LARGE VOLUME FIBRED KNOTS OF FIXED GENUS

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For Alan Reid

Abstract. We show that, for hyperbolic fibred knots in the three-sphere, the volume and the genus are unrelated. Furthermore, for such knots, the volume is unrelated to strong quasipositivity and Seifert form.

In this note we prove the following.

Theorem 1. *For every $g > 1$ and every $V > 0$ there is a knot $w \subset S^3$ with the following properties.*

1. *$S^3 - w$ is fibred over the circle, with fibre of genus g.*
2. *$S^3 - w$ is hyperbolic, with volume at least V.*
3. *Longitudinal surgery on w is hyperbolic, with volume at least V.*

Remark 2. In the opposite direction, the two-bridge knots with continued fraction $[2g - 1, 1, 2]$ are hyperbolic and fibred of genus g, with bounded volume as $g \to \infty$. ◇

Theorem 1 answers a question posed by Reid. Our proof also provides a family of fibred hyperbolic knots in $S^2 \times S^1$, of fixed genus, whose double branched covers have unbounded volume. This answers a special case of a question posed by Hirose, Kalfagianni, and Kin [10, Question 4]. For more details, see Remark 9.

Proof of Theorem 1. Fix $g > 1$. Let $\mathcal{B}_{2g+1}$ be the braid group on $2g + 1$ strands. Let σ_i be the positive half-twist between the i^{th} and $(i + 1)^{\text{th}}$ strands. We define the following braids.

$$\Pi = \sigma_{2g} \cdot \sigma_{2g-1}^{-1} \cdot \sigma_{2g-2} \cdot \sigma_{2g-3}^{-1} \cdots \sigma_4 \cdot \sigma_3^{-1} \cdot \sigma_2$$
$$\Phi = \sigma_2 \cdot \sigma_3^{-1}$$
$$\beta_n = \Pi \cdot \Phi^n \cdot \sigma_1^{-1} \cdot \Phi^{-n}$$

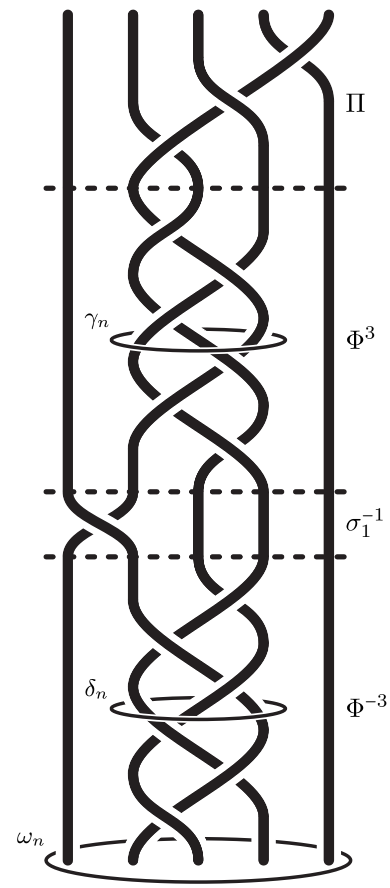

Figure 1

See Figure 1, where we take $g = 2$ and $n = 3$.

Let $\hat{\beta}_n$ be the braid closure of β_n, taken in S^3. Let ω_n be its augmenting braid axis. Let $\Lambda_n = \left(\hat{\beta}_n \cup \omega_n\right)$ be the resulting two-component link. Note that ω_n bounds a disk Ω_n in S^3 meeting $\hat{\beta}_n$ in $2g + 1$ points. This is shown at the very bottom of Figure 1. We deduce that $S^3 - \Lambda_n$ is a punctured disk bundle over the circle, with monodromy β_n.

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Claim 3. Both ω_n and $\hat{\beta}_n$ are unknots in S^3.

Proof. Since ω_n bounds the disk Ω_n, it is an unknot.

Note that $\hat{\beta}_n$ is stabilised along its first strand. Destabilising has the effect of smoothing the crossing at σ_1^{-1} and deleting the first strand. The factors Φ^n and Φ^{-n} now cancel, leaving only the ($2g$–strand) braid closure $\hat{\Pi}$. This is an iterated stabilisation, proving the claim. □

Let γ_n and δ_n be the augmentations of β_n, taken before and after the factor σ_1^{-1}, each containing the second, third, and fourth strands. Again, see Figure 1. Thus γ_n and δ_n bound disks Γ_n and Δ_n in S^3 each meeting $\hat{\beta}_n$ in three points.

