VOLUMES OF CHAIN LINKS

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Abstract. Agol has conjectured that minimally twisted n–chain links are the smallest volume hyperbolic manifolds with n cusps, for $n \leq 10$. In his thesis, Venzke mentions that these cannot be smallest volume for $n \geq 11$, but does not provide a proof. In this paper, we give a proof of Venzke's statement for a number of cases. For $n \geq 60$ we use a formula from work of Futer, Kalfagianni, and Purcell to obtain a lower bound for volume. The proof for n between 12 and 25 inclusive uses a rigorous computer computation that follows methods of Moser and Milley. Finally, we prove that the n–chain link with $2m$ or $2m+1$ half–twists cannot be the minimal volume hyperbolic manifold with n cusps, provided $n \geq 60$ or $|m| \geq 8$, and we give computational data indicating this remains true for smaller n and $|m|$.

1. Introduction

An *n–chain link* consists of n unknotted circles embedded in S^3, linked together in a closed chain. Notice that links of a chain can be connected with an arbitrary amount of twisting. In particular, if we embed the first link in the plane of projection, the next perpendicular to the plane of projection, the next again in the plane of projection, and so on, then the last link may include any integer number of half–twists. See, for example, Figure 1.

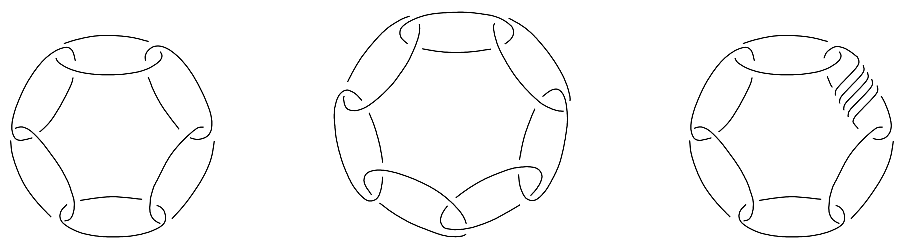

Figure 1. Left: Minimally twisted 6–chain link. Middle: Minimally twisted 7–chain link. Right: 6–chain link with more half–twists.

Hyperbolic structures on n–chain link complements have been studied, for example, by Neumann and Reid [8]. They show any n–chain link complement with $n \geq 5$ admits a hyperbolic structure. In this paper, we are primarily interested in hyperbolic manifolds, so we restrict our attention to $n \geq 5$.

A *minimally twisted n–chain link* is an n–chain link such that, if n is even, each link component alternates between lying embedded in the projection plane and lying perpendicular to the projection plane. If n is odd, the link may be arranged such that each component alternates between lying in the projection plane and perpendicular to it, except a single link component which connects a link which is embedded in the projection plane to one which is perpendicular, with no twisting. See Figure 1.

Notice that there are actually two choices for the minimally twisted n–chain link for n odd, depending on which way the last links are connected. However, these are isometric by an orientation reversing isometry, so we will not distinguish between them.

In [1], Agol conjectures that minimally twisted n–chain link complements, for $n \leq 10$, are the smallest volume hyperbolic 3–manifolds with exactly n cusps, but notes that Venzke has pointed out they cannot be smallest for $n \geq 11$, as the $(n-1)$–fold cyclic cover over one component of the Whitehead link has smaller volume. This statement is included in Venzke's thesis [10]. However, Venzke does not give a proof. In this paper, we give a rigorous proof for $n \geq 60$. The following theorem is the main result of this paper.

Theorem 3.3. *For $n \geq 60$, the minimally twisted n–chain link complement has volume strictly greater than that of the $(n-1)$–fold cyclic cover over one component of the Whitehead link. Hence the minimally twisted n–chain link complement cannot be the smallest volume hyperbolic 3–manifold with n cusps, $n \geq 60$.*

For n between 11 and 59, inclusive, we present computer tabulation of volumes, compared with the volumes of the $(n-1)$–fold cyclic cover of the Whitehead link. See Table 1. By inspection, Theorem 3.3 also holds for these manifolds. In Section 4, we explain how these computations can be made completely rigorous — at least for those values of n for which the minimally twisted n–chain link complement is triangulated with fewer than 100 tetrahedra. In this case, this includes n between 12 and 25, inclusive.

Finally, in Section 5 we evaluate volumes of arbitrarily twisted n–chain link complements. The main result of that section is Theorem 5.3, which states that no n–chain link complement can be the minimal volume n–cusped hyperbolic 3–manifold, provided either $n \geq 60$, or the chain link contains at least 17 half–twists. We present computational data to show that similarly, for $11 \leq n \leq 59$, no n–chain link complement can be minimal volume. When $5 \leq n \leq 10$, we rigorously prove, by computer, that no n–chain link which is *not* minimally twisted can be the minimal volume n–cusped hyperbolic 3–manifold.

Remark 1.1. Since a version of this paper was made public, Hidetoshi Masai has pointed out to us that for even chain links, more can be said. In [9, Chapter 6], Thurston finds a formula for the volumes of minimally twisted chain links with an even number of link components. Namely,

$$\mathrm{vol}(S^3 \setminus C_{2n}) = 8n \left(\Lambda\left(\frac{\pi}{4} + \frac{\pi}{2n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{2n}\right)\right),$$

where Λ is the Lobachevsky function. Masai notes that the difference of this volume and the volume of the $(2n-1)$–cyclic cover over a component of the Whitehead link is an increasing function in n, for $n \geq 6$. This result will give a rigorous proof that minimally twisted n–chain links are not minimal volume for 17 additional chain links, namely for $n = 26, 28, 30, \ldots, 56, 58$. In addition, this gives an alternate proof that minimally twisted $2n$–chain links are not minimal volume for larger n. However, the result for odd links requires other techniques, for instance those in this paper.

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2. Slope lengths on covers

To prove the main theorems of this paper, for $n \geq 60$, we will obtain the complement of the minimally twisted n–chain link by Dehn filling a manifold $\widehat{W}_n$ which is geometrically explicit,

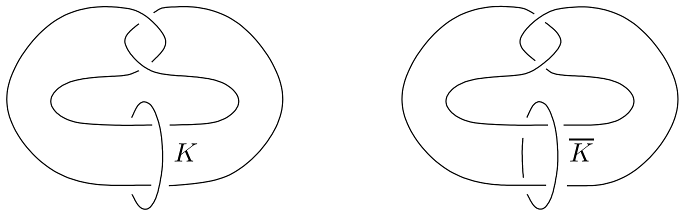

FIGURE 2. The Whitehead link (left), and its reflection (right), with component labeled K and $\overline{K}$, respectively.

