GEODESICS AND COMPRESSION BODIES

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Abstract. We consider hyperbolic structures on the compression body C with genus 2 positive boundary and genus 1 negative boundary. Note that C deformation retracts to the union of the torus boundary and a single arc with its endpoints on the torus. We call this arc the core tunnel of C. We conjecture that, in any geometrically finite structure on C, the core tunnel is isotopic to a geodesic. By considering Ford domains, we show this conjecture holds for many geometrically finite structures. Additionally, we give an algorithm to compute the Ford domain of such a manifold, and a procedure which has been implemented to visualize many of these Ford domains. Our computer implementation gives further evidence for the conjecture.

1. Introduction

For a hyperbolic manifold M with torus boundary component ∂M_0, every homotopically nontrivial arc in M with endpoints on ∂M_0 is homotopic to a geodesic. However, it seems to be a difficult problem to identify arcs in M which are isotopic to a geodesic, given only a topological description of M.

One place this problem arises is in the study of unknotting tunnels. An *unknotting tunnel* for a 3–manifold M with torus boundary components is defined to be an arc τ from ∂M to ∂M such that $M \setminus N(\tau)$ is a handlebody. Manifolds (other than a solid torus) that admit unknotting tunnels are *tunnel number one* manifolds. Adams asked whether the unknotting tunnel of a hyperbolic tunnel number one manifold is always isotopic to a geodesic [1]. This has been shown to be the case for many classes of hyperbolic tunnel number one manifolds ([2], [19]). Recently, Cooper, Futer, and Purcell showed that the conjecture is true for a generic manifold, in an appropriate sense of generic [9]. The original question still remains open, however.

The purpose of this paper is to present and motivate a related question. Any tunnel number one manifold is built by attaching a compression body C to a handlebody, and the unknotting tunnel corresponds to an arc τ in the compression body. We call τ the *core tunnel* of C. Given Adams' question on whether an unknotting tunnel is isotopic to a geodesic, it seems natural to ask whether the arc τ is isotopic to a geodesic under a complete hyperbolic structure on C.

The compression body C admits many complete hyperbolic structures. Here, we examine those that are geometrically finite, and show that for many such structures, the core tunnel is isotopic to a geodesic. In order to investigate such structures, we develop algorithms to find the Ford domains for geometrically finite structures on C. We present one algorithm that is guaranteed to find the Ford domain in finite time and terminate, but which is impractical in practice, and a procedure which has been implemented for the computer, which will find the Ford domain and terminate for large families of geometrically finite structures, and which we conjecture will always find the Ford domain.

Computer investigation and the theorems proven for families of geometrically finite hyperbolic structures lead us to the following conjecture.

Conjecture 1.1. *Let C be a compression body with $\partial_- C$ a torus, and $\partial_+ C$ a genus two surface. Suppose C is given a geometrically finite hyperbolic structure. Then the core tunnel of C is isotopic to a geodesic.*

In fact, we conjecture something stronger. We conjecture that the core tunnel is not only isotopic to a geodesic, but always dual to a face of the Ford domain. This is Conjecture 5.11, explained in Section 5.

The techniques of this paper can be seen as an extension of work of Jørgensen [16], who found Ford domains of geometrically finite structures on $S \times \mathbb{R}$, where S is a once–punctured torus. Jørgensen's work was extended and expanded by others, including Akiyoshi, Sakuma, Wada, and Yamashita [3, 4]. Wada implemented an algorithm to determine Ford domains of these manifolds [21].

A complete understanding of the geometry of compression bodies, for example through a study of Ford domains, could lead to many interesting applications, since compression bodies are building blocks of more complicated manifolds via Heegaard splitting techniques. With Cooper, we have already applied some of the ideas in this paper to build tunnel number one manifolds with arbitrarily long unknotting tunnels [10].

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2. Background and preliminary material

In this section we review terminology and results used throughout the paper. Our intent is to make this paper as self–contained as possible, and also to emphasize relations between the geometry and topology of compression bodies.

First, we review definitions and results on compression bodies, which are the manifolds we study. Next, we review what it means for these manifolds to admit a geometrically finite hyperbolic structure. We then recall the definition of a Ford domain, since we will be using Ford domains to examine geometrically finite hyperbolic structures on compression bodies. We also give a few definitions relevant to Ford domains, such as visible isometric spheres, Ford spines, and complexes dual to Ford spines. Ford domains of geometrically finite manifolds are finite sided polyhedra; thus we can often identify a Ford domain using the Poincaré polyhedron theorem. Finally, we review this theorem and some of its relevant consequences.

2.1. **Compression bodies.** The manifolds we study in this paper are compression bodies with negative boundary a single torus, and positive boundary a genus 2 surface.

Recall that a *compression body* C is either a handlebody, or the result of taking the product $S \times I$ of a closed, oriented (possibly disconnected) surface S and the interval $I = [0, 1]$, and attaching 1–handles to $S \times \{1\}$ such that the result is connected. The *negative boundary* is $S \times \{0\}$ and is denoted $\partial_- C$. When C is a handlebody, $\partial_- C = \emptyset$. The *positive boundary* is $\partial C \setminus \partial_- C$, and is denoted $\partial_+ C$.

Let C be the compression body for which $\partial_- C$ is a torus and $\partial_+ C$ is a genus 2 surface. We will call this the $(1; 2)$*–compression body*, where the numbers $(1; 2)$ refer to the genus of the boundary components. Note the $(1; 2)$–compression body is formed by taking a torus T^2 crossed with $[0, 1]$ and attaching a single 1–handle to $T^2 \times \{1\}$. The 1–handle retracts to a single arc, the core of the 1–handle.

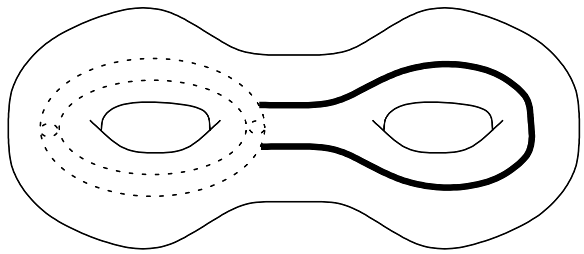

FIGURE 1. The $(1;2)$–compression body. The core tunnel is the thick line shown, with endpoints on the torus boundary.

Let τ be the union of the core of the 1–handle with two vertical arcs in $S \times [0,1]$ attached to its endpoints. Thus, τ is a properly embedded arc in C, and C is a regular neighborhood of $\partial_- C \cup \tau$. We refer to τ as the *core tunnel* of C. See Figure 1, which first appeared in [10].

The fundamental group of a $(1;2)$–compression body C is isomorphic to $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$. We will denote the generators of the $\mathbb{Z} \times \mathbb{Z}$ factor by α, β, and we will denote the generator of the second factor by γ.

2.2. **Hyperbolic structures.** We are interested in the isotopy class of the arc τ when we put a complete hyperbolic structure on the interior of the $(1;2)$–compression body C. We obtain such a structure by taking a discrete, faithful representation $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ and considering the manifold $\mathbb{H}^3/\rho(\pi_1(C))$.

Definition 2.1. A discrete subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is *geometrically finite* if $\mathbb{H}^3/\Gamma$ admits a finite–sided, convex fundamental domain. In this case, we will also say that the manifold $\mathbb{H}^3/\Gamma$ is *geometrically finite*.

The following gives a useful fact about geometrically finite groups in $\mathrm{PSL}(2,\mathbb{C})$.

Theorem 2.2 (Bowditch, Proposition 5.7 [6])**.** *If a subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is geometrically finite, then every convex fundamental domain for $\mathbb{H}^3/\Gamma$ has finitely many faces.*

Definition 2.3. A discrete subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is *minimally parabolic* if it has no rank one parabolic subgroups.

Thus for a discrete, faithful representation $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, the image $\rho(\pi_1(M))$ will be minimally parabolic if for all $g \in \pi_1(C)$, the element $\rho(g)$ is parabolic if and only if g is conjugate to an element of the fundamental group of a torus boundary component of M.

Definition 2.4. A discrete, faithful representation $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ is a *minimally parabolic geometrically finite uniformization of M* if $\rho(\pi_1(M))$ is minimally parabolic and geometrically finite, and $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M.

2.3. **Isometric spheres and Ford domains.** To examine structures on C, we examine paths of Ford domains. This is similar to the technique of Jørgensen [16], developed and expanded by Akiyoshi, Sakuma, Wada, and Yamashita [4], to study hyperbolic structures on punctured torus bundles. Much of the basic material on Ford domains which we review here can also be found in [4].

Throughout this subsection, let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic manifold with a single rank 2 cusp, for example, the $(1;2)$–compression body. In the upper half space model for $\mathbb{H}^3$, assume the point at infinity in $\mathbb{H}^3$ projects to the cusp. Let H be any horosphere about infinity. Let $\Gamma_\infty < \Gamma$ denote the subgroup that fixes H. By assumption, $\Gamma_\infty \cong \mathbb{Z} \times \mathbb{Z}$.

Definition 2.5. For any $g \in \Gamma \setminus \Gamma_\infty$, $g^{-1}(H)$ will be a horosphere centered at a point of $\mathbb{C}$, where we view the boundary at infinity of $\mathbb{H}^3$ to be $\mathbb{C} \cup \{\infty\}$. Define the set $I(g)$ to be the set of points in $\mathbb{H}^3$ equidistant from H and $g^{-1}(H)$. Then $I(g)$ is the *isometric sphere* of g.

Note that $I(g)$ is well–defined even if H and $g^{-1}(H)$ overlap. It will be a Euclidean hemisphere orthogonal to the boundary $\mathbb{C}$ of $\mathbb{H}^3$.

The following is well known, and follows from standard calculations. We include a proof for completeness.

Lemma 2.6. *If*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

then the center of the Euclidean hemisphere $I(g^{-1})$ is $g(\infty) = a/c$. Its Euclidean radius is $1/|c|$.

Proof. The fact that the center is $g(\infty) = a/c$ is clear.

Consider the geodesic running from ∞ to $g(\infty)$. It consists of points of the form $(a/c, t)$ in $\mathbb{C} \times \mathbb{R}^+ \cong \mathbb{H}^3$. It will meet the horosphere H about infinity at some height $t = h_1$, and the horosphere $g(H)$ at some height $t = h_0$. The radius of the isometric sphere $I(g^{-1})$ is the height of the point equidistant from points $(a/c, h_0)$ and $(a/c, h_1)$.

