $S \cap S_{i,j}$  for all  $\{i, j\}$ . The components are disjoint simple closed curves embedded both in  $S$  and in  $S_{i,j}$ . Each  $S_{i,j}$  meets  $K$  four times. By a general position argument, we may assume each component of  $S \cap S_{i,j}$  is disjoint from  $K$ . Then on  $S_{i,j}$ , such a component bounds disks on either side.

**Step 1.** Suppose a component of  $S \cap S_{i,j}$  bounds a disk on  $S_{i,j}$  that is disjoint from  $K$ . Then an innermost such component bounds an embedded open disk  $E \subset S_{i,j}$  disjoint from  $S$  and  $K$ . Because  $S$  is incompressible (by Theorem 2.10, restricting to the case  $S'$  is a vertical 2-sphere),  $\partial E$  must also bound an embedded disk in  $S$ ; call it  $D$ . The union of  $E$  and  $D$  is a sphere in  $S^3$  disjoint from  $K$ , hence it bounds a ball  $B$ . We will use the ball to isotope  $S$  through  $S_{i,j}$ , removing the intersection; however we must take some care to preserve the conclusion of Lemma 3.2.

If  $\partial E$  lies on a single horizontal level, i.e.  $\partial E \subset \Sigma_t$  for some  $t$ , then the curve  $\partial E$  must be exactly the curve  $S \cap \Sigma_t$ , since  $S$  meets any level in a single simple closed curve by Lemma 3.2. Then the disk  $D \subset S$  must also meet each level in a single simple closed curve or a single point. We may isotope  $D$  by pushing vertically to  $E$ , removing all intersections with levels above (or below)  $\Sigma_t$ , and then finally pushing off  $E$  to remove the intersection  $S \cap S_{i,j}$ , preserving the conclusions of Lemma 3.2.

So suppose  $\partial E$  does not lie in a single level, but lies in a region  $\Sigma \times [a, b]$  for a minimal interval  $[a, b]$ . Note  $E \cap \Sigma_a$  is a finite collection of points; by isotoping  $S$  slightly in a level-preserving manner in a small neighborhood of all but one of the points of  $E \cap \Sigma_a$ , we may assume  $E \cap \Sigma_a$  is a single point. By continuity, either  $D \cap \Sigma_a$  is also a single point, or it is a simple closed curve bounding a disk in  $B$ . In the latter case, an argument similar to that of the previous paragraph implies we may isotope  $S$ , preserving conclusions of Lemma 3.2, to be disjoint from levels below  $\Sigma_a$ . Similarly, we may isotope  $S$  so that  $D$  is disjoint from levels above  $\Sigma_b$ . Thus we may assume that  $D$  lies in the region  $\Sigma \times [a, b]$ . Then the ball  $B$  also lies in this range. For each level,  $B \cap \Sigma_t$  is a disk, and we may isotope  $S$  horizontally through each such disk to remove the intersection  $S \cap S_{i,j}$  while preserving the conclusions of Lemma 3.2. Repeating this a finite number of times, we remove all intersections  $S \cap S_{i,j}$  that bound a disk in  $S_{i,j}$  disjoint from  $K$ .

**Step 2.** Suppose now a component of intersection of  $S \cap S_{i,j}$  bounds a disk in  $S_{i,j}$  that meets  $K$  exactly once. Again we will argue that we can isotope the intersection away, preserving the conclusions of Lemma 3.2. For this argument, we need to use the fact that a vertical 2-sphere is *meridionally incompressible*. That is, if a disk  $E$  is embedded in  $S^3$  with  $\partial E \subset S$  and  $E$  meets  $K$  in a single meridional curve on  $K$ , then  $\partial E$  bounds a disk on  $S$  with the same property: it meets  $K$  exactly once in a meridional curve. This result cannot be found in the literature, however its proof is nearly identical to Wu's proof that vertical 2-spheres are essential [26]. We include it in the appendix, Section 5.

Assuming this fact, the proof in this step proceeds as above. If  $E \subset S_{i,j}$  is a disk meeting  $K$  in a single puncture, then the fact that  $S$  is meridionally incompressible implies that  $\partial E$  also is the boundary of a disk  $D \subset S$  meeting  $K$  in a single puncture. Then  $D \cup E$  bounds a ball in  $S^3$  meeting  $K$  exactly twice in meridians. The conditions on the knot imply (by Theorem 2.7) that  $d(\Sigma) \geq n/2 \geq 2n(m-2) > 7$ . So applying ([1, Corollary 6.2]) of Bachman and Schleimer we conclude that  $\mathcal{K}$  is hyperbolic and hence prime. Therefore  $D \cup E$  is a boundary parallel annulus in the knot complement. Consider the intersection of the ball with horizontal levels. An argument similar to the previous step implies that we may isotope through horizontal levels, preserving the conclusions of Lemma 3.2, removing intersections  $S \cap S_{i,j}$  that bound a disk in  $S_{i,j}$  meeting  $K$  once.

**Step 3.** At this stage, we may assume that all intersections in  $S \cap S_{i,j}$  are curves on  $S_{i,j}$  bounding disks with exactly two punctures on either side. Because  $S$  is embedded, if there are multiple components of  $S \cap S_{i,j}$  then all such intersection curves must be parallel on  $S_{i,j}$ . Then it follows that  $S \cap B_{i,j}$  is either empty, all of  $S$ , or a collection of disks and annuli in  $S$ : Note the fact that  $S \cap S_{i,j}$  consists of parallel curves implies that  $S \cap B_{i,j}$  cannot consist of planar surfaces in  $S$  with more than two boundary components.

First note  $S \cap B_{i,j} \neq S$ , for in this case,  $S$  is a punctured 2-sphere contained entirely in the tangle  $B_{i,j}$ . There is a disk embedded vertically in the tangle that separates the two strands of  $K$ ; denote this disk by  $D$ . If we cut along  $D$ , we obtain two trivial

tangles, each containing a single strand of  $K$ , which we will denote by  $B_1$  and  $B_2$ . The disk  $D$  must intersect  $S$  in simple closed curves. An innermost component of intersection bounds a disk on  $D$ , hence it bounds a disk on  $S$ . Thus it can be isotoped away in a level-preserving manner as above; we conclude that  $S$  is disjoint from  $D$ . But then  $S$  is an essential sphere with  $n$  punctures embedded in a trivial tangle  $B_i$  with a single knot strand. This is impossible.

Now we show that we can rule out annulus components of  $S \cap B_{i,j}$ . For suppose  $A$  is an annulus of  $S \cap B_{i,j}$ . Then  $\partial A$  bounds an annulus  $A'$  on  $S_{i,j}$ , and together  $A$  and  $A'$  bound a solid torus  $V$  in  $S^3$ . If  $V$  is disjoint from  $K$ , then we may isotope  $A$  through  $V$  in a level-preserving manner (as above), to  $A'$  and slightly past, removing two components of intersection  $S \cap S_{i,j}$ . So  $K$  must meet  $A$ .

Now consider intersections of the disk  $D$  and tangles  $B_1$  and  $B_2$  with the annuli and solid torus. If  $D$  is disjoint from  $A$  and  $A'$ , then  $A \cup A'$  and the solid torus  $V$  lie entirely in a single trivial tangle, say  $B_1$ . We are assuming  $A$  meets  $K$ . Then it must meet  $K$  at least twice in  $B_1$ . But  $K$  is unknotted in  $B_1$ , so between some two points of entry and exit of  $K$ , we have a boundary compressing disk for  $A$ , hence  $S$ . But  $S$  is boundary incompressible, Theorem 2.10, so this is a contradiction.