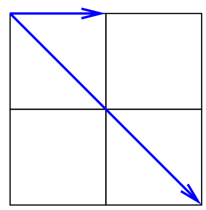

FIGURE 3. Cusp shape of component K of Whitehead link. Meridian runs horizontally across the top, longitude runs diagonally.

constructed by gluing together manifolds isometric to the Whitehead link complement, cut along 2–punctured disks.

We work with the diagram of the Whitehead link as in Figure 2, left, with a link component denoted K. Note that by switching the direction of a pair of crossings, we obtain a link whose complement is isometric to that of the Whitehead link complement by an orientation reversing isometry. The isometry takes K to a link component we denote $\overline{K}$, as in Figure 2, right.

Lemma 2.1. *The shape of the cusp of K is a parallelogram with meridian and longitude meeting at angle $-\pi/4$ (measured from meridian to longitude).*

Similarly, the shape of the cusp of $\overline{K}$ is a parallelogram with meridian and longitude meeting at angle $\pi/4$ (measured from meridian to longitude).

When we take a maximal horocusp about K or $\overline{K}$, the meridian has length $\sqrt{2}$, and the longitude has length 4.

Lemma 2.1 is illustrated in Figure 3.

Proof. The first part of the lemma, and lengths of slopes on K, are well known for the Whitehead link. See, for example [8].

As for $\overline{K}$, the orientation reversing isometry taking the Whitehead link to its reflection takes a meridian of K to a meridian of $\overline{K}$, and reflects the longitude. Since this is an isometry, the meridian and longitude of $\overline{K}$ have the same lengths as those of K, but the angle from the meridian to longitude is reflected across the meridian, to be $\pi/4$. □

Now, the manifold $\widehat{W_n}$ can be described as the minimally twisted n–chain link embedded in a standard solid torus. Therefore, to obtain the complement of the minimally twisted n–chain link, we will Dehn fill $\widehat{W_n}$ along a standard longitude of the solid torus boundary component.

To build $\widehat{W_n}$ from the Whitehead link complement, proceed as follows. First, cut the Whitehead link complement along the 2–punctured disk bounded by K to get a clasp in a cylinder, which we call W_1, shown second from left in Figure 4. Similarly, cut the reflected

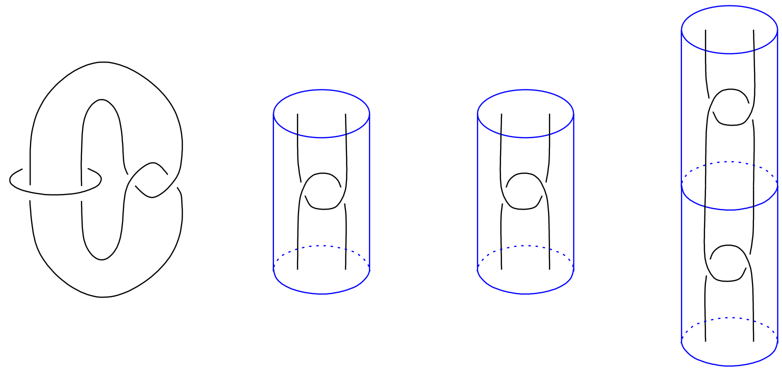

FIGURE 4. Constructing the manifold $\widehat{W}_n$: cut the Whitehead link complement along a 2–punctured disk to obtain W_1 (second from left). Its reflection is $\overline{W}_1$ (third from left). Attach these to form a link in a solid cylinder, right.

Whitehead link complement along the 2–punctured disk bounded by $\overline{K}$ to get a clasp in the opposite direction in a cylinder, which we call $\overline{W}_1$, shown third from left in Figure 4.

Now, attach a copy of W_1 to $\overline{W}_1$ via an isometry of the 2–punctured disk as on the right in Figure 4. In particular, boundary components are glued as shown without twisting. Call the resulting link in a solid cylinder W.

For n even, glue $n/2$ copies of W together end to end, without twisting, followed by gluing the remaining two ends. For n odd, glue $(n-1)/2$ copies of W together without twisting, then glue a single copy of W_1, and attach the ends without twisting. This completes the construction of $\widehat{W}_n$.

Lemma 2.2. *Let $\epsilon = n \mod 2$. The minimally twisted n–chain link in a solid torus, $\widehat{W}_n$, has solid torus boundary component comprised of $\lfloor n/2 \rfloor + \epsilon$ copies of the cusp K coming from W_1, and $\lfloor n/2 \rfloor$ copies of the cusp $\overline{K}$ coming from $\overline{W}_1$. The standard longitude of the solid torus follows a meridian of each copy of K and $\overline{K}$, where the meridians of each copy of K are orthogonal to the meridians of each copy of $\overline{K}$. The length of the longitude of the solid torus boundary component is $\sqrt{n^2+\epsilon}$.*

Proof. This follows from the construction of $\overline{W}_n$ and Lemma 2.1.

The boundary component corresponding to the solid torus comes from $n/2$ copies of the cusp K and $n/2$ copies of the cusp $\overline{K}$ for n even, and $(n-1)/2+1$ copies of the cusp K and $(n-1)/2$ copies of the cusp $\overline{K}$, for n odd. These are glued together along their respective longitudes. Since a copy of the cusp of K is glued to one of $\overline{K}$ along the longitude of each, the meridians meet at right angles. See Figure 5.