Note that $g^{-1}(g(H)) = H$, and hence h_1 is given by the height of $g^{-1}(a/c, h_0)$, which can be computed to be $(-d/c, 1/(|c|^2 h_0))$. Thus $h_1 = 1/(|c|^2 h_0)$. Then the point equidistant from $(a/c, h_0)$ and $(a/c, 1/(|c|^2 h_0))$ is the point of height $h = 1/|c|$. □

Definition 2.7. Let $B(g)$ denote the *open* half ball bounded by $I(g)$, and define $\mathcal{F}$ to be the set

$$\mathcal{F} = \mathbb{H}^3 \setminus \bigcup_{g \in \Gamma \setminus \Gamma_\infty} B(g).$$

Note $\mathcal{F}$ is invariant under Γ_∞, which acts by Euclidean translations on $\mathbb{H}^3$. We call $\mathcal{F}$ the *equivariant Ford domain.*

When H bounds a horoball H_∞ that projects to an embedded horoball neighborhood about the rank 2 cusp of M, $\mathcal{F}$ is the set of points in $\mathbb{H}^3$ which are at least as close to H_∞ as to any of its translates under $\Gamma \setminus \Gamma_\infty$. Provided Γ is discrete, such an embedded horoball neighborhood of the cusp always exists, by the Margulis lemma.

Definition 2.8. A *vertical fundamental domain for* Γ_∞ is a fundamental domain for the action of Γ_∞ cut out by finitely many vertical geodesic planes in $\mathbb{H}^3$.

Definition 2.9. A *Ford domain* of M is the intersection of $\mathcal{F}$ with a vertical fundamental domain for the action of Γ_∞.

A Ford domain is not canonical because the choice of fundamental domain for Γ_∞ is not canonical. However, the equivariant Ford domain $\mathcal{F}$ in $\mathbb{H}^3$ is canonical, and for purposes of this paper, $\mathcal{F}$ is often more useful than the actual Ford domain.

Note that Ford domains are convex fundamental domains (*cf.* [4, Proposition A.1.2]). Thus we have the following corollary of Bowditch's Theorem 2.2.

Corollary 2.10. *$M = \mathbb{H}^3/\Gamma$ is geometrically finite if and only if a Ford domain for M has a finite number of faces.*

FIGURE 2. Left: Schematic picture of the Ford domain of Example 2.11. Right: Three dimensional view of $\mathcal{F}$ in $\mathbb{H}^3$, for $c = 2 + i$, $a = 6 + 2i$, and $b = 4.5i$.

Example 2.11. Let $c \in \mathbb{C}$ be any complex number such that $|c| > 2$, and let a and b in $\mathbb{C}$ be linearly independent over $\mathbb{R}$ with $|a| > 2|c|$, $|b| > 2|c|$. Let $\rho\colon\ \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ be the representation defined by

$$\rho(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \rho(\gamma) = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

(Recall that α and β denote the generators of the $\mathbb{Z} \times \mathbb{Z}$ factor of $\pi_1(C)$, and γ denotes an additional generator of $\pi_1(C)$.)

By Lemma 2.6, $I(\rho(\gamma))$ has center 0, radius 1, and $I(\rho(\gamma^{-1}))$ has center $c \in \mathbb{C}$, radius 1. Since $|c| > 2$, $I(\rho(\gamma))$ will not meet $I(\rho(\gamma^{-1}))$. By choice of $\rho(\alpha)$, $\rho(\beta)$, all translates of $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$ under Γ_∞ are disjoint.

We will see in Lemma 2.27 that ρ gives a minimally parabolic geometrically finite uniformization of C, and that for this example, $\mathcal{F}$ consists of the exterior of (open) half–spaces $B(\rho(\gamma))$ and $B(\rho(\gamma^{-1}))$, bounded by $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$, respectively, as well as translates of these two isometric spheres under Γ_∞. Thus we will show that the Ford domain for this example is as shown in Figure 2. Before proving this fact, we need additional definitions and lemmas. We use this example to illustrate these definitions and lemmas.

2.4. **Visible faces and Ford domains.** Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic manifold with a single rank two cusp, and let $\Gamma_\infty < \Gamma$ denote a maximal rank two parabolic subgroup, which we may assume fixes the point at infinity in $\mathbb{H}^3$. Notice that $\mathcal{F}$, the equivariant Ford domain of M, has a natural cell structure.

Definition 2.12. Let $g \in \Gamma \setminus \Gamma_\infty$. We say $I(g)$ is *visible* if there exists a 2–dimensional cell of the cell structure on $\mathcal{F}$ contained in $I(g)$.

Similarly, we say the intersection of isometric spheres $I(g_1) \cap \cdots \cap I(g_n)$ is *visible* if there exists a cell of $\mathcal{F}$ contained in $I(g_1) \cap \cdots \cap I(g_n)$ of the same dimension as $I(g_1) \cap \cdots \cap I(g_n)$.

Thus in Example 2.11, we claim that the only visible isometric spheres are $I(\rho(\gamma))$, $I(\rho(\gamma^{-1}))$, and the translates of these under Γ_∞. There are no visible edges for this example.

There is an alternate definition of visible, Lemma 2.13. Let H be a horosphere about infinity that bounds a horoball which is embedded under the projection to M.

Lemma 2.13. *For $g \in \Gamma \setminus \Gamma_\infty$, $I(g)$ is visible if and only if there exists an open set $U \subset \mathbb{H}^3$ such that $U \cap I(g)$ is not empty, and for every $x \in U \cap I(g)$ and every $h \in \Gamma \setminus \Gamma_\infty$, the* *hyperbolic distances satisfy*

$$d(x, h^{-1}(H)) \geq d(x, H) = d(x, g^{-1}H).$$

Similarly, if $I(g) \cap I(h)$ is not empty, then it is visible if and only if there exists an open $U \subset \mathbb{H}^3$ such that $U \cap I(g) \cap I(h)$ is not empty, and for every $x \in U \cap I(g) \cap I(h)$ and every $k \in \Gamma \setminus \Gamma_\infty$,

$$d(x, k^{-1}H) \geq d(x, H) = d(x, g^{-1}H) = d(x, h^{-1}H).$$

Proof. An isometric sphere, or interesction of isometric spheres, is visible if and only if it contains a cell of $\mathcal{F}$ of the same dimension. This will happen if and only if there is some open set U in $\mathbb{H}^3$ which intersects the isometric sphere, or intersections of isometric spheres, in the cell of $\mathcal{F}$ in $\mathbb{H}^3$. The result follows now by definition of $\mathcal{F}$: a point x is in $\mathcal{F}$ if and only if it is not contained in any open half space $B(k)$, $k \in \Gamma \setminus \Gamma_\infty$, if and only if $d(x, H) \leq d(x, k^{-1}H)$. □

We can say something even stronger for isometric spheres:

Lemma 2.14. *For Γ discrete, the following are equivalent.*

1. *The isometric sphere $I(g)$ is visible.*
2. *There exists an open set $U \subset \mathbb{H}^3$ such that $U \cap I(g)$ is not empty and for any $x \in U \cap I(g)$ and any $h \in \Gamma \setminus (\Gamma_\infty \cup \Gamma_\infty g)$,*
$$d(x, h^{-1}H) > d(x, H) = d(x, g^{-1}H).$$
3. *$I(g)$ is not contained in $\bigcup_{h \in \Gamma \setminus (\Gamma_\infty \cup \Gamma_\infty g)} \overline{B(h)}$.*

Proof. If (2) holds, then Lemma 2.13 implies $I(g)$ is visible. Conversely, suppose $I(g)$ is visible. Let U be as in Lemma 2.13, so that for all $x \in U \cap I(g)$, and all $h \in \Gamma \setminus \Gamma_\infty$, $d(x, h^{-1}H) \geq d(x, H) = d(x, g^{-1}H)$. Suppose there is some $h \in \Gamma \setminus \Gamma_\infty$ such that for all $x \in U \cap I(g)$ we have equality: $d(x, h^{-1}H) = d(x, H) = d(x, g^{-1}H)$. Then the isometric spheres $I(h)$ and $I(g)$ must agree on an open subset, hence they must agree everywhere. In particular, their centers must agree: $g^{-1}(\infty) = h^{-1}(\infty)$.

Now, notice that $g^{-1}\Gamma_\infty g$ is the subgroup of Γ fixing $g^{-1}(\infty)$, since α fixes $g^{-1}(\infty)$ if and only if $g\alpha g^{-1}$ fixes infinity, so lies in Γ_∞. Next note that since $I(g) = I(h)$, $g^{-1}h$ fixes $g^{-1}(\infty)$. So $g^{-1}h \in g^{-1}\Gamma_\infty g$. Thus $h \in \Gamma_\infty g$. We have shown (1) if and only if (2).

Finally, (2) clearly implies (3). If $I(g)$ is not visible, then for any $x \in I(g)$, either $x \notin \mathcal{F}$, which implies $x \in \bigcup_{h \in \Gamma \setminus (\Gamma_\infty \cup \Gamma_\infty g)} \overline{B(h)}$, or x is in a cell of $\mathcal{F}$ with dimension at most 1. In this case, $x \in I(h)$ for some $h \in \Gamma \setminus (\Gamma_\infty \cup \Gamma_\infty g)$. Thus (3) implies (1). □

Notice that in the above proof, we showed that if two isometric spheres $I(g)$ and $I(h)$ agree, then $h \in \Gamma_\infty g$. It is clear that if $h \in \Gamma_\infty g$, then $I(g) = I(h)$.

We now present two results on visible faces of the Ford domain. Again these are well known, but we include proofs for completeness.

Lemma 2.15. *Let Γ be a discrete, torsion free subgroup of $\mathrm{PSL}(2, \mathbb{C})$ with a rank two parabolic subgroup Γ_∞ fixing the point at infinity, and let $g \in \Gamma \setminus \Gamma_\infty$. Then $I(g)$ is visible if and only if $I(g^{-1})$ is visible. Moreover, g takes $I(g)$ isometrically to $I(g^{-1})$, sending the half space $B(g)$ bounded by $I(g)$ to the exterior of the half space $B(g^{-1})$.*

Proof. Let H be a horosphere about infinity in $\mathbb{H}^3$ that bounds a horoball which projects to an embedded neighborhood of the cusp of M.

First, note that under g, $I(g)$ is mapped isometrically to $I(g^{-1})$, since g takes H to $g(H)$, and $g^{-1}(H)$ to H, and hence takes $I(g)$ to the set of points equidistant from these two horospheres. This is the isometric sphere $I(g^{-1})$. Note the half space $B(g)$, which contains $g^{-1}(H)$, must be mapped to the exterior of $B(g^{-1})$, which contains H, as claimed.