If  $A'$  is disjoint from  $D$ , but  $A$  meets  $D$ , then  $\partial A'$  must be parallel to  $\partial D$ , hence  $A$  meets  $D$  only in simple closed curves. An innermost curve on  $D$  bounds a disk in  $D$ . By incompressibility of  $S$ , it bounds a disk in  $S$  as well, and as in Step 1 we may isotope it away.

So suppose  $A'$  meets  $D$ . Then  $\partial D$  intersects  $A'$  nontrivially. A component of intersection of  $\partial D \cap A'$  cannot be an arc from one boundary component of  $A'$  to itself, because such an arc bounds a disk in  $A'$ ; we may isotope away an innermost such arc in a level-preserving manner. A component of  $\partial D \cap A'$  could be an essential simple closed curve in  $A'$ . Then, after isotopy, we may assume  $D$  meets  $A$  in an essential simple closed curve. But an innermost such curve bounds a disk on  $D$ , and as above we may isotope it away. So a component of  $\partial D \cap A'$  must be an essential arc from one side of  $\partial A'$  to the other. Then  $D$  cuts  $A'$  into disks, and after level-preserving isotopy, cuts the solid torus into balls, each embedded in a trivial one-tangle. Again intersections with  $K$  give boundary compression disks for  $S'$ , a contradiction.

**Step 4.** So the only remaining case is that  $S \cap B_{i,j}$  consists of disk components, each with boundary encircling two punctures of  $K$  on  $S_{i,j}$ .

Let  $E$  be a disk component of  $S \cap B_{i,j}$ . If  $E$  is disjoint from  $K$ , then it must separate the two strands of  $K$  and we are done. So suppose  $E$  is not disjoint from  $K$ . Then consider its intersections with the disk  $D$  that divides  $B_{i,j}$  into one-tangles  $B_1$  and  $B_2$ . If  $E$  is disjoint from  $D$ , then  $E$  lies entirely in one of the tangles  $B_1$  or  $B_2$ , say  $B_1$ . Because  $\partial E$  encircles two punctures of  $K$  on  $S_{i,j}$ , it must run parallel to  $\partial D$ . Then if  $K$  intersects  $E$ , it must do so in two points in  $B_1$ . Then we may form a boundary compressing disk for  $S$  by connecting adjacent points of intersection. Again this contradicts Theorem 2.10. So  $E$  cannot be disjoint from  $D$ .

Now  $D \cap E$  consists of arcs and closed curves of intersection. Suppose first there is a closed curve of intersection  $D \cap E$ . Then there is one which is innermost in  $D$ . This curve bounds a disk  $D' \subset D$ , and hence by incompressibility of  $S$  bounds a disk in  $S$ .

Again as in Step 1 we may isotope away this intersection in a level-preserving manner. So we may assume there are no closed curves of intersection, and thus  $D \cap E$  consists only of arcs with both endpoints on  $S_{i,j}$ . Then intersections of  $S$  with  $B_1$  and  $B_2$  are disks whose boundary consist of arcs of  $D \cap E$  and arcs on  $S_{i,j}$ . The disks cut  $B_1$  and  $B_2$  into balls. Any such ball must contain an arc of  $K$ , else we can use it to form an isotopy removing an intersection of  $S \cap D$ , and as above we may ensure the isotopy is horizontal with respect to bridge spheres, i.e. preserves the conclusions of Lemma 3.2. Again no disk of  $E \cap B_1$  meets  $K$  more than once, otherwise we can obtain, as before, a boundary compression disk for  $S'$ , contradicting Theorem 2.10. Similarly for  $E \cap B_2$ . Thus, for each  $\ell = 1, 2$ , each disk of intersection of  $E \cap B_\ell$  meets  $K$  exactly once, with boundary consisting of an arc on  $D$  and an arc on  $S_{i,j}$ . It follows that  $E$  is a disk meeting  $K$  exactly twice, meeting  $D$  in a single arc, and  $S_{i,j} \cap E$  separates the four punctures of  $S_{i,j}$  into pairs.

As before, we will slide intersections of  $S$  with  $B_{i,j}$  up or down, reducing the number of such intersections. However, we need to take some care to ensure this can be done while preserving the conclusions of Lemma 3.2. Assume the row of twist regions runs from height  $a$  to  $b$ , i.e. the horizontal bridge sphere  $\Sigma_a$  lies just below the twist regions, and the horizontal bridge sphere  $\Sigma_b$  lies just above.

Let  $t \in [a, b]$  be the minimal value such that  $\Sigma_t \cap S \cap S_{i,j} \neq \emptyset$ . If  $t \neq a$ , then by stretching and compressing vertically, there is an isotopy of  $(S^3, K)$  taking the sub-manifold  $\Sigma \times (a, t)$  to the sub-manifold  $\Sigma \times (a - \varepsilon, a)$ , keeping  $\Sigma_b$  fixed. The isotopy preserves the conclusions of Lemma 3.2, while taking the lowest intersection point  $S \cap S_{i,j}$  to a point at the very bottom of that ball  $B_{i,j}$ . Then  $S \cap B_1$  contains a disk  $E'$  with boundary an arc on  $S_{i,j}$  and an arc on  $D$ , meeting  $\Sigma_a$ . The arc on  $D$  bounds a disk  $D'$  with boundary on  $S_{i,j}$  and on  $D$  meeting  $\Sigma_a$ . The union  $D' \cup E'$  is a disk with boundary on  $S_{i,j}$ ; it bounds a disk  $F$  on  $S_{i,j}$  meeting a single puncture of  $S_{i,j}$ . Then  $F \cup D' \cup E'$  is a sphere, bounding a ball  $B$  in  $S^3$  with the property that  $B \cap \Sigma_t$  is either empty, or is a disk whose boundary contains a single arc  $\alpha$  on  $S$ , and an arc  $\beta$  on  $S_{i,j}$  and  $D$ . Use this ball to isotope  $S$  horizontally to  $D$ , and then slightly past, removing an arc of intersection with  $D$ . Now we are in previous Step 2, and the result follows.  $\square$

We now use the previous lemmas to show that  $\varphi(S')$  does not meet twist regions.

**Proposition 3.4.** *Let  $K'$  and  $K$  be two  $2m$ -plat projections of the knot or link  $\mathcal{K} \subset S^3$ , where  $m \geq 3$  and the length of  $K'$  is greater than  $4m(m - 2)$ . Assume  $K'$  is 3-highly twisted and  $K$  is 1-highly twisted. Suppose  $S'$  is a vertical 2-sphere for  $K'$ . Then the isotopy of Lemma 3.3 takes  $S'$  to a 2-sphere  $S$  that does not meet any twist regions.*

*Proof.* Let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy satisfying the conclusions of Lemmas 3.2 and 3.3 for  $S'$ . Then  $S = \varphi(S')$  meets each horizontal level  $\Sigma_t$  in  $K$  in a single simple closed curve,  $S$  intersects  $K$  outside of twist regions, and if  $S$  does intersect a twist region, it runs between the two strands of the twist region.

Suppose that  $S$  runs through a twist region. Let  $\Sigma_c$  and  $\Sigma_d$  be horizontal bridge spheres just below and above the twist region, respectively. The region between  $\Sigma_c$

and  $\Sigma_d$  is homeomorphic to  $\Sigma \times [c, d]$ , where  $\Sigma$  is a  $2m$ -punctured disk. Because  $S$  does not meet  $K$  in twist regions, we can choose  $c$  and  $d$  so that  $S$  does not meet  $K$  in the region  $\Sigma \times [c, d]$ . There is an isotopy  $\zeta$  from  $\Sigma_c$  to  $\Sigma_d$  given by sweeping through the horizontal bridge spheres. The isotopy can be arranged to be the identity away from disks bounding points of  $K$  in a twist region. It is a nontrivial Dehn twist, twisting an amount equal to the number of crossings in the twist region, inside a disk bounding points of  $K$  in a twist region. In particular, the isotopy  $\zeta$  takes the simple closed curve  $\gamma = S \cap \Sigma_c$  in  $\Sigma \times \{c\}$  to a distinct curve  $\tau(\gamma)$  in  $\Sigma \times \{d\}$ , given by a Dehn twist.