The longitude of the solid torus of $\widehat{W}_n$ is given by following each meridian of the copies of K and $\overline{K}$ that glue to give the solid torus boundary component. Since these meridians always meet at right angles, the length of the longitude of the solid torus may be determined by the Pythagorean theorem. By Lemma 2.1, the length of the meridian of the cusp K, and that of the cusp $\overline{K}$, is $\sqrt{2}$. We see that the length of the longitude of the solid torus boundary component of $\widehat{W}_n$ is $\sqrt{\left(\sqrt{2}\left(\lfloor \frac{n}{2} \rfloor + \epsilon\right)\right)^2 + \left(\sqrt{2}\lfloor \frac{n}{2} \rfloor\right)^2} = \sqrt{n^2+\epsilon}$. □

Notice that while the construction of $\widehat{W}_n$ as described above uses $\lfloor n/2 \rfloor + \epsilon$ copies of W_1 and $\lfloor n/2 \rfloor$ copies of $\overline{W}_1$, we could have constructed it using $\lfloor n/2 \rfloor$ copies of W_1 and $\lfloor n/2 \rfloor + \epsilon$

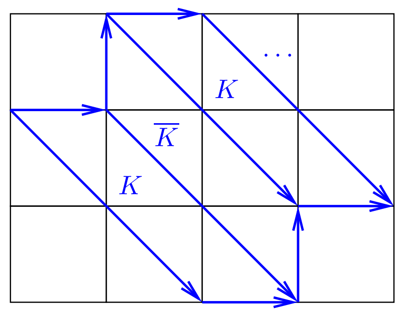

FIGURE 5. The cusp construction of $\widehat{W_n}$.

copies of $\overline{W}_1$ instead. The result of this modified method of construction would be isometric to that of the original construction, by an orientation reversing isometry.

To obtain C_n from $\widehat{W_n}$ we Dehn fill a slope on the solid torus boundary component of $\widehat{W_n}$ that follows one standard longitude of the solid torus. Hence we Dehn fill along a slope of length $\sqrt{n^2 + (n \mod 2)}$.

3. Volumes, large minimally twisted chains

Using the information on slopes above, we may deduce geometric information on minimally twisted chain links by applying appropriate theorems bounding change in geometry under Dehn filling. This will give the desired result when $n \geq 60$.

3.1. **Dehn filling and volume.** We use the following theorem, which is a slightly simpler version of the main theorem in [4].

Theorem 3.1 (Futer–Kalfagianni–Purcell [4]). *Let M be a complete, finite–volume hyperbolic manifold with (at least one) cusp, with horoball neighborhood C about that cusp, and let s be a slope on ∂C with length $\ell(s) > 2\pi$. Then the manifold $M(s)$ obtained by Dehn filling M along s is hyperbolic, with volume:*

$$\mathrm{vol}(M(s)) \;\geq\; \left(1 - \left(\frac{2\pi}{\ell(s)}\right)^2\right)^{3/2} \mathrm{vol}(M).$$

Putting this theorem together with Lemma 2.2, we obtain the following.

Theorem 3.2. *For $n \geq 7$, the volume of the complement of the minimally twisted n–chain link C_n is at least:*

$$\mathrm{vol}(S^3 \smallsetminus C_n) \geq n\, v_8 \left(1 - \frac{4\pi^2}{n^2 + \epsilon}\right)^{3/2},$$

where $v_8 = 3.66386\ldots$ is the volume of a hyperbolic regular ideal octahedron, and $\epsilon = n \mod 2$.

Proof. The chain link complement is obtained by Dehn filling the longitude of $\widehat{W_n}$. This is obtained by gluing copies of the Whitehead link and its reflection along totally geodesic 3–punctured spheres, hence has volume n times the volume of the Whitehead link, $n \cdot v_8$.

By Lemma 2.2, we know the length of the Dehn filling slope is $\sqrt{n^2 + \epsilon}$. The result follows by putting this data into Theorem 3.1. $\square$

We now give a proof of the main theorem.

Theorem 3.3. *For $n \geq 60$, the minimally twisted n–chain link complement has volume strictly greater than that of the $(n-1)$–fold cyclic cover over one component of the Whitehead link. Hence the minimally twisted n–chain link complement cannot be the smallest volume hyperbolic 3–manifold with n cusps, $n \geq 60$.*

Proof. The volume of the $(n-1)$–fold cyclic cover of the Whitehead link is $(n-1)\, v_8$. By Theorem 3.2, the volume of the complement of the minimally twisted n–chain link is

$$\mathrm{vol}(S^3 \smallsetminus C_n) \geq n\, v_8 \left(1 - \frac{4\pi^2}{n^2+\epsilon}\right)^{3/2} \geq n\, v_8 \left(1 - \frac{4\pi^2}{n^2}\right)^{3/2}.$$

We want to find n for which the following inequality holds:

$$n\, v_8 \left(1 - \frac{4\pi^2}{n^2}\right)^{3/2} > (n-1)\, v_8,$$

or

$$\left(\frac{n}{n-1}\right)\left(1 - \frac{4\pi^2}{n^2}\right)^{3/2} - 1 > 0. \tag{1}$$

Let $f(n)$ be the function on the left side of inequality (1). Using calculus, one sees that $\lim_{n\to\infty} f(n) = 0$, f is increasing between $n = 7$ and $n = 6\pi^2 + 2\pi\sqrt{9\pi^2 - 2} \approx 117.8$, and decreasing for larger n, which implies f has at most one root for $n \geq 7$, and that f is positive to the right of any root. The Intermediate Value Theorem implies that there is a root between $n = 59$ and $n = 59.1$. Hence the inequality is satisfied for $n \geq 60$. □

4. Computations of volume, smaller minimally twisted chains

Now we analyze volumes of minimally twisted n–chain links for n between 11 and 59, since the main method of proof of Theorem 3.3 will not apply to these manifolds.

For n between 11 and 59 inclusive, in Table 1 we present computational data using SnapPea (SnapPy) [11, 3] that shows that the minimally twisted n–chain link complement cannot be the minimal volume hyperbolic 3–manifold with n cusps. In particular, W_{n-1}, the $(n-1)$–fold cyclic cover over one component of the Whitehead link, has smaller volume. The volume of W_{n-1} is always $(n-1)\, v_8$, where $v_8 = 3.66386\ldots$ is the volume of a hyperbolic regular ideal octahedron, which is the volume of the Whitehead link complement. Notice that for $n \geq 11$, the volume of $S^3 \smallsetminus C_n$ is strictly larger than that of W_{n-1}.

It would be nice to turn this data into a rigorous proof that the minimally twisted n–chain links for n between 11 and 59 cannot be minimal volume. One way to do this would be to use the methods of Moser [7] and Milley in [6]. Milley has written a program to rigorously prove that a hyperbolic 3–manifold with hyperbolic structure computed by Snap [5] has volume greater than some constant. This program, which is available as supplementary material with [6], is in theory exactly what we need for these chain link examples.