Suppose $I(g)$ is visible. Then there exists an open set $U \subset \mathbb{H}^3$, with $U \cap I(g)$ not empty, so that for every x in $I(g) \cap U$, and for every $h \in \Gamma \backslash \Gamma_\infty$, $d(x, h^{-1}(H)) \geq d(x, H) = d(x, g^{-1}(H))$.

Now consider the action of g on this picture. The set $g(U)$ is open in $\mathbb{H}^3$, and for all $y \in g(U) \cap I(g^{-1})$, we have $y = g(x)$, for some $x \in U \cap I(g)$, so the distance $d(y, H) = d(g(x), gg^{-1}(H)) \leq d(g(x), gh^{-1}(H)) = d(y, gh^{-1}(H))$, for all $h \in \Gamma \setminus \Gamma_\infty$. So $I(g^{-1})$ is visible.

To finish, apply the same proof to g^{-1}. □

Lemma 2.16. *Gluing isometric spheres corresponding to $\rho(\gamma)$ and $\rho(\gamma^{-1})$ of Example 2.11 gives a manifold homeomorphic to the interior of the $(1;2)$–compression body C.*

Proof. In the example, first glue sides of the vertical fundamental domain via the parabolic transformations fixing infinity. The result is homeomorphic to the cross product of a torus and an open interval $(0,1)$. Next glue the face $I(\rho(\gamma))$ to $I(\rho(\gamma^{-1}))$ via γ. The result is topologically equivalent to attaching a 1–handle, yielding a manifold homeomorphic to C. □

Lemma 2.17. *Let Γ be a discrete, torsion free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with a rank two parabolic subgroup Γ_∞ fixing the point at infinity. Suppose $g, h \in \Gamma \setminus \Gamma_\infty$, with $I(g)$ and $I(h)$ visible, and suppose $I(g) \cap I(h)$ is visible. Then $I(gh^{-1}) \cap I(h^{-1})$ is visible, and h maps the visible portion of $I(g) \cap I(h)$ isometrically to the visible portion of $I(gh^{-1}) \cap I(h^{-1})$. In addition, there must be some visible isometric sphere $I(k)$, not equal to $I(h^{-1})$, such that $I(k) \cap I(h^{-1}) = I(gh^{-1}) \cap I(h^{-1})$.*

Notice that in Lemma 2.17, $I(k)$ may be equal to $I(gh^{-1})$, but is not necessarily so. In fact, $I(gh^{-1})$ may not be visible, such as in the case that there is a quadrilateral dual to $I(g) \cap I(h)$. We discuss dual faces later.

Proof. Let H be a horosphere about infinity which bounds a horoball that projects to an embedded neighborhood of the cusp of M. Suppose $I(g) \cap I(h)$ is visible. By Lemma 2.13, there exists an open set $U \subset \mathbb{H}^3$ such that for all $x \in U \cap (I(g) \cap I(h))$, and all $k \in \Gamma \setminus \Gamma_\infty$, the hyperbolic distance $d(x, H)$ is less than or equal to the hyperbolic distance $d(x, k^{-1}(H))$. Since $x \in I(g) \cap I(h)$, we also have $d(x, g^{-1}H) = d(x, h^{-1}H) = d(x, H)$.

Apply h to this picture. We obtain:

$$d(h(x), hg^{-1}H) = d(h(x), H) = d(h(x), hH) \leq d(h(x), hk^{-1}H)$$

for all $k \in \Gamma \backslash \Gamma_\infty$. Thus for all y in the intersection of the open set $h(U)$ and $I(gh^{-1}) \cap I(h^{-1})$, $y = h(x)$ satisfies the inequality of Lemma 2.13, and so $I(gh^{-1}) \cap I(h^{-1})$ is visible. Since this works for any such open set U, and the 1–cell of $\mathcal{F}$ contained in $I(g) \cap I(h)$ may be covered with such open sets, h maps visible portions isometrically.

Finally, since $I(gh^{-1}) \cap I(h^{-1})$ is visible, it contains a 1–dimensional cell of $\mathcal{F}$. There must be two 2–dimensional cells of $\mathcal{F}$ bordering $I(gh^{-1}) \cap I(h^{-1})$. One of these is contained in $I(h^{-1})$, using the fact that $I(h)$ is visible and Lemma 2.15. The other must be contained in some $I(k)$ (possibly, but not necessarily $I(gh^{-1})$), and so this $I(k)$ is visible. □

The first part of Lemma 2.17 is a portion of what Akiyoshi, Sakuma, Wada, and Yamashita call the *chain rule for isometric circles* [4, Lemma 4.1.2].

Additionally, we present a result that allows us to identify geometrically finite uniformizations that are minimally parabolic.

Lemma 2.18. *Suppose $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ is a geometrically finite uniformization. Suppose none of the visible isometric spheres of the Ford domain of $\mathbb{H}^3/\rho(\pi_1(C))$ are visibly tangent on their boundaries. Then $\rho(\pi_1(C))$ is minimally parabolic.*

By *visibly* tangent, we mean the following. Set $\Gamma = \rho(\pi_1(C))$, and assume a neighborhood of infinity in $\mathbb{H}^3$ projects to the rank two cusp of $\mathbb{H}^3/\Gamma$, with $\Gamma_\infty < \Gamma$ fixing infinity in $\mathbb{H}^3$. For any $g \in \Gamma \setminus \Gamma_\infty$, the isometric sphere $I(g)$ has boundary that is a circle on the boundary $\mathbb{C}$ at infinity of $\mathbb{H}^3$. This circle bounds an open disk $D(g)$ in $\mathbb{C}$. Two isometric spheres $I(g)$ and $I(h)$ are *visibly tangent* if their corresponding disks $D(g)$ and $D(h)$ are tangent on $\mathbb{C}$, and for any other $k \in \Gamma \setminus \Gamma_\infty$, the point of tangency is not contained in the open disk $D(k)$.

Proof. Suppose $\rho(\pi_1(C))$ is not minimally parabolic. Then it must have a rank 1 cusp. Apply an isometry to $\mathbb{H}^3$ so that the point at infinity projects to this rank 1 cusp. The Ford domain becomes a region P meeting this cusp, with finitely many faces. Take a horosphere about infinity sufficiently small that the intersection of the horosphere with P gives a subset of Euclidean space with sides identified by elements of $\rho(\pi_1(C))$, conjugated appropriately.

The side identifications of this subset of Euclidean space, given by the side identifications of P, generate the fundamental group of the cusp. But this is a rank 1 cusp, hence its fundamental group is $\mathbb{Z}$. Therefore, the side identification is given by a single Euclidean translation. The Ford domain P intersects this horosphere in an infinite strip, and the side identification glues the strip into an annulus. Note this implies two faces of P are tangent at infinity.

Now apply an isometry, taking us back to our usual view of $\mathbb{H}^3$, with the point at infinity projecting to the rank 2 cusp of the $(1;2)$–compression body $\mathbb{H}^3/\rho(\pi_1(C))$. The two faces of P tangent at infinity are taken to two isometric spheres of the Ford domain, tangent at a visible point on the boundary at infinity. □

We will see that the converse to Lemma 2.18 is not true. There exist examples of geometrically finite representations for which two visible isometric spheres are visibly tangent, and yet the representation is still minimally parabolic. Such an example is given, for example, in Example 4.1, with $t = \sqrt{3}$.

Remark 2.19. In Example 2.11, we claimed that the only visible isometric spheres are those of $I(\rho(\gamma))$, $I(\rho(\gamma^{-1}))$, and their translates under Γ_∞. Since none of these isometric spheres are visibly tangent, provided the claim is true, Lemma 2.18 will imply that this representation is minimally parabolic.

2.5. **The Ford spine.** Let Γ be discrete and geometrically finite. When we glue the Ford domain into the manifold $M = \mathbb{H}^3/\Gamma$, the faces of the Ford domain will be glued together in pairs to form M.

Definition 2.20. The *Ford spine* of M is defined to be the image of the faces, edges, and 0–cells of $\mathcal{F}$ under the covering $\mathbb{H}^3 \to M$.

A spine usually refers to a subset of the manifold for which there is a retraction of the manifold. Using that definition, the Ford spine is not strictly a spine. However, the union of the Ford spine and the non-toroidal boundary components will be a spine for a manifold M with a single rank 2 cusp.

To make that last sentence precise, recall that given a geometrically finite uniformization ρ, the *domain of discontinuity* Ω is the complement of the limit set of $\rho(\pi_1(M))$ in the boundary at infinity $\partial_\infty \mathbb{H}^3$. See, for example, Marden [17, section 2.4].

Lemma 2.21. *Let ρ be a minimally parabolic geometrically finite uniformization of a 3–manifold M with a single rank 2 cusp. Then the manifold $(\mathbb{H}^3 \cup \Omega)/\rho(\pi_1(M))$ retracts onto the union of the Ford spine and the boundary at infinity $(\overline{\mathcal{F}} \cap \mathbb{C})/\Gamma_\infty$.*

Proof. Let H be a horosphere about infinity in $\mathbb{H}^3$ that bounds a horoball which projects to an embedded horoball neighborhood of the cusp of $\mathbb{H}^3/\rho(\pi_1(M))$. Let x be any point in $\mathcal{F} \cap \mathbb{H}^3$. The nearest point on H to x lies on a vertical line running from x to infinity. These vertical lines give a foliation of $\mathcal{F}$. All such lines have one endpoint on infinity, and the other endpoint on $\overline{\mathcal{F}} \cap \mathbb{C}$ or an isometric sphere of $\mathcal{F}$. We obtain our retraction by mapping the point x to the endpoint of its associated vertical line, then quotienting out by the action of $\rho(\pi_1(M))$. □

To any face F_0 of the Ford spine, we obtain an associated collection of visible elements of Γ: those whose isometric sphere projects to F_0 (or more carefully, a subset of their isometric sphere projects to the face F_0).

Definition 2.22. We will say that an element g of Γ *corresponds* to a face F_0 of the Ford spine of M if $I(g)$ is visible and (the visible subset of) $I(g)$ projects to F_0. In this case, we also say F_0 corresponds to g. Notice the correspondence is not unique: if g corresponds to F_0, then so does g^{-1} and $w_0 g^{\pm 1} w_1$ for any words $w_0, w_1 \in \Gamma_\infty$.