Now consider the effect of  $\varphi^{-1}$  on  $\Sigma \times [c, d]$ . The isotopy takes this to a region  $\Sigma' \times [a, b]$ , with  $\Sigma'_a$  the image of  $\Sigma_c$ , and  $\Sigma'_b$  the image of  $\Sigma_d$ . The portion of the surface  $S$  between levels  $c$  and  $d$  is taken to a portion of the surface  $S'$  between levels  $a$  and  $b$ . Note that the surface  $S'$  does not meet any twist regions, and moreover, since  $S$  avoids  $K$  in this region,  $S'$  must avoid  $K'$  in this region.

But again, there is an isotopy  $\zeta'$  from  $\Sigma'_a$  to  $\Sigma'_b$  given by sweeping through horizontal bridge spheres, and again this isotopy can be taken to be the identity away from twist regions. Because  $S'$  does not meet any twist regions, it must take the curve  $\gamma' = S' \cap \Sigma'_a$  on  $\Sigma' \times \{a\}$  to the same curve  $\zeta'(\gamma') = \gamma'$  on  $\Sigma' \times \{b\}$ .

On the other hand, we know

$$\tau(\gamma) = \varphi \circ \zeta' \circ \varphi^{-1}(\gamma) = \varphi \circ \zeta'(\gamma') = \varphi(\gamma') = \gamma.$$

But this implies that the Dehn twist  $\tau$  is trivial. This contradicts the fact that  $K$  is 1-highly twisted.  $\square$

**Remark 3.5.** Note that the argument in the previous lemma uses in a crucial way the fact that the diagram of  $K'$  is fixed, and the twist regions of the diagram are all reduced. In general, an isotopy between bridge surfaces in an arbitrary diagram could first spin around a twist region, then spin in the opposite direction, undoing the twists in that twist region. However, this would result in a twist region that is not reduced. The map  $\zeta'$  above is completely determined by the reduced diagram, and cannot create crossings that can be removed in a different level.

**Remark 3.6.** In the previous lemmas, we have obtained isotopies giving nice properties for  $K$  and a fixed  $S$ . From now on, compose the initial isotopy  $\varphi : (S^3, K') \rightarrow (S^3, K)$  with these isotopies, and denote the composition by  $\varphi$ . Thus we assume that  $\varphi$  satisfies the conclusions of previous lemmas for a fixed  $S'$ .

**Corollary 3.7.** *If  $S'$  is a vertical sphere, and  $S = \varphi(S')$ , then the intersection of  $S$  with the projection plane  $P$  contains precisely one simple closed curve  $\delta$  so that  $\delta \cap K \neq \emptyset$  and contains all  $n$  points of intersection with  $K$ .*

*Proof.* We can assume that the knot or link  $K$  is contained in an  $\varepsilon/2$  neighborhood of  $P$ , i.e.  $K \subset P \times [-\varepsilon/2, \varepsilon/2]$ , and that  $S$  intersects  $P$  transversally. If there is more than one simple closed curve in the intersection  $S \cap P$  which meets  $K$  then consider the annuli  $S \cap (P \times [-\varepsilon, \varepsilon])$ . These annuli can be capped off by disks behind and in front of  $P$  at the  $P \times -\varepsilon$  and  $P \times \varepsilon$  level which do not meet  $K$ . This fact is preserved by the

inverse isotopy  $\varphi^{-1}$  which sends the disks to disks in  $S^3 \setminus \mathcal{N}(K')$ . One of these disks must be essential since it separates intersection points with  $K$ . Thus it is a compressing disk for  $S'$ , in contradiction to Theorem 2.10.  $\square$

**Lemma 3.8.** *Let  $K', K$  be two  $2m$ -plat projections of the same knot or link  $\mathcal{K} \subset S^3$ , where  $m > 3$  and the length  $n'$  of  $K'$  is greater than  $4m(m-2)$ . Moreover, assume that  $K'$  is 3-highly twisted and  $K$  is 1-highly twisted. Let  $S'$  be a vertical 2-sphere for  $K'$ . Let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy (satisfying conclusions of isotopies above), with  $S = \varphi(S')$ . Then  $S$  meets the diagonal segments of  $K$  in any level between twist regions at most once. In particular,  $S$  cannot meet a segment of  $K$  in a point on the outside of an odd level if it has an additional intersection point on either of the two levels that the outside arc spans.*

*Proof.* First, by Lemma 3.2, we may assume that each horizontal bridge sphere for  $K$  meets  $S$  in a single essential curve, and each such curve meets  $K$  at most once. Also by that lemma, we may assume that the isotopy  $\varphi$  takes horizontal bridge spheres for  $K'$  to horizontal bridge spheres for  $K$ .

Consider now the diagonal segments of  $K$  between the  $i$ -th and  $(i+1)$ -st row of twist regions, and assume in contradiction that  $S$  intersects  $K$  more than once in this region. Let  $p_1$  and  $p_2$  be two points of intersection. Then  $p_1$  lies on a horizontal bridge sphere  $\Sigma_u$ , and  $p_2$  lies on a horizontal bridge sphere  $\Sigma_w$ . We may assume  $u > w$ , and without loss of generality, we may assume that the difference  $|u - w|$  is minimal among all such points and bridge spheres. That is, assume there are no points of intersection of  $S \cap K$  between  $\Sigma_u$  and  $\Sigma_w$ .

Now consider the preimage of the region between  $\Sigma_u$  and  $\Sigma_w$  under the map  $\varphi$ . Both  $\Sigma_u$  and  $\Sigma_w$  are mapped to horizontal bridge spheres under  $\varphi^{-1}$ , and the region  $\Sigma \times [u, w]$  between them is mapped to a region  $\varphi^{-1}(\Sigma \times [u, w])$  foliated by horizontal bridge spheres. Since  $\varphi^{-1}(\Sigma_u)$  and  $\varphi^{-1}(\Sigma_w)$  meet  $S' \cap K'$ , they must lie on different levels, separated by twist regions. Then the isotopy given by the horizontal foliation of  $\varphi^{-1}(\Sigma \times [u, w])$ , coming from the fixed, reduced diagram of  $K'$ , gives a nontrivial homeomorphism from  $\varphi^{-1}(\Sigma_u)$  to  $\varphi^{-1}(\Sigma_w)$ ; it is nontrivial because moving through the twist regions performs Dehn twists on the bridge sphere. Also note that sweeping above or below the interval does not cancel these Dehn twists, as the diagram of  $K'$  is reduced.

On the other hand, the isotopy given by the horizontal foliation of  $\Sigma \times [u, w]$  in  $K$ , coming from the diagram of  $K$ , just gives the identity homeomorphism between  $\Sigma_u$  and  $\Sigma_w$ , since  $p_1$  and  $p_2$  lie in the same region, so there are no twist regions between  $\Sigma_u$  and  $\Sigma_w$ . But this is a contradiction: the homeomorphism must be the same in both  $S^3 \setminus K$  and  $S^3 \setminus K'$ .