However, in practice, making Moser and Milley's programs work with the chain links has proven to be difficult, due to the computational complexity of the chain links. While Milley worked with small manifolds, for example with less than 10 tetrahedra, and Moser's largest manifold included 57 tetrahedra, our triangulations of minimally twisted n–chain link complements include between 40 and 236 tetrahedra. We were successfully able to run Moser's algorithm for n between 11 and 25, inclusive, which gives results for those manifolds triangulated with up to 100 tetrahedra, but then the program failed. We were able to run

n	$\mathrm{vol}(S^3 \smallsetminus C_n)$	$\mathrm{vol}(W_{n-1})$
5	10.14941606	14.65544951
6	14.65544951	18.31931188
7	19.79685462	21.98317426
8	24.09218408	25.64703664
9	28.47566906	29.31089901
10	32.55154031	32.97476139
11	36.64918655	36.63862377
12	40.59766426	40.30248614
13	44.5536682	43.96634852
14	48.42519197	47.63021090
15	52.29990219	51.29407327
16	56.12184477	54.95793565
17	59.94533184	58.62179803
18	63.73354269	62.28566041
19	67.52257845	65.94952278
20	71.28681886	69.61338516
21	75.05153335	73.27724753
22	78.79813245	76.94110991
23	82.54502011	80.60497229
24	86.27825885	84.26883466
25	90.01168157	87.93269704
26	93.73455871	91.59655942
27	97.45755771	95.26042179
28	101.1722364	98.92428417
29	104.8869984	102.5881465
30	108.5950782	106.2520089
31	112.3032167	109.9158713
32	116.0059062	113.5797337

n	$\mathrm{vol}(S^3 \smallsetminus C_n)$	$\mathrm{vol}(W_{n-1})$
33	119.708638	117.2435961
34	123.4068675	120.9074584
35	127.1051279	124.5713208
36	130.7996249	128.2351832
37	134.494145	131.8990456
38	138.1854868	135.5629079
39	141.8768462	139.2267703
40	145.5654969	142.8906327
41	149.2541611	146.5544951
42	152.9404979	150.2183574
43	156.6268453	153.8822198
44	160.3111779	157.5460822
45	163.995519	161.2099446
46	167.6781044	164.8738070
47	171.3606965	168.5376693
48	175.0417493	172.2015317
49	178.7228075	175.8653941
50	182.4025087	179.5292565
51	186.0822143	183.1931188
52	189.7607173	186.8569812
53	193.4392239	190.5208436
54	197.1166599	194.1847060
55	200.7940988	197.8485683
56	204.4705802	201.5124307
57	208.1470642	205.1762931
58	211.8226885	208.8401555
59	215.4983149	212.5040178
60	219.1731666	216.1678802

TABLE 1. Volumes of the complement of the minimally twisted n–chain link C_n, compared to volumes of W_{n-1}, the $(n-1)$–fold cyclic cover over a component of the Whitehead link, for $5 \leq n \leq 60$. Note that $S^3 \smallsetminus C_n$ has greater volume for $n \geq 11$.

Milley's algorithm for all values of n for which Moser's algorithm applied. However, Milley's algorithm only returned a positive result for n between 12 and 25, inclusive. Therefore, we have the following result.

Theorem 4.1. *For n between 12 and 25, inclusive, the minimally twisted n–chain link complement has volume strictly greater than that of the $(n-1)$–fold cyclic cover over one component of the Whitehead link, hence cannot be the smallest volume hyperbolic 3–manifold with n cusps.*

Proof. The proof is identical to that of Milley [6], and uses his code, included as supplementary material with that reference [6], modified to read in minimally twisted n–chain links rather than Dehn fillings of census manifolds. The first step is to feed the triangulations of the minimally twisted n–chain links into Snap, and ensure that the triangulations used in the computation are geometric, that is, all tetrahedra are positively oriented. This is true for all minimally twisted n–chain links, $11 \leq n \leq 59$.

Next, use Moser's algorithm [7] to find a value δ which measures the maximal error between Snap's computed solution and the true solution. Moser's algorithm gave us such a value for

$11 \leq n \leq 25$, but failed thereafter, presumably due to computational complexity of the chain link complements.

Finally, for each n between 12 and 25, inclusive, input the Snap triangulation data and Moser's value δ into Milley's program `rigorous_volume.C`, along with the constant value $(n-1) * 3.66386237670888$. The program checks rigorously whether the volume of the given n–chain link is larger than the given constant. For $12 \leq n \leq 25$, the program definitively proved that the volumes of the minimally twisted n–chain link complement are strictly larger than that of the $(n-1)$–fold cyclic cover over a component of the Whitehead link. □

Remark 4.2. Note that the above theorem does not hold for $n = 11$. Although the triangulation of the minimally twisted 11–chain link is positively oriented, and Moser's algorithm returns a value of δ for this link, Milley's program `rigorous_volume.C` is unable to verify that its volume is larger than that of the 10–fold cyclic cover of the Whitehead link. When $n = 11$, the volumes of these manifolds are too close for rigorous checking.

What about the volumes output by SnapPea for $26 \leq n \leq 59$? Note in Table 1 that the minimally twisted n–chain link for these n has volume greater than 2 plus the volume of W_{n-1}. It is highly unlikely that SnapPea's computation would be so far off as to make the theorem untrue for any of these values of n. However, since we do not have a rigorous proof at this time, we do not include the result as a theorem.

5. Arbitrary chain links

In this section, we extend our results to chain links with an arbitrary amount of twisting.

Consider again the manifold $\widehat{W_n}$, which is a minimally twisted n–chain link in a solid torus. Let λ_n denote the standard longitude of the solid torus, and let μ_n denote the meridian. In the previous section, we performed Dehn filling along the slope λ_n to obtain the complement of the minimally twisted n–chain link in S^3. Notice that Dehn filling along any slope of the form $\lambda_n + m\,\mu_n$ will also yield the complement of a chain link in S^3, where the resultant chain has $|m|$ additional full twists (or $2|m|$ additional crossings). The twisting will be positive or negative depending on the sign of m.