Remark 2.23. Consider again the unifomization of C given in Example 2.11. We will see that the Ford domain of this example has faces coming from a vertical fundamental domain and the two isometric spheres $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$. Hence the Ford spine of this manifold consists of a single face, corresponding to $\rho(\gamma)$.

2.6. **Poincaré polyhedron theorem.** We need a tool to identify the Ford domain of a hyperbolic manifold. This tool will be Lemma 2.26. The proof of that lemma uses the Poincaré polyhedron theorem, which we use repeatedly in this paper. Those results we use most frequently are presented in this subsection. Our primary reference is Epstein and Petronio [13], which contains a version of the Poincaré theorem that does not require finite polyhedra.

The setup for the following theorems is the same. We begin with a finite number of elements of $\mathrm{PSL}(2,\mathbb{C})$, $g_1, g_2, \dots, g_n$, as well as a parabolic subgroup $\Gamma_\infty \cong \mathbb{Z}\times\mathbb{Z}$ of $\mathrm{PSL}(2,\mathbb{C})$, fixing the point at infinity. Let P be a polyhedron cut out by isometric spheres corresponding to $\{g_1, \dots, g_n\}$ and $\{g_1^{-1}, \dots, g_n^{-1}\}$, as well as either:

1. all isometric spheres given by translations of g_i and g_i^{-1} under Γ_∞, or
2. a vertical fundamental domain for the action of Γ_∞.

An example of the former would be an equivariant Ford domain, $\mathcal{F}$. An example of the latter would be a Ford domain. Note that in both cases, we allow P to contain an open neighborhood of a point on the boundary at infinity of $\mathbb{H}^3$, so it will not necessarily have finite volume.

Let M be the object obtained from P by gluing isometric spheres corresponding to g_j and g_j^{-1} via the isometry g_j, for all j, and then, if applicable, gluing faces of the vertical fundamental domain by parabolic isometries in Γ_∞.

Theorem 2.24 (Poincaré polyhedron theorem, weaker version)**.** *For P, M as above, if M is a smooth hyperbolic manifold, then*

- *the group* Γ *generated by face pairings is discrete,*
- $\pi_1(M) \cong \Gamma$.

Proof. The result will follow essentially from [13, Theorem 5.5]. First we check the conditions of this theorem. Since M is a smooth hyperbolic manifold, the condition *Pairing*, requiring faces to meet isometrically, will hold. Similarly, the condition *Cyclic* must hold, requiring the monodromy around an edge in the identification to be the identity, and sums of dihedral angles to be 2π. Condition *Connected* is automatically true for P a single polyhedron (rather than a collection of polyhedra). Finally, note that since we have a finite number of original isometric spheres corresponding to $g_1, \dots, g_n$ and their inverses, and translation by an element in Γ_∞ moves an isometric sphere a fixed positive distance, any isometric sphere of P can meet only finitely many other isometric spheres. This is sufficient to imply condition *Locally finite*.

We need to show the universal cover $\widetilde{M}$ of M is complete. Since M is a smooth hyperbolic manifold and P is complete, M will be complete if and only if the link of its ideal vertex inherits a Euclidean structure coming from horospherical cross sections to P, by [20, Theorem 3.4.23]. In the case that P is cut out only by isometric spheres and their translates under Γ_∞, there is nothing to show. In the case that P is cut out by a vertical fundamental domain, we know the holonomy of the link of this vertex is given by the group Γ_∞, which is a rank 2 subgroup of $\mathrm{PSL}(2, \mathbb{C})$ fixing the point at infinity. Thus it acts on a horosphere about infinity by Euclidean isometries, and so M is indeed complete. It follows that $\widetilde{M}$ is complete.

Thus all the conditions for [13, Theorem 5.5] hold, and the developing map $\widetilde{M} \to \mathbb{H}^3$ is a covering map, with covering transformations generated by Γ. It follows that Γ is discrete, and $\pi_1(M) \cong \Gamma$. □

Theorem 2.25 (Poincaré polyhedron theorem)**.** *For* P, M *as above, and* Γ *the group generated by face pairings, suppose each face pairing maps a face of* P *isometrically to another face of* P*, and that for each edge* e *of* M*, i.e. for each equivalence class of intersections of isometric spheres under the equivalence given by the gluing, the sum of dihedral angles about* e *is* 2π*, and the monodromy around the edge is the identity. Then*

- M *is a smooth hyperbolic manifold with* $\pi_1(M) \cong \Gamma$*, and*
- Γ *is discrete.*

Proof. Again this follows from various results in [13]. Because faces of P are mapped isometrically, we have the condition *Pairing*. The fact that dihedral angles sum to 2π and the monodromy is the identity implies condition *Cyclic*. Again because isometric spheres can meet only finitely many others in P, we have condition *Locally finite*, and because we have a single polyhedron, we have condition *Connected*. When we send P to $\mathbb{H}^3$ via the developing map, we may find a horosphere about infinity disjoint from the isometric spheres forming faces of P. In the case that P is cut out by a vertical fundamental domain, since Γ_∞ preserves this horosphere and acts on it by Euclidean transformations, in the terminology of Epstein and Petronio, the universal cover of the boundary of M has a consistent horosphere. This is true automatically if P is not cut out by a vertical fundamental domain. Then by [13, Theorem 6.3], the universal cover $\widetilde{M}$ of M is complete. Now Poincaré's Theorem [13, Theorem 5.5] implies the developing map $\widetilde{M} \to \mathbb{H}^3$ is a covering map, hence $M \cong \mathbb{H}^3/\Gamma$ is a smooth, complete hyperbolic manifold with $\pi_1(M) \cong \Gamma$ a discrete group. □

Our first application of Poincaré's theorem is the following lemma, which helps us identify Ford domains.

Lemma 2.26. *Let Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with a rank 2 parabolic subgroup Γ_∞ fixing the point at infinity.*

Suppose the isometric spheres corresponding to a finite set of elements of Γ, as well as their translates under Γ_∞, cut out a region $\mathcal{G}$ so that the quotient under face pairings and the group Γ_∞ yields a smooth hyperbolic manifold with fundamental group Γ. Then Γ is discrete and geometrically finite, and $\mathcal{G}$ must be the equivariant Ford domain of $\mathbb{H}^3/\Gamma$.

Similarly, suppose the isometric spheres corresponding to a finite set of elements of Γ, as well as a vertical fundamental domain for Γ_∞, cut out a polyhedron P, so that face pairings given by the isometries corresponding to isometric spheres and to elements of Γ_∞ yield a smooth hyperbolic manifold with fundamental group Γ. Then Γ is discrete and geometrically finite, and P must be a Ford domain of $\mathbb{H}^3/\Gamma$.

Proof. In both cases, Theorem 2.24 immediately implies that Γ is discrete. The fact that Γ is geometrically finite follows directly from the definition.

In the case of the polyhedron P, suppose P is not a Ford domain. Since the Ford domain is only well–defined up to choice of fundamental region for Γ_∞, there is a Ford domain F with the same choice of vertical fundamental domain for Γ_∞ as for P. Since P is not a Ford domain, F and P do not coincide. Because both are cut out by isometric spheres corresponding to elements of Γ, there must be a visible face that cuts out the domain F that does not agree with any of those that cut out the domain P. Hence F is a strict subset of P, and there is some point x in $\mathbb{H}^3$ which lies in the interior of P, but does not lie in the Ford domain.

Now consider the covering map $\varphi\colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$. This map φ glues both P and F into the manifold $\mathbb{H}^3/\Gamma$, since both are fundamental regions for the manifold. Now consider φ applied to x. Because x lies in the interior of P, and P is a fundamental domain, there is no other point of P mapped to $\varphi(x)$. On the other hand, x does not lie in the Ford domain F. Thus there is some preimage y of $\varphi(x)$ under φ which does lie in F. But F is a subset of P. Hence we have $y \neq x$ in P such that $\varphi(x) = \varphi(y)$. This contradiction finishes the proof in the case of the polyhedron P.

The proof for $\mathcal{G}$ is nearly identical. Again if $\mathcal{G}$ is not the equivariant Ford domain $\mathcal{F}$, then there is an additional visible face of $\mathcal{F}$ besides those that cut out $\mathcal{G}$, and again there is some point x in $\mathbb{H}^3$ which lies in the interior of $\mathcal{G}$, but does not lie in $\mathcal{F}$. Again the covering map $\varphi\colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ glues $\mathcal{G}$ and $\mathcal{F}$ into the manifold $\mathbb{H}^3/\Gamma$, and again since a point x lies in $\mathcal{G}$ but not in $\mathcal{F}$, we have some $y \neq x$ in $\mathcal{F}$ such that $\varphi(x) = \varphi(y)$. Again this is a contradiction. □

We may now complete the proof that the Ford domain of the representation of Example 2.11 is as shown in Figure 2.

Lemma 2.27. *Let $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ be the representation given in Example 2.11. Then ρ gives a minimally parabolic geometrically finite uniformization of C, and a Ford domain is given by the intersection of a vertical fundamental domain for Γ_∞ with the half–spaces exterior to the two isometric spheres $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$.*

Proof. We have seen that $I(\rho(\gamma))$, $I(\rho(\gamma^{-1}))$, and the translates of these isometric spheres under Γ_∞ are all disjoint. Select a vertical fundamental domain for Γ_∞ which contains the isometric spheres $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$. This is possible by choice of $\rho(\alpha)$ and $\rho(\beta)$, particularly because the translation lengths $|a|$ and $|b|$ are greater than $2|c|$.

Let P be the region in the interior of the vertical fundamental domain, exterior to the half–spaces $B(\rho(\gamma))$ and $B(\rho(\gamma^{-1}))$ bounded by $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$, respectively. Then when we identify vertical sides of P via elements of Γ_∞, and identify $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$

via $\rho(\gamma^{-1})$, the object we obtain is a smooth hyperbolic manifold, by Theorem 2.25, since P has no edges. Lemma 2.26 now implies that P is a Ford domain for $\mathbb{H}^3/\Gamma$, and that Γ is geometrically finite. Lemma 2.18 implies Γ is minimally parabolic. Finally, Lemma 2.16 shows $\mathbb{H}^3/\Gamma$ is homeomorphic to the interior of C, so this is indeed a uniformization of C. $\square$

We conclude this section by stating a lemma that will help us identify representations which are *not* discrete. It is essentially the Shimizu–Leutbecher lemma [18, Proposition II.C.5].