Finally, note that if  $S$  meets a segment of  $K$  on the outside of an odd level, then we may isotope  $S$  vertically, preserving the conclusions of Lemmas 3.2 and 3.3, to meet either the diagonal segment above the odd row or the one below, so the above argument again gives a contradiction if  $S$  meets either of these levels elsewhere.  $\square$



4. IMAGES OF PLATS

4.1. **Images of vertical 2-spheres.** In this subsection we prove the following theorem, which allows us to control the images of almost vertical 2-spheres.

**Theorem 4.1.** *Let  $K', K$  be  $2m$ -plat projections of the same knot or link  $\mathcal{K}$ . Assume  $m \geq 3$ , the length  $n'$  of  $K'$  satisfies  $n' > 4m(m - 2)$ , and  $K'$  is 3-highly twisted. Then we may take the isotopy  $\varphi : (S^3, K') \rightarrow (S^3, K)$  to map all almost vertical 2-spheres  $S' \subset (S^3, K')$  to almost vertical 2-spheres  $S \subset (S^3, K)$  simultaneously.*

In order to prove the theorem, we will be discussing different regions of the plat diagram, and how arcs meet these regions. For ease of reference, we give regions of the diagram the following labels.

**Definition 4.2.** Let  $K \subset S^3$  be a knot or link with a  $2m$ -plat projection. Enclose every twist region in a box which we also denote by  $t_{i,j} = (\text{twist region}) \times [-\varepsilon, \varepsilon]$ . Then  $P \setminus (K \cup (\cup_{i,j} t_{i,j}))$  is a collection of regions, as in Figure 2.

The regions of the graph come in four different types up to symmetry:

- (1) A unique unbounded region, denoted by  $\mathcal{U}$ .
- (2) Regions with four edges that are segments of  $K$ . Such a region will be called a *generic region*, denoted by  $\mathcal{Q}$ .
- (3) Regions with three edges that are segments of  $K$ . They appear on the top or bottom or on the left-most or right-most sides of the plat, but not in a corner. These regions will be called *triangular* and denoted by  $\mathcal{T}^t, \mathcal{T}^b, \mathcal{T}^r$ , and  $\mathcal{T}^l$  respectively.
- (4) Four bigon regions which appear in the corners and are denoted  $\mathcal{B}^{t,l}, \mathcal{B}^{t,r}, \mathcal{B}^{b,l}$  and  $\mathcal{B}^{b,r}$ .

Let  $\Delta$  be the portion of the projection plane  $P$  outside the unbounded region. That is  $\Delta = P \setminus \mathcal{U}$ .

**Lemma 4.3.** *Let  $S'$  be a vertical 2-sphere for  $K'$  and let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy satisfying the conclusions of Lemmas 3.2 and 3.3 for  $S'$ . The image  $S = \varphi(S')$  must intersect each row of twist regions so that it has at least one twist region on each side in that row.*

*Proof.* By Lemma 3.2, we may take the isotopy so that  $S$  intersects horizontal bridge spheres in each row of twist regions. By Proposition 3.4, it avoids all twist regions. By Corollary 3.7,  $S$  meets  $P$  in a single simple closed curve which, by Lemma 3.8, meets diagonals on each level no more than once. Note this implies that  $S$  consists of exactly one arc in  $\mathcal{U}$  and one arc in  $\Delta$ , for otherwise  $S$  would exit  $\Delta$  by crossing an arc of the diagram on the outside of an odd level, as well as one of the two diagonal levels spanned by that outside arc, contradicting Lemma 3.8.

Assume that the image  $S$  does not have one twist region on both sides of the vertical arc  $\alpha = S \cap \Delta$  on some level. Without loss of generality we can assume that there is no twist region to the right side of the arc  $\alpha$  on that level. If the level is even, then there is a disk with boundary on  $S$  contained in the unbounded face. If such a disk exists, then its image under  $\varphi^{-1}$  is a disk on a horizontal bridge sphere  $\Sigma'_t$  for some  $t \in [0, 1]$ , and boundary on  $S'$ . But for all  $t \in [0, 1]$ ,  $S' \cap \Sigma'_t$  bounds points of  $K'$  on both sides. So this cannot happen.

It remains to consider an odd level such that the arc  $\alpha$  has no twist regions on the right side in this level. The only way this can happen is if the arc  $\alpha$  passes through some region  $\mathcal{T}^r$ , or one of the four  $\mathcal{B}$ 's, as in Definition 4.2. Suppose first that  $\alpha$  meets  $\mathcal{B}$ . Then  $\alpha$  must meet the segment of  $K$  forming a bridge, and the segment of  $K$  between twist regions. But we may isotope  $S$  along the bridge, preserving the conclusions of Lemmas 3.2 and 3.3 and Proposition 3.4, moving the intersection on the bridge to lie on the level between the first and second rows of twist regions. Then we obtain a contradiction to Lemma 3.8. So  $\alpha$  meets a region  $\mathcal{T}^r$ .

Let  $\beta \subset K$  denote the rightmost arc in  $\mathcal{T}^r$ . Let  $T_r$  denote the tangle in the component of  $S^3 \setminus S$  which contains  $\beta$ . The tangle  $T_r$  contains a *cutting point*  $c$ : a point  $c$  on the sub-arc  $\beta$  so that there is a disk  $D$ , meeting  $c$ , embedded in  $T_r$  that meets  $K$  in a single point. Moreover, we can choose the disk  $D$  to lie in a horizontal bridge sphere for  $K$ , so that the boundary of  $D$  is the intersection of that horizontal bridge sphere with  $S$ .

Now consider the preimage  $S'$  of the sphere  $S$ . Since horizontal bridge spheres go to horizontal bridge spheres, the preimage  $D'$  of  $D$  is a disk on a horizontal bridge sphere for  $K'$  with  $\partial D'$  on  $S'$ . There is no such disk which intersects  $K'$  in a single point.  $\square$

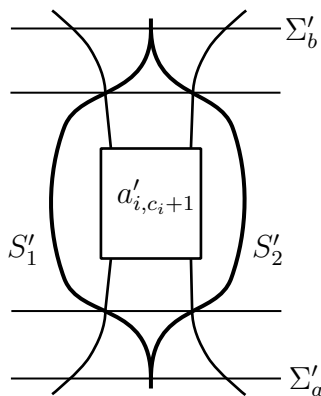
**Proposition 4.4.** *Let  $K'$  and  $K$  be two  $2m$ -plat projections of the same knot or link  $K \subset S^3$  such that  $m \geq 3$  and  $K'$  is 3-highly twisted, and  $K$  is 1-highly twisted. If  $n, n'$  are the respective lengths of the plats  $K$  and  $K'$ , with  $n'$  satisfying  $n' > 4m(m - 2)$ , then  $n = n'$ .*

*Proof.* Let  $S'$  be a vertical 2-sphere for  $K'$ . Let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy between  $K'$  and  $K$  satisfying the conclusions of Lemmas 3.2 and 3.3. Let  $S = \varphi(S')$  be the image of  $S'$  under  $\varphi$ . Note that  $S$  intersects  $K$  in  $n'$  points.

If  $n < n'$  then by the pigeonhole principal, there is some level where  $S$  must intersect  $K$  more than once. This contradicts Lemma 3.8. So  $n \geq n'$ .