Lemma 5.1. *For any integer m, the slope $\lambda_n + m\,\mu_n$ on the solid torus boundary component of $\widehat{W_n}$ has length $\sqrt{n^2 + 16m^2 + (n \mod 2)(1 + 8m)}$.*

Proof. The solid torus boundary component of $\widehat{W_n}$ is tiled by regular ideal octahedra coming from the Whitehead link, and these appear as squares of side length $\sqrt{2}$ by Lemma 2.1. When we place the corner of one such square at $(0, 0)$ in the Euclidean plane, we see that the slope λ_n runs from $(0, 0)$ to $(\sqrt{2}(\lfloor n/2 \rfloor + (n \mod 2)), \sqrt{2}\lfloor n/2 \rfloor)$, as in Lemma 2.2.

The slope μ_n runs along exactly one of the 2–punctured disks we sliced in the Whitehead link complement (or its reflection) to build $\widehat{W_n}$. Hence by Lemma 2.1 it runs from $(0, 0)$ to $(2\sqrt{2}, -2\sqrt{2})$.

Thus the slope $\lambda_n + m\mu_n$ runs from $(0, 0)$ to $(\sqrt{2}(\lfloor n/2 \rfloor + (n \mod 2) + 2m, \sqrt{2}(\lfloor n/2 \rfloor - 2m)$, hence has length as follows.

For n even, $n = 2k$,

$$\ell(\lambda_n + m\mu_n) = \left(2(k + 2m)^2 + 2(k - 2m)^2\right)^{1/2} = \sqrt{n^2 + 16m^2}.$$

For n odd, $n = 2k + 1$,

$$\ell(\lambda_n + m\mu_n) = \left(2(k + 1 + 2m)^2 + 2(k - 2m)^2\right) = \sqrt{n^2 + 16m^2 + (1 + 8m)}.$$

□

In order to obtain any n–chain link, in addition to considering Dehn filling on the manifold $\widehat{W}_n$, we must also consider Dehn filling on an n–chain link in the solid torus which differs from $\widehat{W}_n$ by the insertion of a single crossing, or half–twist, at a 2–punctured disk. We call this manifold $\overline{W}_n$. Recall that we constructed $\widehat{W}_n$ by gluing together alternating copies of W_1 and $\overline{W}_1$ along their 2–punctured disk boundaries, without twisting, to form a link in a solid cylinder, and then gluing the cylinder end to end without twisting. To form $\overline{W}_n$, we may glue by a single half–twist when we connect the final solid cylinder end to end. Equivalently, if n is even, replace the last copy of $\overline{W}_1$ with W_1 and glue end to end without twisting. If n is odd, replace the last W_1 with $\overline{W}_1$ and glue end to end without twisting. This gives the desired half–twist in both cases.

Now, denote the standard longitude of the solid torus boundary component of $\overline{W}_n$ by $\bar{\lambda}_n$, and the meridian by $\bar{\mu}_n$. Dehn filling along a slope of the form $\bar{\lambda}_n + m\,\bar{\mu}_n$ yields the complement of an n–chain link in S^3, which differs from the minimally twisted n–chain link by the insertion of $2m+1$ half–twists, where the direction of half–twist is determined by the sign of m.

Lemma 5.2. *For any integer m, the slope $\bar{\lambda}_n + m\,\bar{\mu}_n$ on the solid torus boundary component of $\widehat{W}_n$ has length $\sqrt{n^2+4(1+2m)^2}$, if n is even, and $\sqrt{n^2+16m^2+(1-8m)}$, if n is odd.*

Proof. Again the solid torus boundary component of $\widehat{W}_n$ is tiled by squares of side length $\sqrt{2}$, by Lemma 2.1, which we view with sides parallel to the x and y axes in the Euclidean plane. The slope $\bar{\mu}_n$ still runs once along a 2–punctured disk, which came from W_1 or $\overline{W}_1$, hence runs from $(0,0)$ to $(2\sqrt{2},-2\sqrt{2})$, on the Euclidean plane, also by Lemma 2.1.

First suppose n is even, $n=2k$. The slope $\bar{\lambda}_n$ will be formed by stepping $k+1$ times horizontally (following the meridian of K in the cusp tiling), and stepping $k-1$ times vertically (following the meridian of $\overline{K}$). Hence it runs from $(0,0)$ to $(\sqrt{2}(k+1),\sqrt{2}(k-1))$.

Thus in the case $n=2k$, $\bar{\lambda}_n + m\,\bar{\mu}_n$ runs from $(0,0)$ to $(\sqrt{2}(k+1+2m),\sqrt{2}(k-1-2m))$, so has length

$$\left(2(k+1+2m)^2+2(k-1-2m)^2\right)^{1/2}=\sqrt{n^2+4(1+2m)^2}.$$

When n is odd, $n=2k+1$, the slope $\bar{\lambda}_n$ is formed by stepping k times horizontally and $k+1$ times vertically, hence runs from $(0,0)$ to $(\sqrt{2}k,\sqrt{2}(k+1))$. So the slope $\bar{\lambda}_n + m\,\bar{\mu}_n$ runs from $(0,0)$ to $(\sqrt{2}(k+2m),\sqrt{2}(k+1-2m))$, and thus has length

$$\left(2(k+2m)^2+2(k+1-2m)^2\right)^{1/2}=\sqrt{n^2+16m^2+(1-8m)}.$$

□

Theorem 5.3. *For $n \geq 7$, or $n \geq 5$ and $|m| \geq 1$, the volume of the complement of the n–chain link with r (signed) half–twists is at least:*

$$n\,v_8\left(1-\frac{4\pi^2}{n^2+16m^2}\right)^{3/2} \quad \text{if } n \text{ is even and } r=2m \text{ is even,}$$

$$n\,v_8\left(1-\frac{4\pi^2}{n^2+16m^2+16m+4}\right)^{3/2} \quad \text{if } n \text{ is even and } r=2m+1 \text{ is odd,}$$

$$n\,v_8\left(1-\frac{4\pi^2}{n^2+16m^2+(1+8m)}\right)^{3/2} \quad \text{if } n \text{ is odd and } r=2m \text{ is even,}$$

$$n\,v_8\left(1-\frac{4\pi^2}{n^2+16m^2+(1-8m)}\right)^{3/2} \quad \text{if } n \text{ is odd and } r=2m+1 \text{ is odd.}$$

In all cases, the volume of the complement of that n–chain link is larger than the volume of W_{n-1} whenever $n \geq 60$ or $|m| \geq 8$.