Appealing to Claim 3, the branched double cover of S^3 along $\hat{\beta}_n$ is again homeomorphic to S^3. Let c_n, d_n, and w_n be the preimages of γ_n, δ_n, and ω_n, respectively; let C_n, D_n, and W_n be the preimages of Γ_n, Δ_n, and Ω_n, respectively. Since the disks meet $\hat{\beta}_n$ in an odd number of points, each of c_n, d_n, and w_n is connected and equals the boundary of the corresponding surface C_n, D_n, and W_n. An Euler characteristic calculation shows that C_n and D_n are homeomorphic to $S_{1,1}$: a torus with one boundary component; see, for example, [5, Figure 9.13]. Similarly, W_n is homeomorphic to $S_{g,1}$: a surface of genus g with one boundary component. We deduce that the knot complement $M_n = S^3 - w_n$ is an $S_{g,1}$–bundle over S^1. Thus w_n is a genus g fibred knot in S^3. This is the family of knots promised in Theorem 1(1).

By a result of Birman and Hilden [5, Theorem 9.2], the monodromy of M_n is a product of Dehn twists, one for each half-twist generator in $\mathcal{B}_{2g+1}$. To fix notation, let s_i be the Dehn twist in $S_{g,1}$ lifting σ_i. Lifting Π, Φ, and β_n in this way gives the following mapping classes.

$$\begin{aligned} P &= s_{2g} \cdot s_{2g-1}^{-1} \cdot s_{2g-2} \cdot s_{2g-3}^{-1} \cdots s_4 \cdot s_3^{-1} \cdot s_2 \\ F &= s_2 \cdot s_3^{-1} \\ b_n &= P \cdot F^n \cdot s_1^{-1} \cdot F^{-n} \end{aligned}$$

Thus b_n is the monodromy of $M_n = S^3 - w_n$.

Claim 4. For all g, the knot complement M_0 is hyperbolic. The same holds for its longitudinal filling.

Proof. Since $n = 0$, the monodromy of M_0 simplifies to

$$b_0 = s_{2g} \cdot s_{2g-1}^{-1} \cdot s_{2g-2} \cdot s_{2g-3}^{-1} \cdots s_4 \cdot s_3^{-1} \cdot s_2 \cdot s_1^{-1}$$

which is (cyclically) conjugate to

$$(s_{2g} \cdot s_{2g-2} \cdots s_4 \cdot s_2)(s_{2g-1} \cdot s_{2g-3} \cdots s_3 \cdot s_1)^{-1}.$$

Let $\alpha_i \subset S_{g,1}$ be the core curve of the Dehn twist s_i. We isotope the curves α_i to intersect minimally. This done, α_i and α_j intersect (and then intersect in a single point) if and only if $|i - j| = 1$. Thus, the union $\bigcup_i \alpha_i$ is connected. Also, the complement of $\bigcup_i \alpha_i$ is a peripheral annulus. We now apply a criterion of Thurston [18, Theorem 7] (see also Veech [19, pages 578–579]). Let N be the matrix with $N_{ij} = |\alpha_i \cap \alpha_j|$. Let μ be the largest real eigenvalue of NN^t; note that μ is positive. Let A be the union of the α_i for i even; let B be the union of the α_i for i odd. Let T_A and T_B be the corresponding multi-twists. The image of $T_A \cdot T_B^{-1}$ under Thurston's representation has trace $2 + \mu$, hence b_0 is pseudo-Anosov. Appealing to Thurston's hyperbolisation theorem for mapping tori [17, Theorem 5.6], we find that M_0 is hyperbolic.

If we fill, replacing $S_{g,1}$ by S_g, then the complement of $\bigcup_i \alpha_i$ is a disk. Thus the same proof shows that the longitudinal filling of M_0 is hyperbolic. This proves the claim. $\square$

Remark 5. By an observation of Gabai and Kazez [7, Proposition 2], the knots w_0 are two-bridge. In genus g, the continued fraction for w_0 is $[2, 2, \ldots, 2]$ with $2g$ terms. This gives another proof that M_0 is hyperbolic. $\diamond$

We now prove Theorem 1(2).

Claim 6. In fixed genus g, as n tends to infinity, the knot complements M_n are eventually hyperbolic, with volumes tending to infinity.

Proof. We fix a genus g and reuse the notation above, taking $N = M_0 - (c_0 \cup d_0)$. Since M_0 is hyperbolic (Claim 4), since c_0 and d_0 lie in distinct fibres, and since c_0 and d_0 are not isotopic, we deduce that N is hyperbolic by Thurston's hyperbolisation theorem [17, Theorem 2.3].