On the other hand, if  $n > n'$ , consider a vertical 2-sphere  $S'$  in  $K'$ . By Lemma 4.3, its image  $S$  has at least one twist region on both sides on each level. However,  $n > n'$  means there are levels it cannot reach. Thus  $n = n'$ .  $\square$

**Lemma 4.5.** *Let  $K'$  and  $K$  be two  $2m$ -plat projections of the same knot or link such that  $m \geq 3$ ,  $K'$  is 3-highly twisted, and  $K$  is 1-highly twisted. Suppose  $n', n$  are the respective lengths of  $K'$  and  $K$  with  $n'$  satisfying  $n' > 4m(m - 2)$ . Finally, let  $S' \subset (S^3, K')$  be a vertical 2-sphere, and let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy satisfying Lemmas 3.2 and 3.3. Then  $\varphi(S') = S$  is a vertical 2-sphere in  $(S^3, K)$ .*


 FIGURE 3. An isolating 2-sphere  $\Omega'$ 

*Proof.* By Corollary 3.7,  $S$  intersects the projection plane  $P$  in precisely one simple closed curve  $\delta$ . By Lemma 3.8, that curve  $\delta$  meets diagonal segments of  $K$  at most once in any level between twist regions. By Proposition 4.4, the number of intersections  $|S \cap K| = |S' \cap K'| = n' = n$ , and by Lemma 4.3,  $\delta$  has twist regions on either side. Thus the curve must be exactly the intersection of a vertical 2-sphere for  $K$  with the projection plane, and  $S$  must be obtained by capping off disks in front of and behind the projection plane with boundary on  $\delta$ . This is a vertical 2-sphere.  $\square$

**Definition 4.6.** Let  $S'_1 \subset (S^3, K')$  be a vertical or almost vertical 2-sphere and let  $S'_2 \subset (S^3, K')$  be an almost vertical 2-sphere that is different from  $S'_1$  by a single twist region on the  $i$ -th row. We assume without loss of generality that  $S'_1 = S'_1(c_1, \dots, c_i, \dots, c_{n-1})$  and  $S'_2 = S'_2(c_1, \dots, c_i + 1, \dots, c_{n-1})$ . Thus the bounded component  $B'$  of  $S^3 \setminus (S'_1 \cup S'_2)$  is a rational 2-tangle as in Figure 3 containing only the  $t'_{i, c_i+1}$  twist region inside a tangle  $B'_{i, c_i+1}$ . The intersection  $S'_1 \cap S'_2$  is a disk, and  $S'_1 \setminus (S'_1 \cap S'_2)$  is a disk  $\Delta'_1$  meeting  $K'$  in two points. Similarly,  $S'_2 \setminus (S'_1 \cap S'_2)$  is a disk  $\Delta'_2$  meeting  $K'$  in two points. The union  $\Delta'_1 \cup \Delta'_2$  is a 2-sphere  $\Omega'$  meeting  $K'$  four times as in Figure 3. We call  $\Omega'$  an *isolating 2-sphere*.

Given an isolating 2-sphere  $\Omega'_{i,j} = \Delta'_1 \cup \Delta'_2$  corresponding to the  $\{i, j\}$  twist region, we will consider the region between two horizontal bridge spheres  $\Sigma'_a$  and  $\Sigma'_b$  meeting  $\partial\Delta'_1$  in a single point at the top of  $\partial\Delta'_1$  and a single point at the bottom.

**Lemma 4.7.** *With the above notation, let  $(B', T')$  denote the tangle bounded by  $S'_1$ ,  $\Sigma'_a$ , and  $\Sigma'_b$  on the side of  $S'_1$  containing  $\Delta'_2$ . The punctured disk  $\Delta'_2 \setminus \mathcal{N}(K')$  is essential and meridionally incompressible in  $B' \setminus \mathcal{N}(T')$ .*

*Proof.* On one side of  $\Delta'_2$  in  $(B', T')$  is the tangle  $(B'_{i,j}, t'_{i,j})$  consisting of a single twist region. The fact that  $\Delta'_2 \setminus \mathcal{N}(K')$  is essential and meridionally incompressible to this side follows from [26, Lemma 3] and Lemma 5.1 in the Appendix.

Consider the other side of  $\Delta'_2$ . If  $\Delta'_2$  is contained in a vertical 2-sphere  $S'_2$ , then the punctured  $S'_2$  is essential and meridionally incompressible by Theorem 2.10 and

Lemma 5.3. So if  $E$  is a compressing disk on the other side of  $\Delta'_2$  in  $B' \setminus \mathcal{N}(T')$ , then  $\partial E$  bounds a disk in  $S'_2$ . But  $S'_2$  meets the knot  $K'$  outside of  $\Delta'_2$ ; thus  $\partial E$  must bound a disk in  $\Delta'_2$ . A similar argument shows  $\Delta'_2$  is meridionally incompressible and boundary incompressible.

If  $\Delta'_2$  is contained in an almost vertical 2-sphere  $S'_2$ , then the component of  $(B', T')$  on the side of  $\Delta'_2$  that does not contain  $(B'_{i,j}, t'_{i,j})$  is a trivial tangle, with either two or three trivial strings, two of which meet  $\Delta'_2$ . Therefore  $\Delta'_2 \setminus \mathcal{N}(K')$  must be essential and meridionally incompressible to this side as well.  $\square$

**Proposition 4.8.** *Let  $K'$  and  $K$  be two  $2m$ -plat projections of the same knot or link. Let  $\varphi : (S^3, K') \rightarrow (S^3, K)$  be an isotopy taking a fixed almost vertical 2-sphere  $S'_1$  to an almost vertical 2-sphere  $S_1 = \varphi(S'_1)$ , and satisfying the conclusions of Lemmas 3.2. Let  $S'_2$  be an almost vertical 2-sphere that differs from  $S'_1$  in one twist region in one row. Let  $\Sigma'_a$  and  $\Sigma'_b$  denote horizontal bridge spheres at the top and bottom of that row. Then there exists an isotopy  $\psi : (S^3, K) \rightarrow (S^3, K)$  that satisfies:*

- $\psi$  preserves the conclusions of Lemma 3.2,
- it fixes the tangle outside  $S_1$ ,  $\varphi(\Sigma'_a)$ , and  $\varphi(\Sigma'_b)$ , and
- it takes  $\varphi(S'_2)$  to an almost vertical 2-sphere that differs from  $S_1$  in one twist region in one row.

*Proof.* Let  $\Delta'_1$  and  $\Delta'_2$  be as in Definition 4.6. Because  $\varphi(S'_1) = S_1$  is an almost vertical 2-sphere,  $\varphi(S'_1 \cap S'_2)$  is a disk in an almost vertical 2-sphere.

Note that  $\Delta'_1$  and  $\Delta'_2$  both lie in a region between horizontal bridge spheres  $\Sigma'_a$  and  $\Sigma'_b$ , as indicated in Figure 3. Lemma 3.2 implies that their images lie in a region between horizontal bridge spheres  $\Sigma_c = \varphi(\Sigma'_a)$  and  $\Sigma_d = \varphi(\Sigma'_b)$ . Because  $\Delta_1 = \varphi(\Delta'_1)$  meets  $K$  twice and is a sub-disk of an almost vertical 2-sphere  $S$ , the only possibility is that  $\Sigma_c$  and  $\Sigma_d$  cut off a “slab”  $\Xi$  in  $(S^3, K)$  homeomorphic to  $S^2 \times I$  containing a single row of twist regions. Both  $\Delta_1$  and  $\Delta_2 = \varphi(\Delta'_2)$  lie in  $\Xi$  between  $\Sigma_c$  and  $\Sigma_d$ .