When n lies between 5 and 10, inclusively, the volume of the complement of that n–chain link is larger than the volume of W_{n-1} whenever $|m| \geq 6$.

Proof. The volume estimates come from combining Lemmas 5.1 and 5.2 with Theorem 3.1, using the fact that the volumes of $\widehat{W}_n$ and $\overline{W}_n$ are both $n\, v_8$, as they are both obtained by gluing n copies of manifolds isometric to the Whitehead link complement along totally geodesic 3–punctured spheres.

Note that in all cases, the bound on the volume is minimized for the integer m when $m = 0$, in which case the argument of Theorem 3.3 still shows that the volume is greater than that of W_{n-1} for $n \geq 60$.

Let $\ell(n,m)$ denote the length of the Dehn filling slope, that is, $\ell(n,m)$ is one of the four functions of (n,m) in Lemmas 5.1 and 5.2. The volume of the chain link is guaranteed to be strictly greater than that of W_{n-1} whenever we have

$$n\, v_8 \left(1 - \frac{4\pi^2}{\ell(n,m)^2}\right)^{3/2} \geq (n-1)\, v_8,$$

or whenever the function

$$f(n,m) = \frac{n}{n-1}\left(1 - \frac{4\pi^2}{\ell(n,m)^2}\right)^{3/2} - 1$$

is strictly greater than 0. Notice that in all four cases for $\ell(n,m)$, the function f is increasing with $|m|$. Hence for fixed n, to find where this function is greater than 0, it suffices to set the function equal to zero and solve for m.

We do so, and after a calculation, find that the zeros for m can all be computed in terms of the function

$$R(n) = \frac{1}{2}\sqrt{\frac{\pi^2}{1 - \left(\frac{n-1}{n}\right)^{2/3}} - \frac{n^2}{4}}.$$

That is, the zeros of f for m are given by:

$\pm R(n)$	if n and $r = 2m$ are even,
$-1/2 \pm R(n)$	if n is even and $r = 2m+1$ is odd,
$-1/4 \pm R(n)$	if n is odd and $r = 2m$ is even,
$1/4 \pm R(n)$	if n is odd and $r = 2m+1$ is odd.

Now, check that $R(n) > 0$ for any integer n between 1 and 59, inclusively, and that $R(n)$ is maximized at $n \approx 29.6104$, with maximum value approximately 7.36. Thus the maximum and minimum possible values for the zeros of f lie strictly between -8 and 8.

As for the n–chain links with n between 5 and 10, inclusively, substituting the particular value of n ($n = 5, 6, \ldots, 10$) into $R(n)$ and finding the zeros of f for m, we see that all such zeros lie strictly between -6 and 6. □

In fact, for particular values of n between 5 and 59, inclusively, one can check that often the zeros of m lie in a more restrictive region than between -8 and 8. However, the power of Theorem 5.3 is that it reduces the problem of determining whether any n–chain link may have volume smaller than that of W_{n-1} to the task of checking only finitely many examples.

As in Section 4, we may now check the volumes of only finitely many examples using the computer. These finitely many examples are checked by performing Dehn filling along slopes of the form $(1,m)$, $m = 0, \pm 1, \pm 2, \ldots, \pm 7$, on 98 initial manifolds, namely the manifolds

$\widehat{W}_n$ and $\overline{W}_n$ for $n = 11, 12, \ldots, 59$. In fact, when n is odd, $\widehat{W}_n$ and $\overline{W}_n$ are isometric, by an orientation reversing isometry, and so it suffices to check volumes of Dehn fillings of $\widehat{W}_n$ alone in this case. In the even case, $\widehat{W}_n$ and $\overline{W}_n$ must be checked separately. However, in this case the Dehn fillings of $\widehat{W}_n$ along slopes $(1, m)$ and $(1, -m)$ are isometric, by an orientation reversing isometry, and Dehn fillings of $\overline{W}_n$ along $(1, m)$ and $(1, -(m-1))$ are isometric, by an orientation reversing isometry. Finally, since $(1, 0)$ Dehn filling on $\widehat{W}_n$ gives the minimally twisted n–chain link, which was examined in the previous section, we omit that case. In total, this leaves 14 Dehn fillings to check on the 25 initial manifolds $\widehat{W}_n$ for n odd, 7 Dehn fillings to check on the 24 initial manifolds $\widehat{W}_n$ for n even, and 8 Dehn fillings to check on the 24 initial manifolds $\overline{W}_n$ for n even, or 710 volumes to compute by computer.

We automated the process of computing volumes as follows. Triangulations for initial manifolds $\widehat{W}_n$ and $\overline{W}_n$ were generated by Schleimer using the program Twister [2], which computes SnapPea triangulations for manifolds described from the viewpoint of the mapping class group.

We then ran the triangulations through Snap [5], to find volumes of the manifolds under appropriate Dehn fillings. These volumes were compared with those of W_{n-1}. In all cases, the volume of W_{n-1} was strictly smaller. The data generated is shown in Tables 2, 3, and 4.

Again in order to convert these results into a rigorous proof, the algorithms of Moser [7] and Milley [6] could be applied directly. However, as the triangulations of general n–chain link complements are of similar complexity as those of the minimally twisted chain links, which were too complex for the computer implementation of these algorithms for n larger than 25, we omitted this step.

5.1. **Chain links with 5 through 10 link components.** Our methods can be used to show that of all n–chain links, only the minimally twisted n–chain link can possibly be the minimal volume manifold with n cusps for n between 5 and 10, inclusively.

In fact, because the complexity of these manifolds was comparatively small, we ran them through Milley's algorithm [6], to rigorously check this fact. The algorithm successfully applied, and we have the following theorem.

Theorem 5.4. *Let n be an integer between 5 and 10, inclusively. If C_n is an n–chain link that is* not *minimally twisted, then the complement $S^3 \setminus C_n$ cannot be the minimal volume n–cusped hyperbolic manifold.*

Proof. Theorem 5.3 implies that for these n, and those chain links with at least 11 half–twists, the volume is strictly greater than that of the $(n-1)$–fold cyclic cover over a component of the Whitehead link, which is known to have larger volume than that of the minimally twisted n–chain links in these cases.