Lemma 2.28. *Let Γ be a discrete, torsion free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ such that $M = \mathbb{H}^3/\Gamma$ has a rank two cusp. Suppose that the point at infinity projects to this cusp, and let Γ_∞ be its stabilizer in Γ. Then for all $\zeta \in \Gamma \setminus \Gamma_\infty$, the isometric sphere of ζ has radius at most the minimal (Euclidean) translation length of all non-trivial elements in Γ_∞.*

3. Algorithm to compute Ford domains

We will use Ford domains to study geometrically finite minimally parabolic uniformizations of the $(1;2)$–compression body. To facilitate this study, we have developed algorithms to construct Ford domains. In this section, we present an algorithm which is guaranteed to construct the Ford domain, but is impractical. We also present a practical procedure which we have implemented, which we conjecture will always construct the Ford domain of the $(1;2)$–compression body.

3.1. An initial algorithm.

Let Γ be a discrete, geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{C})$ such that $\mathbb{H}^3/\Gamma$ is homeomorphic to the interior of the $(1;2)$–compression body. We will assume that Γ is given by an explicit set of matrix generators. We now present an (impractical) algorithm to find the Ford domain of $\mathbb{H}^3/\Gamma$. Assume without loss of generality that in the universal cover $\mathbb{H}^3$, the point at infinity is fixed by the rank 2 cusp subgroup, $\Gamma_\infty < \Gamma$.

Algorithm 3.1. Enumerate all elements of the group: $\Gamma = \{g_1, g_2, g_3, \dots\}$. Again we assume that each g_i is given as a matrix with explicit entries. Step through the list of group elements. At the n-th step:

1. Draw isometric spheres corresponding to g_n and g_n^{-1}.
2. If these isometric spheres are visible over other previously drawn isometric spheres (corresponding to $g_1, \dots, g_{n-1}$ and their inverses), check if the object obtained by gluing pairs of currently visible, previously drawn isometric spheres via the corresponding isometries satisfies the hypotheses of Theorem 2.25.
3. If it does satisfy these hypotheses, then by the Poincaré polyhedron theorem, Theorem 2.24, the fundamental group of the manifold is generated by isometries corresponding to face identifications. Therefore, if we can write the generators of Γ as words in the isometries of these faces, we will be done, by Lemma 2.26. Put this manifold into a list of manifolds built by repeating the previous two steps.
4. For each manifold in the list of manifolds built by steps (1) and (2), we have an enumeration of words in the group elements generated by gluing isometries of faces: $L = \{h_1, h_2, \dots\}$.
   - (a) For each generator g of Γ, step through the first n words of L to see if g equals one of these words.
   - (b) If each g can be written as a word in one of the first n elements of L, we are done. The Ford domain is given by the isometric spheres which are the faces of this manifold.

Note that in step (2), if we find that isometric spheres glue to give a manifold, it does not necessarily follow that this manifold is our original compression body. For example, we may have found a non-trivial cover of the original compression body. Therefore, steps (3) and (4) are required.

Since Ford domains of geometrically finite hyperbolic manifolds have a finite number of faces, after a finite number of steps, Algorithm 3.1 will have drawn all isometric spheres corresponding to visible faces. Since identifying a finite number of generators as words in a finite number of generators given by face pairings can be done in a finite number of steps, after a finite number of steps the algorithm will terminate.

3.2. **A practical procedure.** The algorithm above is impractical for computer implementation. In this section we present a practical procedure, which will generate the Ford domain and terminate in many cases for a (1; 2)–compression body. We conjecture it will terminate for all cases.

We have implemented this procedure, and used the images it produced to analyze behavior of paths of Ford domains. The computer images of this paper were generated by this program.

Procedure 3.2. Let α, β be parabolic, fixing a common point at infinity in $\mathbb{H}^3$. Let γ be loxodromic, such that $\langle \alpha, \beta, \gamma \rangle \cong (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$.

Conjugate such that

$$\alpha = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

We will hold two lists: The list of elements to draw, L_0, and the list of elements that have been drawn L_1. These are ordered lists.

Initialization. Replace α and β if necessary, so that the lattice generated by a and b has generators of shortest length.

Replace γ if necessary so that $\gamma(\infty)$ is within the parallelogram with vertices at $0 = \gamma^{-1}(\infty)$, a, b, and $a+b$.

Add γ and γ^{-1} to the list of elements to draw, L_0.

Loop. While the list L_0 is non-empty, do the following.

(1) Remove the first element of L_0, call it ζ. Consider the isometric sphere of ζ. Check $I(\zeta)$ against elements of L_1. If $I(\zeta)$ is no longer visible, discard and start over with the next element of L_0. If $I(\zeta)$ is still visible, draw the isometric sphere determined by ζ to the screen. Add ζ to the end of the list L_1.

    Now also draw isometric spheres of each element of the form $w = \alpha^{\epsilon}\beta^{\delta} I(\zeta)$, where ϵ, δ lie in $\{0, \pm 1, \pm 2, \cdots \pm m\}$, with m chosen so that we draw only those translates of $I(\zeta)$ which are contained in the region of the screen.

(2) For each ξ in the list of drawn elements L_1, find integers p, q such that the center of $\alpha^p \beta^q I(\zeta)$ is nearest the center of ξ.

    For each isometric sphere of the form $\alpha^{p+\epsilon}\beta^{q+\delta} I(\zeta) = I(\zeta \beta^{-q-\delta}\alpha^{-p-\epsilon})$, with ϵ, δ in $\{0, \pm 1\ \pm 2, \pm 3\}$, check if that isometric sphere and $I(\xi)$ intersect visibly. That is, check if they intersect and, if so, if the edge of their intersection is visible from infinity. (In the case of $I(\zeta)$, no need to check for intersections of $I(\zeta)$ and the isometric sphere of the newly added last element ζ of L_1.)

    We claim that if $I(\xi)$ intersects any translate of $I(\zeta)$ under Γ_∞, then that translate will have the form $\alpha^{p+\epsilon}\beta^{q+\delta} I(\zeta)$ where ϵ, δ are in $\{0, \pm 1, \pm 2, \pm 3\}$. See Lemma 3.3 below.

(3) If $\alpha^{p+\epsilon}\beta^{q+\delta}I(\zeta)$ and $I(\xi)$ do intersect visibly, then the isometric sphere of the element ξw^{-1} should be drawn, where $w = \zeta\beta^{-q-\delta}\alpha^{-p-\epsilon}$, so that $I(w) = \alpha^{p+\epsilon}\beta^{q+\delta}I(\zeta)$. Step through the lists L_1 and L_0 to ensure the isometric sphere $I(\xi w^{-1})$ hasn't been drawn already, and is not yet slated to be drawn (to avoid adding the same sequence of faces repeatedly – note there are more time effective ways of ensuring the same thing). If ξw^{-1} is not in either list, then add ξw^{-1}, and $w\xi^{-1}$ to the end of the list L_0 to be drawn.

Lemma 3.3. *Suppose α and β are parabolic fixing the point at infinity, chosen as above such that α has the shortest translation length in the group $\langle\alpha,\beta\rangle \cong \mathbb{Z}\times\mathbb{Z}$, and such that β has the shortest translation length of all parabolics independent from α. Suppose ξ and ζ are loxodromic such that the group $\langle\alpha,\beta,\xi,\zeta\rangle$ is discrete. Choose integers p, q such that the center of $I(\xi)$ is nearer the center of $\alpha^p\beta^q I(\zeta)$ than the center of any other translate of $I(\zeta)$ under $\langle\alpha,\beta\rangle$. Then if $I(\xi)$ intersects any translate of $I(\zeta)$, that translate must be of the form $\alpha^{p+\epsilon}\beta^{q+\delta}I(\zeta)$ for $\epsilon,\delta \in \{0,\pm1,\pm2,\pm3\}$.*

Proof. Apply an isometry to $\mathbb{H}^3$ so that α translates by exactly 1 along the real axis in $\mathbb{C}$. Note that after this isometry, by Lemma 2.28, all isometric spheres have radius at most 1. Hence if two intersect, the distance between their centers is less than 2. Let x denote the center of $\alpha^p\beta^q I(\zeta)$. We may apply another isometry of $\mathbb{H}^3$ so that $x = 0$ in $\mathbb{C}$. Finally, since β is the shortest translation independent of α, β must translate x to be within the hyperbolic triangle on $\mathbb{C}$ with vertices $1/2 + i\sqrt{3}/2$, $-1/2 + i\sqrt{3}/2$, ∞.

Since the center of $I(\xi)$, denote it by y, is closer to x than to any of the translates of x under $\langle\alpha,\beta\rangle$, the real coordinate of y in $\mathbb{C}$ must have absolute value at most 1/2. Similarly, the difference in imaginary coordinates of y and βx is at least $\sqrt{3}/6$, for otherwise the square of the distance between y and some lattice point of the form $\alpha^\epsilon\beta x$ is at most $(1/2)^2 + (\sqrt{3}/6)^2 = 1/3$. Finally, we may assume the imaginary coordinate of y is positive, by symmetry of the lattice.

Suppose $I(\xi)$ meets $\alpha^{p+\epsilon}\beta^{q+\delta}I(\zeta)$, where one of $|\epsilon|$ or $|\delta|$ is greater than 3. Then the distance between y and $\alpha^\epsilon\beta^\delta x$ on $\mathbb{C}$ is at most 2. On the other hand, if $|\delta| \geq 3$, then the difference between the imaginary coordinates of y and $\alpha^\epsilon\beta^\delta x$ is at least $\sqrt{3}+\sqrt{3}/6 > 2$, which is a contradiction. So suppose $|\delta| < 3$ and $|\epsilon| > 3$. Then the difference in real coordinates of $\alpha^\epsilon\beta^\delta x$ and y is at least $4 - 1/2 - \delta\cdot 1/2 > 2$, which is again a contradiction. □

Theorem 3.4. *Suppose each of the spheres drawn by Procedure 3.2 is a face of the Ford domain of a geometrically finite uniformization of the $(1;2)$–compression body C. Then the procedure draws (at least one translate under Γ_∞ of) all visible isometric spheres, and the procedure terminates.*

Proof. The fact that the procedure terminates follows from Corollary 2.10: there are only finitely many visible faces, and each face the procedure draws is visible.

The fact that the procedure draws all visible isometric spheres of the Ford domain will follow from Lemma 2.26 and the Poincaré polyhedron theorem, as follows.

First, suppose the faces corresponding to γ and γ^{-1} are visible, and they do not intersect each other or any other faces. Then the procedure terminates after drawing these faces and a few translates under Γ_∞. Because there are no edges of intersection, the argument of Lemma 2.27 implies that the only visible face of the Ford domain corresponds to γ (and γ^{-1}), and in this case we are done.