There is exactly one almost vertical 2-sphere that differs from  $S_1$  in a single twist region in  $\Xi$ ; denote this almost vertical 2-sphere by  $S$ . We want to show that we may isotope  $S_2 = \varphi(S'_2)$  to  $S$  in the slab  $\Xi$ . We can ensure that  $S$  meets  $K$  exactly in the points in which  $S_2$  meets  $K$ : For those points of  $S_2 \cap K$  that lie on  $S_1 \cap K$ , this follows from the fact that  $S_1 \setminus (S_1 \cap S_2)$  agrees with  $S_1 \setminus (S_1 \cap S)$ . For the two points on  $\Delta_2$ , note that we may assume the intersections  $S'_2 \cap K'$  on  $\Delta'_2$  are on the same levels as the intersections of  $S'_1 \cap K'$  on  $\Delta'_1$ ; see Figure 3. Since the map  $\varphi$  is level preserving, the two points in  $\Delta_1 \cap K$  lie on the same levels as  $\Delta_2 \cap K$ , which we can ensure are the same levels as  $S \cap K$ .

Let  $\Delta = S \setminus (S \cap S_1)$ . Let  $(B_{r,s}, t_{r,s})$  denote the tangle containing the twist region between  $S_1$  and  $S$ . We may assume the sphere  $\partial B_{r,s}$  is disjoint from the sphere  $\Delta_1 \cup \Delta$ , with  $\partial B_{r,s}$  in the interior of the ball bounded by  $\Delta_1 \cup \Delta$ .

First we show that  $\Delta_2$  is disjoint from  $\partial B_{r,s}$ . For suppose not. Then  $\Delta_2$  intersects  $\partial B_{r,s}$  in a collection of closed curves. If any bounds a disk on  $\partial B_{r,s}$  that does not meet  $K$ , then there is an innermost bounding a disk  $E$  on  $\partial B_{r,s}$ . Since  $\Delta'_2$  is incompressible

in the tangle  $(B', T')$  (Lemma 4.7),  $\partial E$  bounds a disk  $D$  on  $\Delta_2$ . Hence  $E \cup D$  bounds a ball in the tangle in  $\Xi$  between  $\Sigma_c$  and  $\Sigma_d$ . We may isotope horizontally in the ball, preserving levels, to remove this intersection. After repeating this procedure a finite number of times, we may assume that closed curves of  $\partial B_{r,s} \cap \Delta_2$  do not bound disks in  $\partial B_{r,s}$  disjoint from  $K$ . Since  $\Delta_2$  is meridionally incompressible and  $\varphi$  is level-preserving, we may rule out curves of intersection that encircle a single point of  $K \cap \partial B_{r,s}$ . So if there are intersection curves  $\Delta_2 \cap \partial B_{r,s}$ , they encircle two points of  $K \cap \partial B_{r,s}$ . Note that  $\Delta_2$  is disjoint from the plat  $K$  in  $(B_{r,s}, t_{r,s})$  as all possible intersections have been accounted for. Thus if  $\partial B_{r,s} \cap \Delta_2$  is nonempty, the only possibility is that  $\Delta_2$  is parallel to the unique compressing disk for the tangle  $(B_{r,s}, t_{r,s})$  separating the tangle into two trivial tangles. But then  $K$  is 1-highly twisted implies that  $\partial \Delta_2$  must meet the interior of  $\Delta_1$ . This contradicts the fact that the interior of  $\Delta_1$  is disjoint from  $\Delta_2$ .

Thus we have shown that  $\Delta_2$  is disjoint from  $B_{r,s}$ , has boundary meeting  $\partial \Delta_1 = \partial \Delta$ , and intersects  $K$  in the points  $K \cap \Delta$ . A similar argument to that above implies that  $\Delta_2$  can be isotoped in a level-preserving manner to be disjoint from other tangles  $(B_{j,s}, t_{j,s})$  in  $\Xi$ . Then  $\Delta_2$  lies in the trivial tangle between  $\Sigma_c, \Sigma_d$ , and disks outside of twist regions in  $\Xi$ . The only possibility is that  $\Delta_2$  is parallel to  $\Delta$ , and since both  $\Delta_2$  and  $\Delta$  meet each level in a single arc, we may isotope  $\Delta_2$  horizontally to  $\Delta$ .  $\square$

*Proof of Theorem 4.1.* Let  $S'_\ell \subset (S^3, K')$  denote the left-most vertical 2-sphere. That is,  $S'_\ell = S'(1, \dots, 1)$ . Lemma 4.5 implies that a level-preserving isotopy  $\varphi$  takes  $S'_\ell$  to a vertical 2-sphere in  $K$ .

Now let  $S'_v$  be an almost vertical 2-sphere that differs from  $S'_\ell$  by a single isolating 2-sphere. By Proposition 4.8, we may adjust the isotopy  $\varphi$  to take  $S'_v$  to an almost vertical 2-sphere differing from  $\varphi(S'_\ell)$  by a single isolating 2-sphere. By induction, adjust  $\varphi$  one twist region at a time to obtain an isotopy taking each vertical 2-sphere to an almost vertical 2-sphere, fixing the previously adjusted vertical and almost vertical 2-spheres, proving the theorem.  $\square$

**4.2. Matching twist regions to twist regions.** It remains to show that twist regions map to identical twist regions. We do this in two steps.

**Definition 4.9.** Twist regions that can be isolated by  $\Omega' = \Delta'_1 \cup \Delta'_2$  for  $\Delta'_1, \Delta'_2$  coming from vertical 2-spheres will be called *allowable twist regions*. Such twist regions are inside the diagram, not on the far left or far right. Twist regions on the far left or far right but not in one of the corners will be called *almost allowable twist regions*. That is, almost allowable twist regions are those  $t'_{i,j} \in K'$  such that either  $i$  is odd with  $3 \leq i \leq n-3$  and  $j \in \{1, m-1\}$  or  $i$  is even with  $4 \leq i \leq n-4$  and  $j \in \{1, m\}$ .

The only twist regions which are not allowable or almost allowable are the extreme right and left twist regions in rows 1, 2,  $n-2$  and  $n-3$ . These twist regions are called *extreme twist regions*.

As a corollary to the proof of Theorem 4.1 we obtain:

**Corollary 4.10.** *Let  $t'_{i,j} \in K'$  be an allowable or almost allowable twist region and let  $S'_1$  and  $S'_2$  be its corresponding vertical 2-spheres. The bounded component of  $S^3 \setminus (\varphi(S'_1) \cup \varphi(S'_2))$  in  $(S^3, K)$  is a rational 2-tangle as in Figure 3 containing a single twist region  $t_{r,s}$ .  $\square$*

**Lemma 4.11.** *If  $t'_{i,j}$  is an allowable or almost allowable twist region in  $(S^3, K')$  that is mapped to  $t_{r,s}$  in  $(S^3, K)$  then the number of crossings in the twist regions agree:  $a_{r,s} = a'_{i,j}$ .*

*Proof.* Since  $\varphi$  is an isotopy, it will map a rational tangle  $1/a'_{i,j}$  corresponding to  $t'_{i,j}$  to a rational tangle  $1/a_{r,s}$  corresponding to  $t_{r,s}$ . The only way that the two tangles might disagree is if the two strings of the tangle are twisted about each other during the isotopy. However such a twisting will induce a homeomorphism of the four punctured sphere  $\Omega'$  that fixes the punctures. Such a homeomorphism will force the images  $\varphi(\Delta'_1)$  and  $\varphi(\Delta'_2)$  to intersect  $P$  in more than in one component, and thus  $S_1$  and  $S_2$  will not be vertical spheres, in contradiction to Theorem 4.1.  $\square$

**Lemma 4.12.** *Let  $t'_{i,j}$  be an allowable twist region in  $K'$ . Then the allowable and almost allowable twist regions adjacent to  $t'_{i,j}$  in  $K'$  are mapped to twist regions adjacent to  $\varphi(t'_{i,j}) = t_{r,s}$  in  $(S^3, K)$ , in the same order, up to rotation along a vertical or horizontal axis through the  $t_{r,s}$  twist region.*

*Proof.* By the definition of an allowable twist region, there are two vertical 2-spheres which isolate the twist region  $t'_{i,j}$ , as in Corollary 4.10. We know by Lemma 4.11 that the twist region  $t_{r,s}$  in  $K$  has the same number of crossings as  $t'_{i,j}$ , and is isolated by images of the two vertical 2-spheres. Given  $t'_{i,j} \in K'$ , each allowable adjacent twist region (at most four in the general case) shares a segment of  $K'$ . This segment intersects only one of the vertical 2-spheres used in the isolation of  $t'_{i,j}$ .