The remaining cases to check are Dehn fillings $(1, \pm 1), (1, \pm 2), \ldots, (1, \pm 5)$ on manifolds $\widehat{W}_n$ for n odd, Dehn fillings $(1, 1), \ldots, (1, 5)$ on manifolds $\widehat{W}_n$ for n even, and Dehn fillings $(1, 0), (1, 1), \ldots, (1, 5)$ on manifolds $\overline{W}_n$ for n even. These cases were run through algorithms of Moser [7] and Milley [6], and their programs rigorously proved that the volumes of these chain links were larger than that of the corresponding minimally twisted n–chain link. Programs are available from [6] or the second author. □

In Table 5, we show the volumes of n–chain link complements, n between 5 and 10, whose volumes are not automatically larger than the minimally twisted n–chain link by Theorem 5.3. These are compared with the volume of the minimally twisted chain link.

n	-7	-6	-5	-4	-3	-2	-1	vol(W_{n-1})
11	39.792	39.635	39.405	39.057	38.532	37.780	36.924	36.639
12	43.456	43.309	43.098	42.790	42.340	41.714	40.991	40.302
13	47.059	46.897	46.666	46.338	45.879	45.291	44.715	43.966
14	50.731	50.580	50.371	50.080	49.683	49.182	48.670	47.630
15	54.338	54.175	53.952	53.651	53.260	52.802	52.402	51.294
16	58.015	57.865	57.663	57.396	57.054	56.654	56.284	54.958
17	61.628	61.468	61.258	60.988	60.657	60.300	60.014	58.622
18	65.309	65.162	64.972	64.731	64.439	64.120	63.847	62.286
19	68.927	68.773	68.579	68.340	68.063	67.782	67.571	65.950
20	72.612	72.471	72.294	72.079	71.831	71.575	71.369	69.613
21	76.234	76.089	75.912	75.701	75.469	75.246	75.087	73.277
22	79.922	79.788	79.626	79.436	79.225	79.019	78.859	76.941
23	83.549	83.413	83.252	83.068	82.873	82.695	82.572	80.605
24	87.238	87.113	86.965	86.797	86.618	86.450	86.325	84.269
25	90.869	90.744	90.598	90.438	90.274	90.129	90.033	87.933
26	94.559	94.443	94.309	94.161	94.009	93.871	93.771	91.597
27	98.195	98.079	97.949	97.809	97.670	97.551	97.474	95.260
28	101.885	101.777	101.657	101.527	101.397	101.282	101.202	98.924
29	105.524	105.418	105.301	105.179	105.061	104.963	104.900	102.588
30	109.214	109.115	109.006	108.892	108.781	108.685	108.619	106.252
31	112.857	112.760	112.655	112.549	112.448	112.366	112.314	109.916
32	116.546	116.455	116.358	116.257	116.162	116.081	116.025	113.580
33	120.192	120.104	120.010	119.916	119.830	119.760	119.718	117.244
34	123.881	123.797	123.710	123.621	123.538	123.469	123.423	120.907
35	127.529	127.448	127.365	127.282	127.208	127.149	127.113	124.571
36	131.217	131.141	131.062	130.984	130.911	130.852	130.813	128.235
37	134.868	134.794	134.719	134.646	134.582	134.531	134.500	131.899
38	138.554	138.485	138.414	138.344	138.281	138.230	138.197	135.563
39	142.207	142.140	142.073	142.009	141.952	141.908	141.882	139.227
40	145.893	145.829	145.765	145.704	145.648	145.604	145.575	142.891
41	149.547	149.486	149.426	149.369	149.319	149.281	149.259	146.554
42	153.232	153.174	153.116	153.061	153.013	152.974	152.949	150.218
43	156.888	156.833	156.778	156.727	156.684	156.650	156.631	153.882
44	160.572	160.519	160.466	160.417	160.374	160.340	160.319	157.546
45	164.229	164.179	164.129	164.084	164.045	164.016	163.999	161.210
46	167.912	167.863	167.816	167.772	167.734	167.704	167.685	164.874
47	171.570	171.524	171.480	171.439	171.405	171.379	171.364	168.538
48	175.252	175.208	175.165	175.125	175.091	175.064	175.048	172.202
49	178.912	178.869	178.829	178.792	178.762	178.739	178.726	175.865
50	182.592	182.552	182.512	182.477	182.446	182.422	182.408	179.529
51	186.253	186.214	186.177	186.144	186.117	186.096	186.085	183.193
52	189.933	189.895	189.859	189.827	189.800	189.778	189.765	186.857
53	193.593	193.558	193.525	193.495	193.470	193.452	193.441	190.521
54	197.273	197.238	197.206	197.176	197.151	197.133	197.121	194.185
55	200.934	200.901	200.871	200.844	200.822	200.805	200.796	197.849
56	204.612	204.581	204.551	204.524	204.502	204.485	204.474	201.512
57	208.275	208.245	208.217	208.192	208.172	208.157	208.149	205.176
58	211.952	211.923	211.896	211.871	211.851	211.836	211.826	208.840
59	215.615	215.587	215.561	215.539	215.521	215.507	215.500	212.504

TABLE 2. Volumes of chain links obtained by Dehn filling $\widehat{W}_n$ along slope $s=(1,m)$, where m is the integer at the top of the column, compared with $\mathrm{vol}(W_{n-1})$.