Let $C, D \subset N$ be the images of C_0 and D_0 respectively. We cut the cusps off of N and then cut along small open regular neighbourhoods of C and D to obtain a compact three-manifold P. Let $\rho \subset \partial P$ be the remains of the cusp tori about w_0, c_0, and d_0. That is, ρ consists of a torus, coming from w_0, and two *paring annuli*, coming from c_0 and d_0. The annular components of ρ separate the genus two components of ∂P into one-holed tori $C^\pm$ and $D^\pm$ respectively. These form the *horizontal boundary* of (P, ρ)

Since C and D were contained in distinct fibres in M_0, the horizontal boundary of (P, ρ) is incompressible in (P, ρ). Thus (P, ρ) is a compact, oriented, irreducible, atoroidal three-manifold with incompressible horizontal boundary and with non-abelian fundamental group. Since c_0 and d_0 are not homotopic in M_0, we deduce that (P, ρ) is a *pared manifold* as in [2, Section 2.4].

For each horizontal boundary component E of (P, ρ) we now chose *any* complete marking $\mu(E)$; see [2, Section 2.1]. Define $N_n = M_n - (c_n \cup d_n)$. Let $m(c_n)$ and $m(d_n)$ be the meridional slopes in the associated torus cusps of N_n. We note that N_n is obtained from (P, ρ) by gluing C^+ to C^-, and D^- to D^+, using the n^{th} power of F, the monodromy of the figure-eight knot complement. Since F is pseudo-Anosov, we deduce that the gluings N_n have *R–bounded combinatorics* and increasing *height* in the sense of [2, Section 2.12].

We now apply a theorem of Brock, Minsky, Namazi, and Souto [2, Theorem 8.1] to find that, for sufficiently large n, the gluing N_n is hyperbolic and has a *bilipschitz model* $\mathbb{M}_n$. We deduce that the volumes of the manifolds N_n tend to infinity coarsely linearly with n. Likewise, the lengths of the meridional slopes $m(c_n)$ and $m(d_n)$ tend to infinity. Applying a Dehn surgery result [6, Theorem 1.1], we find that the manifolds M_n are hyperbolic, and have volume tending to infinity coarsely linearly with n. This proves the claim. $\square$

The proof of Theorem 1(3) is similar. Fix the genus g. By Claim 4, the longitudinal filling M' of M_0 is hyperbolic. Taking N' to be the longitudinal filling of N along w_0, we deduce from the above that N' is also hyperbolic. We cut along the surfaces C' and D' (the images of C_0 and D_0) to obtain a pared manifold (P', ρ'). We now again apply the machinery of [2] and of [6]. This completes the proof of Theorem 1. $\square$

Remark 7. There is another, related, proof of parts (2) and (3) of Theorem 1. One uses the work of Clay, Leininger, and Mangahas [4, Theorem 5.2] to show that the monodromies b_n have linearly growing subsurface projections to C_n and D_n. Pairing this with the Masur-Minsky distance formula [12, Section 8] and results of Brock [3, Theorems 1.1 and 2.1], we find that M_n has linearly growing volume. $\diamond$

Remark 8 (An enhancement of Theorem 1). Our knots w_n, produced via a branched double covering construction, are morally similar to a family of hyperbolic knots produced by Misev [13, Section 3], using a plumbing construction. However, in any fixed genus Misev's family has bounded hyperbolic volume. Nevertheless, with a bit more work one may adapt the construction of Theorem 1 so that the family of knots has volume going to infinity while satisfying Misev's conclusions. In particular, for each $g \geq 2$, there is a family of knots w_n with the following properties.

(1) $S^3 - w_n$ is fibred over the circle, with fibre of genus g.
(2) $S^3 - w_n$ is hyperbolic, with volume at least V.
(3) Longitudinal surgery on w_n is hyperbolic, with volume at least V.
(4) w_n is strongly quasipositive.
(5) w_n has the same Seifert form as the torus knot $T(2g+1, 2)$.

Following Rudolph [15, 16], we call a knot K *strongly quasipositive* if it is the closure of a braid that lies in the monoid generated by all elements of the following form.

$$(\sigma_j \sigma_{j-1} \ldots \sigma_{i+1}) \sigma_i (\sigma_j \sigma_{j-1} \ldots \sigma_{i+1})^{-1}$$

By work of Livingston [11], such knots K have $\tau(K) = g_4(K) = g_3(K)$: here g_4 is the four-ball genus, g_3 is the Seifert genus, and τ is the Ozsváth–Szabó concordance invariant. By a result of Hedden [9, Proposition 2.1], if K is strongly quasipositive and fibred then it serves as the binding of an open book decomposition that supports the tight contact structure on S^3. If K has the same Seifert form as $T = T(2g+1, 2)$ then K has the same Alexander module and signature function as T. We refer to Misev [13] for further discussion and implications of these properties.