So suppose two isometric spheres drawn by the procedure intersect. Say isometric spheres $I(g)$ and $I(h)$ intersect. Then the procedure will draw $I(gh^{-1})$. Since the procedure only draws visible isometric spheres, $I(gh^{-1})$ must be visible. By Lemma 2.17, it intersects $I(h^{-1})$ in an edge which is mapped isometrically to the edge of $I(g) \cap I(h)$. Changing roles of g and h in the same lemma, the isometric sphere $I(hg^{-1})$ must be visible, and $I(hg^{-1}) \cap I(g^{-1})$ is mapped isometrically to $I(g) \cap I(h)$.

Now notice that the faces of the Ford domain corresponding to the pairs $I(g)$ and $I(g^{-1})$, $I(h)$ and $I(h^{-1})$, and $I(gh^{-1})$ and $I(hg^{-1})$ are the only faces that meet the edge class of $I(g) \cap I(h)$ (up to translation by Γ_∞). This can be seen by noting that g takes $I(g) \cap I(h)$ and $I(h)$ to $I(g^{-1}) \cap I(hg^{-1})$ and $I(hg^{-1})$, respectively. Then apply hg^{-1}. This sends $I(g^{-1}) \cap I(hg^{-1})$ and $I(g^{-1})$ to $I(h^{-1}) \cap I(gh^{-1})$ and $I(h^{-1})$, respectively. Finally apply h^{-1}, which sends $I(h^{-1}) \cap I(gh^{-1})$ and $I(gh^{-1})$ to $I(h) \cap I(g)$ and $I(g)$, respectively. Thus the monodromy is given by $h^{-1} \circ hg^{-1} \circ g = 1$. As for dihedral angles around this edge class, because the monodromy is the identity, the sum of the dihedral angles must be a multiple of 2π. Since there are only three faces in the edge class, and the dihedral angle between any two faces is less than π, the sum of the dihedral angles around the edge $I(h) \cap I(g)$ must be exactly 2π. Now we have the hypotheses of the Poincaré polyhedron theorem, Theorem 2.25. That theorem tells us that the gluing of the faces our procedure has drawn gives a smooth hyperbolic manifold. Lemma 2.26 implies that the procedure has drawn the entire Ford domain, as desired. □

One way the hypotheses of Theorem 3.4 might not hold is if there is an edge class of the cell structure on the Ford domain that meets more than three visible faces. When two of the visible faces intersect, say corresponding to $I(g)$ and $I(h)$, our procedure will draw $I(gh^{-1})$. However, if the edge class meets more than three visible faces, the isometric sphere $I(gh^{-1})$ will not be visible, and so the hypotheses of the theorem are not satisfied. In practice, we were unable to find a structure on the $(1;2)$–compression body for which this situation arose. S. Burton found such a structure on a $(1;3)$–compression body [8]. However, even in this higher genus case the above procedure drew all visible isometric spheres for the example, since the isometric sphere covering $I(gh^{-1})$ arose as the intersection of other visible isometric spheres. Based on experimental evidence in the case of the $(1;2)$–compression body, we offer the following conjecture.

Conjecture 3.5. *Procedure 3.2 always draws the Ford domain for a geometrically finite uniformization for the* $(1;2)$*–compression body, and terminates.*

The generalization of Procedure 3.2 to $(1;n)$–compression bodies, for $n \geq 3$ has been shown to fail by S. Burton [8]. That is, the procedure will not necessarily draw the full Ford domain. This is because in the higher genus case, a choice of loxodromic generators may give an isometric sphere which is completely covered by some visible isometric sphere. As long as that visible isometric sphere is not one of our generators, and as long as the isometric spheres of our generators remain disjoint from that visible isometric sphere, the visible isometric sphere will never be drawn by the above procedure. However in the $(1;2)$–compression body case, up to translation by Γ_∞ there is only one choice for loxodromic generator, and so this issue does not seem to arise.

4. Examples of Ford domains

Recall that we are interested in isotopy classes of the core tunnel of a $(1;2)$–compression body. We use the computer program implementing Procedure 3.2 to study isotopy classes of the core tunnel for many different geometrically finite uniformizations. To identify core tunnels in Ford domains, we will examine the dual structure to a Ford domain. In this section, we define the dual structure and present several examples. The examples were obtained by computer using the procedure of the previous section.

4.1. **Paths of Ford domains.** Recall that if C denotes the $(1;2)$–compression body, then $\pi_1(C) \cong (\mathbb{Z}\times\mathbb{Z}) * \mathbb{Z}$ with generators we denote α and β for the $(\mathbb{Z}\times\mathbb{Z})$ factor, and γ. Let $\rho_0\colon \pi_1(C) \to PSL(2,\mathbb{C})$ be the representation of Example 2.11. Keeping the images of α and β parabolic, allow the images of the three generators α, β, and γ to vary smoothly. We obtain a smooth path of discrete, faithful representations ρ_t. For some amount of time, these will be minimally parabolic geometrically finite uniformizations of C. As ρ_t changes smoothly, the visible isometric spheres of $\mathbb{H}^3/\rho_t(\pi_1(C))$ will change smoothly. In particular, we can change the images of the generators such that two isometric spheres bump into each other. By Lemma 2.17, if two visible isometric spheres intersect, then a new visible face must arise when they meet. We present two examples to illustrate some of the behavior that may occur.

Example 4.1. Consider the smooth path of representations $\rho_t\colon \pi_1(C) \to PSL(2,\mathbb{C})$ given by

$$\rho_t(\alpha) = \begin{pmatrix} 1 & 5+i \\ 0 & 1 \end{pmatrix}, \quad \rho_t(\beta) = \begin{pmatrix} 1 & 5.5i \\ 0 & 1 \end{pmatrix}, \quad \rho_t(\gamma) = \begin{pmatrix} -1+i\,t & -1 \\ 1 & 0 \end{pmatrix},$$

where t runs from 2 down to 1.2.

Note here that $\rho_t(\alpha)$ and $\rho_t(\beta)$ are constant. They were chosen somewhat arbitrarily to be parabolics fixing infinity, with large enough Euclidean translation distance that nontrivial translations under $\Gamma_\infty = \langle \rho_t(\alpha), \rho_t(\beta)\rangle$ of the isometric spheres corresponding to $\rho_t(\gamma^{\pm 1})$ and $\rho_t(\gamma^{\pm 2})$ don't meet any of these original isometric spheres.

Consider the isometric spheres corresponding to $\rho_t(\gamma)$. By Lemma 2.6, these have radius 1 throughout the path. When $t = 2$, the isometric spheres of $\rho_t(\gamma)$ and $\rho_t(\gamma^{-1})$, which have centers 0 and $-1+i\,t$ respectively, do not intersect, so we have the simple Ford spine with a single face as above. However, as t decreases, these two isometric spheres first become tangent, at $t = \sqrt{3}$, and then overlap for $t < \sqrt{3}$. As these spheres meet, the isometric spheres corresponding to $\rho_t(\gamma^2)$ and $\rho_t(\gamma^{-2})$ emerge, and their intersections with isometric spheres of $\rho_t(\gamma)$ and $\rho_t(\gamma^{-1})$, respectively, become visible, as predicted by Lemma 2.17. We can compute explicitly that for these particular representations, for $1.2 < t < \sqrt{3}$, the region cut out by the isometric spheres of $\rho_t(\gamma^{\pm 1})$ and $\rho_t(\gamma^{\pm 2})$ and a vertical fundamental domain for Γ_∞ is a fundamental polyhedron for a manifold, using Poincaré's theorem 2.25. By Lemma 2.26, these isometric spheres must define the Ford domain for the manifold $\mathbb{H}^3/\rho_t(\pi_1(C))$. Thus our Ford spine has two faces, corresponding to $\rho_t(\gamma)$ and $\rho_t(\gamma^2)$. Figure 3 illustrates this particular example.

We claim this is still a uniformization of C, i.e. that $\mathbb{H}^3/\rho(\pi_1(C))$ is homeomorphic to the interior of C. The Ford spine of $\mathbb{H}^3/\rho(\pi_1(C))$ has two faces, one of which has boundary which is the union of the 1–cell of the spine and an arc on ∂_+C (corresponding to $\gamma^{\pm 2}$). Collapse the 1–cell and this face. The result is a new complex with the same regular neighborhood. It now has a single 2–cell attached to ∂_+C. Thus $\mathbb{H}^3/\rho(\pi_1(C))$ is obtained by attaching a 2–handle to $\partial_+C \times I$, and then removing the boundary. So $\mathbb{H}^3/\rho(\pi_1(C))$ is homeomorphic to the interior of C.

Example 4.2. Consider the same path as in Example 4.1, only now allow t to run from 1.2 down to 0.8. As t decreases, the isometric spheres corresponding to $\rho_t(\gamma^{\pm 2})$ slide towards those corresponding to $\rho_t(\gamma^{\pm 1})$, as illustrated in Figure 4. At approximately time $t = 1$, these isometric spheres meet visibly, and for $1 > t > 0.8$, these isometric spheres overlap.

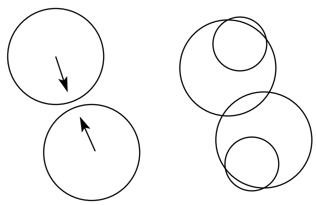
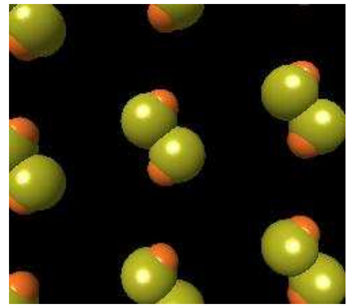

FIGURE 3. Faces of the Ford domain meet. Left: schematic picture for $t = 2$ down to $t = 1.2$. Right: Computer generated image for $t = 1.2$.

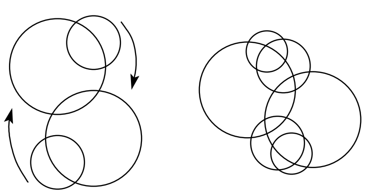
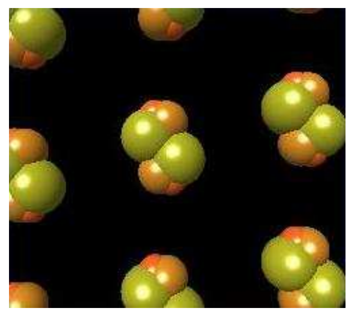

FIGURE 4. Left: Schematic picture of path for $t = 1.2$ down to $t = 0.8$. Right: Computer generated image, $t = 0.8$.