This 2-sphere is shared by the twist region adjacent to  $t'_{i,j}$ , so it must be mapped to the 2-sphere separating the images of these adjacent twist regions in  $(S^3, K)$ . Thus adjacent allowable and almost allowable twist regions are mapped to adjacent allowable and almost allowable twist regions. Since twist regions are symmetric with respect to reflections in the vertical and horizontal axis, so is the relative position of the corresponding twist regions in  $K$ ; see Figure 4.  $\square$

Recall the definition of extreme twist regions in Definition 4.9. We still need to show that the extreme twist regions are fixed. We will define an isolating sphere for an extreme twist region on the top left corner of the  $K'$  plat; the others are defined by symmetry. Consider the intersection of  $S'(1, 1, \dots, 1) \cap P'$  with the second top external region, i.e. the left-most region  $\mathcal{T}^t$ , and the generic region  $\mathcal{Q}$  on the second row counted from the left. A sub-arc of this arc of intersection can be completed to a simple closed curve  $\gamma'$  circling both the  $t'_{1,1}$  and  $t'_{2,1}$  twist regions. The curve  $\gamma'$  is indicated in Figure 5. Cap  $\gamma'$  by disks in front and behind  $P$  to obtain a 2-sphere  $S_1^{\ell}$  bounding the rational 2-tangle corresponding to the  $1/(a'_{1,1} + 1/a'_{2,1})$  continued fraction expansion.

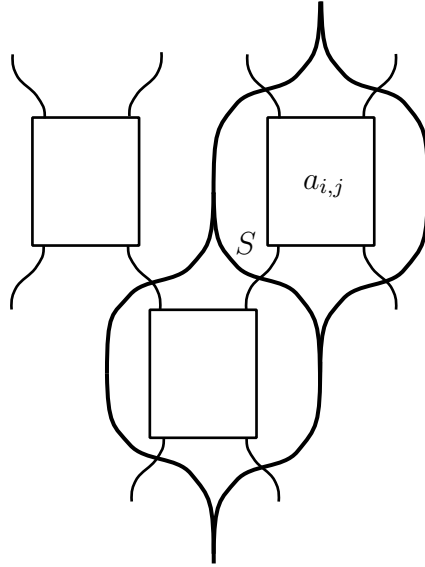


FIGURE 4. Two adjacent twist regions with a shared vertical 2-sphere  $S$

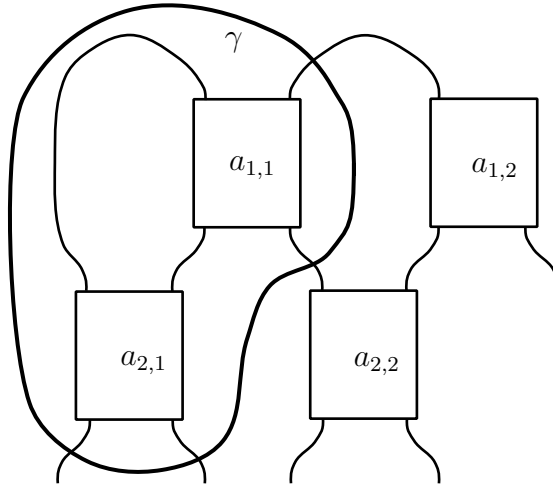


FIGURE 5. The  $\gamma$  curve for an isolating sphere for the extreme twist regions

**Lemma 4.13.** *The 2-sphere  $\varphi(S_1^\ell)$  bounds a rational 2-tangle corresponding to the  $1/(a_{1,1} + 1/a_{2,1})$  continued fraction expansion. Thus, up to rotation,  $a_{1,1} = a'_{1,1}$  and  $a_{2,1} = a'_{2,1}$ , and similarly for the other three corners.*

*Proof.* Since a sub-arc of  $\gamma'$  is also a sub-arc of  $S'(1, \dots, 1)$ , the image  $\varphi(S_1^\ell)$  shares a sub-disk with  $\varphi(S'(1, \dots, 1))$ . By Lemma 4.12, we may assume without loss of generality that  $\varphi(S'(1, \dots, 1))$  is equal to  $S(1, \dots, 1)$ .

It now follows from the same argument as in Corollary 4.10 that  $\varphi(S_1^\ell) \cap P$  is a single simple closed curve  $\gamma$  and  $\varphi(S_1^\ell) = S_1^\ell$ . By Lemma 4.12 the points  $K \cap \varphi(S_1^\ell)$  are fixed.

Thus  $1/(a'_{1,1} + 1/a'_{2,1}) = 1/(a_{1,1} + 1/a_{2,1})$ . It follows that  $a_{1,1} = a'_{1,1}$  and  $a_{2,1} = a'_{2,1}$ , and similarly for the extreme twist regions in the other corners of the plat.  $\square$

**4.3. Proof of Theorem 1.1.** We are now ready to prove the main theorem.

*Proof of Theorem 1.1.* Let  $\mathcal{K} \subset S^3$  be a knot or link with a  $2m'$ -plat projection  $K'$  that is 3-highly twisted, and length  $n' > 4m'(m' - 2)$ . Assume further that  $\mathcal{K}$  has an additional  $2m$ -plat projection or even plat projection  $K$  as well. Then Corollary 2.8 implies  $m \geq m'$ . If  $m' = m$ , and  $K$  is 1-highly twisted, then Proposition 4.4 implies that the length  $n$  of  $K$  equals the length  $n'$ . (So in fact,  $K$  is not an even plat projection.)

It then follows from Corollary 4.10 through Lemma 4.13 that the twist regions in both plats are equal, namely  $t_{i,j} = t'_{i,j}$  up to reflection by a vertical or a horizontal axis, and  $t_{i,j}$  and  $t'_{i,j}$  contain the same number of (signed) crossings, i.e.  $a_{i,j} = a'_{i,j}$ . Thus up to reflection by a vertical or a horizontal axis,  $K = K'$  and the plat is unique.  $\square$

## 5. APPENDIX

In this section, we prove that vertical 2-spheres are meridionally incompressible. The proof follows very closely the proof of Wu that vertical 2-spheres are essential [26].

**Lemma 5.1.** *Suppose  $(B, T)$  is a rational tangle consisting of a single twist region with  $q$  crossings (i.e. a  $1/q$  tangle). Let  $C$  be a circle on  $\partial B$  of slope  $\infty$ ,  $X = B \setminus \mathcal{N}(T)$ , and  $P$  a component of  $(\partial B \cap X) \setminus C$ . If  $q \geq 3$  then  $P$  is meridionally incompressible in  $X$ .*

*Proof.* Suppose  $D$  is a meridionally compressing disk for  $P$  in  $X$ . There exists a disk  $E$  embedded in  $X$  with boundary on  $\partial B$ , separating  $B$  into two trivial one-tangles. Consider intersections of  $D$  and  $E$ . Suppose there is a closed curve component of  $D \cap E$ . Then there is an innermost such component  $\gamma$  on  $D$ , bounding a disk  $D' \subset D$ . Note that  $\gamma$  also bounds a disk  $E'$  on  $E$  thus  $D' \cup E'$  is a sphere in  $B$  which bounds a 3-ball  $B'$ . If  $B'$  is disjoint from the tangle  $T$ , we may isotope  $D$  through  $B'$  to remove the component of intersection  $\gamma$ . If  $T$  is not disjoint from  $B'$ , then  $T$  must meet  $\partial B' = D' \cup E'$ . But  $E$  is disjoint from  $T$ , and  $D'$  can meet  $T$  only in a single meridional puncture. But then  $T$  can puncture the 2-sphere  $\partial B'$  at most once. This is a contradiction. So after a finite number of isotopies, we may assume there are no closed curve components of  $E \cap D$ .