To construct the sequence w_n having properties (1)–(5), we modify the definitions of the braids Φ, Π, and β_n appearing in the proof of Theorem 1. We first require Φ to factor as a product of (conjugates of) squares of full twists on sets of odd numbers of strands. We also require Φ to be a pseudo-Anosov element on the last $2g$ strands. For instance, when $g = 2$, one can take

$$\Phi = \left(\sigma_3 \sigma_4^2 \cdot (\sigma_2 \sigma_3)^6 \cdot \sigma_4^{-2} \sigma_3^{-1}\right) \cdot \left(\sigma_3^{-1} \sigma_4^{-2} \cdot (\sigma_2 \sigma_3)^6 \cdot \sigma_4^2 \sigma_3\right).$$

We define Π and β_n as follows.

$$\Pi = \sigma_{2g} \cdot \sigma_{2g-1} \cdot \sigma_{2g-2} \cdot \sigma_{2g-3} \cdots \sigma_4 \cdot \sigma_3 \cdot \sigma_2$$
$$\beta_n = \Pi \cdot \Phi^n \cdot \sigma_1 \cdot \Phi^{-n}$$

Property (1) now holds, for the same reason as above.

The choices of Π and β_n ensure that the fibre of w_n is a plumbing of a positive Hopf band onto the fibre of the torus link $T(2, 2g)$. The strong quasipositivity of w_n now follows from the strong quasipositivity of $T(2, 2g)$ by [15, Proposition 4.2], yielding property (4).

Since Φ factors as product of squares of full twists (each about an odd number of strands), its lift F factors as a product of Dehn twists along null-homologous curves. Since we are twisting along null-homologous curves, the Seifert form of w_n is independent of n. This yields property (5), as w_0 is the torus knot $T(2, 2g+1)$.

Observe that β_n factors as follows.

$$\beta_n = (\Pi\, \sigma_1) \cdot (\sigma_1^{-1} \Phi^n \sigma_1 \cdot \Phi^{-n})$$

The first term in parentheses is periodic; in fact, $(\Pi\, \sigma_1)^{2g+1}$ is a Dehn twist about the boundary curve ω_n (and thus trivial in the mapping class group). We now claim that β_n is pseudo-Anosov. It suffices to prove that β_n^{2g+1} is pseudo-Anosov. To see this, note that we can pull all copies of the first term $\Pi\sigma_1$ to the front by conjugating the copies of the second term, $\sigma_1^{-1}\Phi^n\sigma_1 \cdot \Phi^{-n}$. Thus β_n^{2g+1} is a Dehn twist about ω_n, composed with a product of conjugates of the second term. These are themselves a product of one conjugate of Φ^n and one conjugate of Φ^{-n}. Each conjugate is supported in a sub-disk containing $2g$ punctures, so every pair of supporting domains is neither disjoint nor nested. Furthermore, each conjugate has large translation distance acting on the curve complex of its supporting domain. Thus [4, Theorem 6.1] implies that β_n^{2g+1} is pseudo-Anosov when n is large.

Since β_n is pseudo-Anosov, so is its lift b_n, hence M_n is hyperbolic. Since Φ is pseudo-Anosov on a sub-disk containing the last $2g$ strands, it follows that its lift F is pseudo-Anosov on the branched double cover, a copy of $S_{g-1,2}$. As in the previous paragraph, b_n^{2g+1} factors as a product of conjugates of large powers of F or F^{-1}. Thus [4, Theorem 5.2] implies that b_n^{2g+1} has linearly growing translation distances in curve complexes of the corresponding supporting domains. Properties (2) and (3) now follow exactly as in Remark 7. ◇

Remark 9 (Other work). Baker [1, Theorem 4.1] finds among the Berge knots a sub-collection which have unbounded volume. However, as observed by Goda and Teragaito [8, page 502], all Berge knots are closures of positive (or negative) braids, hence there are only finitely many Berge knots of any given genus. Positivity also implies that the Berge knots are fibred.

Hirose, Kalfagianni, and Kin [10, Theorem 2] give a construction of branched double covers (of any fixed closed, connected, oriented three-manifold M) that are fibred of genus $g \gg 0$ and hyperbolic with volume tending to infinity with g. They ask [10, Question 4] whether, for every M, there is such a sequence with genus fixed and volume unbounded.

Our work gives a positive answer for $M = S^2 \times S^1$, as follows. By Claim 3, the longitudinal filling of $\omega_n \subset S^3$ is $M = S^2 \times S^1$. Let $\omega_n' \subset M$ be the core of the filling solid torus. Then the longitudinal fillings of the knots w_n, as in Theorem 1(3), are branched double covers of M with branch locus $\hat{\beta}_n \cup \omega_n'$. Finally, by using our Theorem 1(1)(2), Hirose, Kalfagianni, and Kin give a positive answer to their Question 4 with $M = S^3$ and genus even [10, Corollary 11].

Very recently, Oakley proved a version of our Theorem 1 for knots in arbitrary closed three-manifolds [14]. Combined with [10, Theorem 10], this gives a complete positive answer to [10, Question 4]. ◇

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