The isometric spheres corresponding to $\rho_t(\gamma^{\pm 3})$ are visible during these times, and emerge out from under the intersection between faces corresponding to $\rho_t(\gamma^{\pm 1})$ and $\rho_t(\gamma^{\pm 2})$, as illustrated in Figure 4. Again one may show that these isometric spheres, as well as a vertical fundamental domain for Γ_∞, cut out a polyhedron which glues up to give our manifold $\mathbb{H}^3/\rho_t(\pi_1(C))$, so again by Lemma 2.26, these isometric spheres cut out the Ford domain for the manifold.

We can show that this is a uniformization of C, i.e. that $\mathbb{H}^3/\rho(\pi_1(C))$ is homeomorphic to the interior of C, this time by considering the face of the Ford spine corresponding to $\gamma^{\pm 3}$. This face has boundary consisting of two 1–cells and an arc on $\partial_+ C$. Collapse this face. In fact, we may collapse the faces in the order they appeared, and we are again left with a single 2–cell attached to $\partial_+ C$ (corresponding to $\gamma^{\pm 1}$). So again this is a uniformization of C.

The examples above illustrate the phenomenon of Lemma 2.17, that is, that new faces emerge when existing faces meet in a path of uniformizations. We will see in Section 5 that this is the only way a new face can emerge.

4.2. **The dual structure.** Recall that we are interested in core tunnels of the $(1; 2)$–compression body C. In many cases, we can identify the core tunnel as an edge of the geometric dual of the Ford spine. This dual is reminiscent of the canonical polyhedral decompositions for finite volume manifolds which were introduced by Epstein and Penner [12]. We build the dual structure as follows.

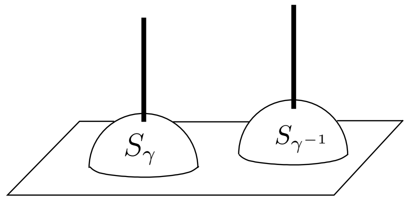

FIGURE 5. The dual to the simplest Ford spine is an edge that lifts to a collection of vertical geodesics in $\mathcal{F}$, shown in bold.

Consider again $\mathcal{F} = \mathbb{H}^3 \setminus \bigcup_{g\in\Gamma\setminus\Gamma_\infty} B(g)$. To each visible isometric sphere $I(g)$ of $\mathcal{F}$, there is an associated edge $e(g)$, which is the geometric dual of $I(g)$ running from the center of $I(g)$ to infinity in $\mathbb{H}^3$.

If two isometric spheres $I(g_1)$ and $I(g_2)$ of $\mathcal{F}$ overlap visibly, then they correspond to a dual face $F(g_1, g_2)$ which is the vertical plane bounded by $e(g_1)$ and $e(g_2)$ intersected with $\mathcal{F}$.

If visible isometric spheres of $\mathcal{F}$ meet (visibly) in a vertex, then their dual is a 3–dimensional region in $\mathcal{F}$ bounded by dual faces.

This forms a complex C. When we take C/Γ, we obtain a complex C_0 which is the geometric dual of the Ford spine.

Example 4.3. Consider Example 2.11, which gives a minimally parabolic geometrically finite uniformization on a $(1;2)$–compression body with only one face of the Ford spine. The geometric dual to the Ford spine for this example is a single edge running through the geometric center of the Ford spine. This edge lifts to a collection of geodesics in $\mathcal{F} \subset \mathbb{H}^3$ running through centers of isometric spheres corresponding to $\rho(\gamma)$, $\rho(\gamma^{-1})$, and their translates under Γ_∞. See Figure 5.

Example 4.4. Consider again Example 4.1, which describes the Ford domain of a geometrically finite unifomization of C in which the isometric spheres corresponding to $\rho(\gamma)$ and $\rho(\gamma^{-1})$ "bump", and only isometric spheres corresponding to $\rho(\gamma^2)$ and $\rho(\gamma^{-2})$ emerge. Consider the geometric dual to this picture. In $\mathcal{F}/\Gamma_\infty$, we see three intersections of isometric spheres: one corresponding to $\rho(\gamma)$ and $\rho(\gamma^{-1})$, one corresponding to $\rho(\gamma^2)$ and $\rho(\gamma)$, and one corresponding to $\rho(\gamma^{-1})$ and $\rho(\gamma^{-2})$. Thus the lift of the geometric dual to $\mathcal{F}$ has the form on the left of Figure 6.

These three lines of intersection in $\mathcal{F}$ are all glued under the action of Γ to the same single line. The dual faces glue up to give a single ideal triangle, as on the right in Figure 6, with two sides on the same edge (dual to the isometric sphere of γ).

Example 4.5. When Γ is the final uniformization in the path of representations considered in Example 4.2, the dual is a single ideal tetrahedron, as shown in Figure 7. Note the tetrahedron has two faces which are identified to each other under the action of Γ.

The dual structure, along with a horoball at infinity, also carries the topological information of the $(1;2)$–compression body.

Lemma 4.6. *For M the interior of any hyperbolizable 3–manifold with a single torus boundary component, let $\rho\colon \pi_1(M) \to PSL(2,\mathbb{C})$ be a minimally parabolic geometrically finite uniformization of M. Then there is a deformation retraction of M onto the union of the geometric dual of its Ford spine and an embedded horoball neighborhood of the rank 2 cusp.*

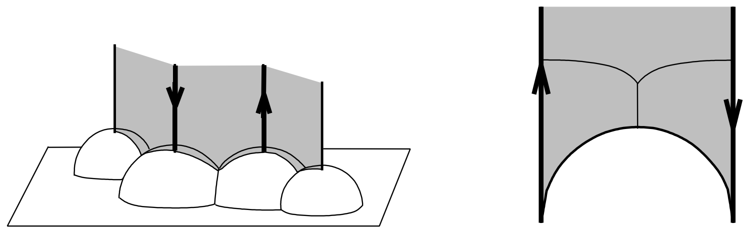

Figure 6. Left: the lift to $\mathcal{F}$ of the geometric dual of a Ford spine as in Example 4.1. Right: in this case the geometric dual to the Ford spine is a single ideal triangle, with two sides on the same edge.

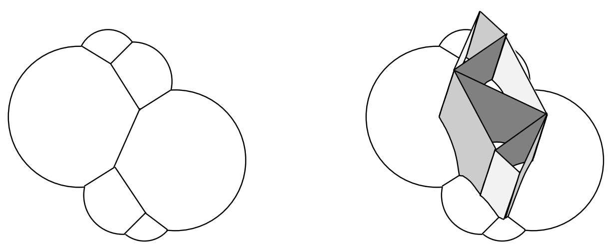

Figure 7. On the right is the lift to $\mathcal{F}$ of the geometric dual of the Ford spine of Example 4.2. The dual structure meets the horosphere about infinity in four triangles, corresponding to the four vertices of the single ideal tetrahedron.

Proof. Because ρ is geometrically finite, there exist finitely many visible isometric spheres in a Ford domain, which we view as $\mathcal{F} \subset \mathbb{H}^3$ intersected with a vertical fundamental domain. The boundaries of these isometric spheres are circles on $\mathbb{C}$, which bound disks on $\mathbb{C}$. There exists some $\epsilon > 0$ such that the ϵ–neighborhood of the union of these disks on $\mathbb{C}$ is embedded in $\mathbb{C}$. Translates by Γ_∞ remain embedded on $\mathbb{C}$. Now let H_∞ be the lift an embedded horoball neighborhood of the rank 2 cusp to $\mathbb{H}^3$. Project the ϵ–neighborhood of the union of disks vertically onto ∂H_∞. For each visible isometric sphere, there is a portion of a Euclidean cone in $\mathbb{H}^3 \setminus H_\infty$ which intersects $\mathbb{C}$ in the boundary of the isometric sphere, and intersects ∂H_∞ in the ϵ–neighborhood. Let S denote the union of all these cones. Note they form a regular neighborhood of the lift of the geometric dual of the Ford spine, intersected with $\mathcal{F} \setminus H_\infty$.

For the first step of the deformation retract, consider a point x in $\mathbb{H}^3 \setminus (S \cup H_\infty)$. Hyperbolic space $\mathbb{H}^3$ is foliated by vertical lines, and the vertical line through x will meet $\partial(H_\infty \cup S)$ in exactly one point. We define a deformation retract on $\mathbb{H}^3 \setminus (S \cup H_\infty)$ by taking x to this unique point on $\partial(H_\infty \cup S)$.

For the second step, since S is a regular neighborhood of the lift of the geometric dual of the Ford spine in $\mathcal{F} \setminus H_\infty$, we deformation retract $S \cup \partial H_\infty$ to the union of the geometric dual and the boundary ∂H_∞. We may choose the deformation retraction to be equivariant with respect to the action of $\rho(\pi_1(C))$. Putting both steps together and taking the quotient under $\rho(\pi_1(C))$, the result is the desired deformation retraction of $\mathbb{H}^3/\rho(\pi_1(C))$. □

With this picture of the dual structure, the fact that the core tunnel is geodesic in the case in which the Ford spine consists of a single face is immediate.

Proposition 4.7. *Suppose the Ford spine of a minimally parabolic geometrically finite hyperbolic uniformization of a* $(1;2)$*–compression body consists of a single face, corresponding to the loxodromic generator. Then the core tunnel is isotopic to a geodesic, dual to this single face.*

Proof. Let $\rho\colon \pi_1(C) \to PSL(2,\mathbb{C})$ be a uniformization of C with one face of the Ford spine, as in the statement of the proposition, and denote $\rho(\pi_1(C))$ by Γ. As in Example 4.3, the dual structure to the Ford spine consists of a single edge.

By Lemma 4.6, we may retract $\mathbb{H}^3/\Gamma$ onto a union of a horoball neighborhood of the cusp and this geodesic. Thus in this case, the single geodesic, which is the edge dual to the single face of the Ford spine, is isotopic to the core tunnel. □

In fact, for any uniformization $\rho\colon \pi_1(C) \to PSL(2,\mathbb{C})$, the core tunnel will always be *homotopic* to the edge dual to the isometric sphere corresponding to $\rho(\gamma)$.