Now suppose  $E \cap D$  consists of arc components. An outermost arc  $\alpha$  on  $D$  bounds a disk  $D' \subset D$ . It also bounds a disk  $E' \subset E$ . Then consider  $\partial D' \setminus \alpha$  and  $\partial E' \setminus \alpha$ . These are arcs on  $\partial B$  disjoint from  $T$  whose endpoints agree; thus they bound a disk  $F' \subset \partial B$ , and the union  $D' \cup E' \cup F'$  is a 2-sphere in  $B$  bounding a 3-ball  $B'$ . Again if  $B'$  is disjoint from  $T$ , we may isotope  $E'$  through  $B'$ , with an arc of its boundary remaining in  $F'$ , until we isotope away the arc of intersection. So suppose  $B'$  does meet  $T$ . Since  $E$  is disjoint from  $T$ , and  $D'$  meets  $T$  at most once in a meridional puncture, it follows that  $T$  must meet  $F'$  exactly once. Because  $B'$  lies in a trivial one-tangle,  $T$  must run through  $B'$  in an unknotted arc from  $F'$  to  $D'$ . But then  $D'$  can be isotoped through  $B'$ , with an arc of its boundary remaining in  $E'$ , to be parallel to  $F'$ , removing



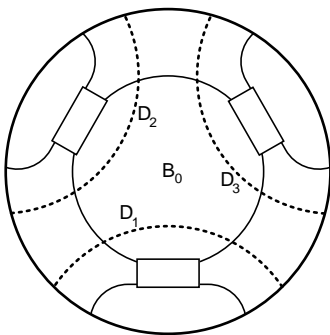


FIGURE 6. Tangle to the left of  $S(1, \dots, 1)$  is equivalent to this tangle when  $k = 3$ ; figure from [26].

the intersection component  $\alpha$ . Repeat this procedure until all other arcs of intersection are removed. It follows that  $D$  is parallel to a disk on  $\partial B$  which meets exactly one puncture of  $T$ ; hence  $P$  is meridionally incompressible.  $\square$

**Lemma 5.2** (See Lemma 5 of [26]). *The vertical 2-sphere  $S(1, \dots, 1)$  is meridionally incompressible.*

*Proof.* Consider the tangle  $(B, T)$  to the left of  $S(1, \dots, 1)$ , and let  $X$  denote  $B \setminus \mathcal{N}(T)$ . By an isotopy of  $(B, T)$ , we can untwist the twist regions of  $T$  which lie in the even rows, so the tangle  $(B, T)$  is equivalent to the one shown in Figure 6, where each twist box corresponds to the first box on an odd row of the diagram. There are  $k$  such boxes, where  $k = n/2$  and  $n$  denotes the height of the plat. (Recall that  $n$  is even.)

Let  $D_1, \dots, D_k$  be disks represented by the dotted lines in Figure 6, which cut  $(B, T)$  into  $k + 1$  sub-tangles  $(B_0, T_0), \dots, (B_k, T_k)$ , where  $(B_0, T_0)$  is the tangle in the middle, which intersects all the  $D_i$ . Let  $P_i = D_i \cap X$  be the twice punctured disk corresponding to  $D_i$ .

Now suppose  $D$  is a meridional compressing disk for  $S(1, \dots, 1)$ ; we may assume the meridian meeting  $D$  lies completely in some  $B_i$ . Consider components  $P_i \cap D$ , and first suppose  $i \geq 1$ . Any innermost curve or outermost arc of  $D \cap P_i$  in  $D$  bounds a disk  $D'$  in  $D$ . If  $D'$  is disjoint from the tangle, then because  $P_i$  is incompressible and boundary incompressible in  $B_i - \mathcal{N}(T_i)$  (by [26, Lemma 3]) and in  $B_0 - \mathcal{N}(T_0)$  (by [26, Lemma 5]), such a curve of intersection can be slid off. Repeat a finite number of times to remove all such intersections. Any remaining innermost disk or outermost arc of intersection with  $P_i$  must bound a meridional disk in  $D$ . But Lemma 5.1 implies  $P_i$  is meridionally incompressible in  $X_i$ , hence the portion of  $D$  in  $B_i$  can be isotoped through  $B_i$  to the boundary, and we may remove this intersection.

It follows that  $D$  must have its boundary completely contained in  $\partial B_0 - \cup D_i = Q$ . But since  $D$  meets  $T_0$  in a single meridian,  $\partial D$  must encircle one component of  $T_0$  on one side and  $2k + n - 1$  components, an odd number, on the other side. Since the disks  $D_i$  meet endpoints of  $T_0$  in pairs, this is impossible.  $\square$

**Lemma 5.3** (See Lemma 6 of [26]). *Any vertical 2-sphere  $S(c_1, \dots, c_{n-1})$  is meridionally incompressible.*

*Proof.* There is a sequence of vertical spheres  $S_1, S_2, \dots, S_{r+1}$  so that  $S_1 = S(1, \dots, 1)$ ,  $S_{r+1} = S(c_1, \dots, c_{n-1})$ , and  $S_i$  and  $S_{i+1}$  differ by a single twist region in the diagram. That is,  $S_i \cup S_{i+1} - \text{int}(S_i \cap S_{i+1}) = \partial B_i$  for some tangle  $(B_i, T_i)$  bounding a single twist region with  $a \neq 0$  twists. By Lemma 5.2, we know  $S_1$  is meridionally incompressible. The proof will be by induction on the length of the sequence. Assume, therefore, that  $S_r$  is meridionally incompressible.

Now suppose that  $D$  is a meridian compressing disk for  $S_{r+1}$ . The disk  $D$  must meet the tangle  $(B_r, T_r)$ , else it would be a meridionally compressing disk for  $S_r$ , which is impossible. Let  $P_r = \partial B_r \cap S_r$ ; this is a twice punctured disk on  $\partial B_k$ . Consider curves and arcs of intersection of  $D$  and  $P_r$ . Innermost curves and outermost arcs bounding disks in  $D$  can be isotoped away, using the incompressibility and boundary incompressibility of  $P_r$  in  $B_r$ , and incompressibility of  $S_r$ .

Any innermost curve or arc of intersection therefore bounds a disk of  $D$  meeting the meridian of the link. But  $P_r$  is meridionally incompressible by Lemma 5.1, hence this intersection can be isotoped outside of  $P_r$ . It follows that  $D$  is disjoint from  $P_r$ , but meets the tangle  $(B_r, T_r)$ . This is impossible unless  $D$  is completely contained in  $(B_r, T_r)$ . But in that case,  $D$  is a meridional compressing disk for  $P_{r+1} = \partial B_r \setminus P_r$ , contradicting Lemma 5.1.  $\square$

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DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, 32000 ISRAEL  
*E-mail address:* ymoriah@tx.technion.ac.il

SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, VIC 3800 AUSTRALIA  
*E-mail address:* jessica.purcell@monash.edu