n	1	2	3	4	5	6	7	$\mathrm{vol}(W_{n-1})$
11	37.340	38.184	38.821	39.249	39.531	39.721	39.851	36.639
13	44.982	45.598	46.126	46.517	46.792	46.985	47.122	43.966
15	52.581	53.034	53.467	53.813	54.072	54.263	54.403	51.294
17	60.139	60.478	60.829	61.131	61.370	61.553	61.693	58.622
19	67.662	67.919	68.205	68.465	68.682	68.855	68.991	65.950
21	75.155	75.354	75.586	75.810	76.005	76.166	76.296	73.277
23	82.623	82.780	82.971	83.162	83.336	83.484	83.607	80.605
25	90.073	90.197	90.355	90.520	90.673	90.809	90.925	87.933
27	97.506	97.607	97.738	97.879	98.015	98.139	98.247	95.260
29	104.926	105.009	105.119	105.240	105.361	105.473	105.573	102.588
31	112.335	112.404	112.497	112.602	112.708	112.810	112.902	109.916
33	119.735	119.792	119.872	119.963	120.057	120.149	120.234	117.244
35	127.127	127.176	127.244	127.323	127.407	127.489	127.567	124.571
37	134.513	134.554	134.613	134.682	134.757	134.831	134.903	131.899
39	141.893	141.928	141.979	142.040	142.106	142.174	142.240	139.227
41	149.268	149.299	149.343	149.397	149.456	149.517	149.577	146.554
43	156.639	156.665	156.704	156.752	156.805	156.860	156.916	153.882
45	164.006	164.029	164.064	164.106	164.154	164.204	164.254	161.210
47	171.370	171.390	171.421	171.459	171.501	171.547	171.594	168.538
49	178.731	178.749	178.776	178.810	178.849	178.890	178.933	175.865
51	186.089	186.106	186.130	186.160	186.195	186.233	186.272	183.193
53	193.446	193.460	193.482	193.509	193.541	193.576	193.612	190.521
55	200.800	200.813	200.832	200.857	200.886	200.918	200.951	197.849
57	208.152	208.164	208.181	208.204	208.230	208.259	208.290	205.176
59	215.503	215.513	215.529	215.550	215.574	215.601	215.629	212.504

TABLE 3. Volumes of chain links obtained by Dehn filling $\widehat{W}_n$ along slope $s = (1, m)$, where m is the integer at the top of the column, compared with $\mathrm{vol}(W_{n-1})$.

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n	-7	-6	-5	-4	-3	-2	-1	0	vol(W_{n-1})
12	43.513	43.389	43.214	42.959	42.586	42.049	41.349	40.709	40.302
14	50.790	50.661	50.484	50.237	49.896	49.443	48.914	48.492	47.630
16	58.076	57.945	57.771	57.539	57.234	56.858	56.456	56.165	54.958
18	65.370	65.241	65.073	64.858	64.591	64.279	63.971	63.763	62.286
20	72.671	72.545	72.387	72.191	71.958	71.701	71.461	71.308	69.613
22	79.979	79.858	79.711	79.534	79.332	79.119	78.930	78.814	76.941
24	87.292	87.178	87.042	86.883	86.708	86.531	86.380	86.290	84.269
26	94.610	94.503	94.378	94.237	94.085	93.937	93.815	93.744	91.597
28	101.933	101.833	101.718	101.592	101.461	101.337	101.237	101.180	98.924
30	109.259	109.166	109.062	108.950	108.836	108.730	108.647	108.601	106.252
32	116.588	116.502	116.407	116.307	116.208	116.119	116.049	116.011	113.580
34	123.919	123.840	123.754	123.665	123.579	123.501	123.443	123.411	120.907
36	131.253	131.179	131.101	131.022	130.946	130.880	130.830	130.803	128.235
38	138.588	138.520	138.449	138.379	138.312	138.254	138.211	138.188	135.563
40	145.924	145.861	145.797	145.734	145.675	145.625	145.588	145.568	142.891
42	153.260	153.203	153.145	153.088	153.036	152.992	152.960	152.943	150.218
44	160.598	160.545	160.492	160.441	160.395	160.356	160.328	160.313	157.546
46	167.936	167.888	167.839	167.793	167.752	167.717	167.693	167.680	164.874
48	175.274	175.230	175.186	175.144	175.107	175.076	175.055	175.043	172.202
50	182.613	182.572	182.532	182.494	182.461	182.433	182.414	182.404	179.529
52	189.951	189.914	189.877	189.843	189.813	189.788	189.771	189.762	186.857
54	197.290	197.255	197.222	197.190	197.163	197.141	197.126	197.118	194.185
56	204.629	204.596	204.566	204.537	204.512	204.493	204.479	204.471	201.512
58	211.967	211.937	211.909	211.883	211.860	211.843	211.830	211.824	208.840

TABLE 4. Volumes of chain links obtained by Dehn filling $\overline{W}_n$ along slope $s = (1, m)$, where m is the integer at the top of the column, compared with vol(W_{n-1}).

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Fill mfld	-5	-4	-3	-2	-1	0	min twist
$\widehat{W}_5$	17.806	17.527	16.973	15.743	12.845	-	10.149
$\widehat{W}_6$	21.438	21.171	20.675	19.678	17.628	-	14.655
$\widehat{W}_7$	24.972	24.641	24.042	22.916	20.924	-	19.797
$\widehat{W}_8$	28.630	28.327	27.809	26.904	25.418	-	24.092
$\widehat{W}_9$	32.172	31.821	31.242	30.301	28.996	-	28.476
$\widehat{W}_{10}$	35.851	35.536	35.041	34.274	33.236	-	32.552
$\overline{W}_6$	21.526	21.324	20.963	20.266	18.832	16.000	14.655
$\overline{W}_8$	28.734	28.498	28.104	27.419	26.237	24.553	24.092
$\overline{W}_{10}$	35.963	35.711	35.317	34.696	33.776	32.759	32.552

Fill mfld	1	2	3	4	5	min twist
$\widehat{W}_5$	14.603	16.485	17.301	17.688	17.894	10.149
$\widehat{W}_6$	17.628	19.678	20.675	21.171	21.438	14.655
$\widehat{W}_7$	22.040	23.567	24.387	24.830	25.082	19.797
$\widehat{W}_8$	25.418	26.904	27.809	28.327	28.630	24.092
$\widehat{W}_9$	29.672	30.824	31.568	32.018	32.294	28.476
$\widehat{W}_{10}$	33.236	34.274	35.041	35.536	35.851	32.552
$\overline{W}_6$	16.000	18.832	20.266	20.963	21.324	14.655
$\overline{W}_8$	24.553	26.237	27.419	28.104	28.498	24.092
$\overline{W}_{10}$	32.759	33.776	34.696	35.317	35.711	32.552

TABLE 5. Volumes of small chain links obtained by Dehn filling $\widehat{W}_n$ or $\overline{W}_n$ along slope $s = (1, m)$, where m is the integer at the top of the column, compared with the volume of the minimally twisted chain link.