Lemma 4.8. *For any uniformization* $\rho\colon \pi_1(C) \to PSL(2,\mathbb{C})$*, the core tunnel will be homotopic to the edge dual to the isometric sphere corresponding to the loxodromic generator of* $\rho(\pi_1(C))$*.*

Proof. Denote the loxodromic generator by $\rho(\gamma)$. Consider the core tunnel in the compression body $\mathbb{H}^3/\rho(\pi_1(C))$. Take a horoball neighborhood H_∞ of the cusp. The core tunnel runs through the horospherical torus ∂H_∞ into the cusp. Denote by $\widetilde{H}_\infty$ a lift of H_∞ to $\mathbb{H}^3$ about the point at infinity in $\mathbb{H}^3$.

There is a homeomorphism from $C \setminus \partial_+ C$ to $(\mathbb{H}^3/\rho(\pi_1(C))) \setminus \mathring{H}_\infty$. Slide the tunnel in C so that it starts and ends at the same point, and so that the resulting loop represents γ. The image of this loop under the homeomorphism to $(\mathbb{H}^3/\rho(\pi_1(C))) \setminus \mathring{H}_\infty$ is some loop. It lifts to an arc in $\mathbb{H}^3$ starting on $\widetilde{H}_\infty$ and ending on $\rho(\gamma)(\widetilde{H}_\infty)$. Extend to an arc in $\mathbb{H}^3/\rho(\pi_1(C))$ by attaching a geodesic in $\widetilde{H}_\infty$ and in $\rho(\gamma)(\widetilde{H}_\infty)$ and projecting. This is isotopic to (the interior of) the core tunnel. Now homotope the arc to a geodesic. It will run through the isometric sphere corresponding to $\rho(\gamma^{-1})$ once. □

5. Paths of structures and tunnels

We have encountered examples of minimally parabolic geometrically finite uniformizations of a $(1;2)$–compression body C for which the core tunnel is geodesic. This was shown explicitly for structures with simple Ford spines in Proposition 4.7. It can also be seen for those with spines as in Examples 4.1 and 4.2, by constructing a deformation retract onto the geodesic dual to the face corresponding to γ.

In this section we investigate Conjecture 1.1 more carefully. We find families of geometrically finite uniformizations of C for which the core tunnel is geodesic. Those structures of Examples 4.1 and 4.2 will fit into these families.

Our method of proof is to consider paths through the space of minimally parabolic geometrically finite uniformizations, and the corresponding Ford spines and their dual structures. We will see that in many cases, under some assumptions on the path, the core tunnel must remain isotopic to a geodesic.

5.1. **Paths and visible isometric spheres.** In this subsection we will work with slightly more general manifolds than C. We let M be the interior of a hyperbolic manifold with only one of its boundary components a torus.

The following follows from work of Bers, Kra, and Maskit (see [5]).

Lemma 5.1. *The space of minimally parabolic geometrically finite uniformizations of M is path connected.*

Proof. Bers, Kra, and Maskit showed that the space of conjugacy classes of minimally parabolic geometrically finite uniformizations may be identified with the Teichmüller space of the higher genus boundary components, quotiented out by $\mathrm{Mod}_0(M)$, the group of isotopy classes of homeomorphisms of M which are homotopic to the identity. Since the Teichmüller space is path connected, the quotient will also be path connected. □

Thus given any minimally parabolic geometrically finite uniformization of C, it is connected by a path of uniformizations to a uniformization admitting a simple Ford spine, as in Lemma 2.27.

Now, we will be taking paths through the interior of the space of geometrically finite, minimally parabolic uniformizations of the manifold M. Technically, such uniformizations are paths of representations $\rho_t\colon \pi_1(M) \to PSL(2,\mathbb{C})$. For any group element $g \in \pi_1(M)$, ρ_t will give a path of isometric spheres corresponding to $\rho_t(g)$.

As the isometric spheres in a Ford domain bump into each other, new isometric spheres become visible, and in turn visible faces may become invisible. We will determine when and how spheres become visible. First, we show that isometric spheres are visible for an open set of time.

Lemma 5.2. *Let Γ be a group with subgroup $\Gamma_\infty \cong \mathbb{Z}\times\mathbb{Z}$, and let $\rho_t\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be a continuous path of minimally parabolic geometrically finite representations of Γ such that $\rho_t(\Gamma_\infty)$ fixes the point at infinity in $\mathbb{H}^3$ for all t. Then any isometric sphere will be visible for an open set of time.*

Proof. Suppose the isometric sphere corresponding to the element $g_0 \in \Gamma$ is visible at time t_0. By Lemma 2.14, there exists x on the hemisphere $I(\rho_{t_0}(g_0))$ which is not contained in the closure of half–spaces $B(\rho_{t_0}(h))$ bounded by any isometric spheres corresponding to elements of $\Gamma \setminus (\Gamma_\infty \cup \Gamma_\infty g_0)$. Let U be a small open ball around x which is disjoint from the closures of these half spaces.

We claim that there is some $\epsilon > 0$ such that for any $t \in (t_0-\epsilon, t_0+\epsilon)$, $B(\rho_t(h)) \cap U = \emptyset$ for all $h \in \Gamma \setminus (\Gamma_\infty \cup \Gamma_\infty g_0)$. We may also choose $\epsilon > 0$ so that $I(\rho_t(h)) \cap U \neq \emptyset$ for all $t \in (t_0-\epsilon, t_0+\epsilon)$. Hence this claim will prove the lemma, because then points in this intersection will be visible.

Suppose that the claim is not true. There is then a sequence of times t_n (where $n \geq 1$) tending to t_0 and a sequence of elements $g_n \in \Gamma\setminus(\Gamma_\infty \cup \Gamma_\infty g_0)$ such that $B(\rho_{t_n}(g_n)) \cap U \neq \emptyset$. So $\rho_{t_n}(g_n)$ lies in the subset V of $\mathrm{PSL}(2,\mathbb{C})$ defined as follows:

$$V = \{g \in \mathrm{PSL}(2,\mathbb{C}) : \overline{B(g)} \cap \overline{U} \neq \emptyset \text{ and } g^{-1}(H) \cap H = \emptyset\},$$

where as usual, H denotes an embedded horoball about infinity.

We wish to argue by compactness. Note that V itself is not compact, for if $g \in V$, then so is wg for any $w \in \Gamma_\infty$. However, we may consider a compact subset of V. Let V_{norm} consist of $wg \in \mathrm{PSL}(2,\mathbb{C})$ where $g \in V$ and $w \in \Gamma_\infty$ is chosen such that $I(wg)$ and $I((wg)^{-1})$ have minimal (Euclidean) distance. That is, for any other $x \in \Gamma_\infty$, the distance between $I(xg)$ and $I((xg)^{-1})$ is at least as large as that between $I(wg)$ and $I(wg)^{-1})$.

Now V_{norm} is a compact subset of $\mathrm{PSL}(2,\mathbb{C})$. By composing with a suitable element of Γ_∞, we may assume that each $\rho_{t_n}(g_n)$ lies in V_{norm}. Hence we may pass to a subsequence where $\rho_{t_n}(g_n)$ converges to some $h \in \mathrm{PSL}(2,\mathbb{C})$. Now the groups $\rho_{t_n}(\Gamma)$ converge algebraically to $\rho_{t_0}(\Gamma)$. Since $\rho_{t_0}(\Gamma)$ is geometrically finite, this convergence is also geometric [7].

So h lies in $\rho_{t_0}(\Gamma)$. Say that $h = \rho_{t_0}(g)$ for some $g \in \Gamma$. Then $\rho_{t_n}(gg_n^{-1})$ is an element of $\rho_{t_n}(\Gamma)$ that can be made arbitrarily close to the identity in $\mathrm{PSL}(2,\mathbb{C})$ by taking large n. Powers of this form a cyclic subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and after passing to a subsequence, these converge geometrically to a non-discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. But this implies that $\rho_{t_0}(\Gamma)$ is not discrete, which is a contradiction. This proves the claim and hence the lemma. □

In what follows, we will analyze how the pattern of visible isometric spheres changes along a path $\rho_t(\Gamma)$ of minimally parabolic geometrically finite uniformizations. The first step is to examine how two Euclidean hemispheres $I(\rho_t(g_1))$ and $I(\rho_t(g_2))$ interact. It would be useful to know that during an interval $[t_-, t_+]$ of time, the set of times where $I(\rho_t(g_1))$ completely covers $I(\rho_t(g_2))$ is a finite collection of closed intervals. However, this need not be the case in general. Although the set of times where $I(\rho_t(g_1))$ covers $I(\rho_t(g_2))$ is a closed subset of $[t_-, t_+]$, this subset can have infinitely many components. To visualise this, imagine a continuous function $[t_-, t_+] \to \mathbb{R}$ which fluctuates between positive and negative values infinitely often near some $t_0 \in [t_-, t_+]$. We may find a path of uniformizations where the distance of $I(\rho_t(g_1))$ below (or above) $I(\rho_t(g_2))$ is equal to this function. Even if we require our path ρ_t of representations to be smooth, this phenomenon can occur. However, it is does not arise when the path of representations $[t_-, t_+] \times \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is real analytic. Note that $\mathrm{PSL}(2,\mathbb{C})$ inherits an obvious real analytic structure from $\mathbb{C}^4$. Moreover, any path of minimally parabolic geometrically finite uniformizations can be approximated by a real analytic path, by the Whitney Approximation Theorem.

Lemma 5.3. *Let Γ be a group with a subgroup $\Gamma_\infty \cong \mathbb{Z} \times \mathbb{Z}$. Let ρ_t be a real analytic path of uniformizations of Γ, where $t \in [t_-, t_+]$, such that $\rho_t(\Gamma_\infty)$ fixes the point at infinity in $\mathbb{H}^3$ for all t. Let g_1 and g_2 be elements of $\Gamma \setminus \Gamma_\infty$. Then, the set of times t where $I(\rho_t(g_1))$ covers $I(\rho_t(g_2))$ is a finite collection of closed intervals and points in $[t_-, t_+]$.*

Proof. Any isometric sphere is a hyperplane. Consider the hyperboloid model for hyperbolic space $\mathbb{H}^3$, which is the positive sheet of $\{v \in \mathbb{R}^{3,1} : \langle v, v\rangle = -1\}$. In this model, any hyperplane is of the form $\{w \in \mathbb{H}^3 : \langle v, w\rangle = 0\}$ for some space–like vector $w \in \mathbb{R}^{3,1}$. We may choose w so that $\langle w, w\rangle = 1$. In other words, the norm of w is 1.

Given two hyperplanes H_1 and H_2 specified by space–like vectors w_1 and w_2 with norm 1, they are tangent if and only if $\langle w_1, w_2\rangle = 1$. So, consider the isometric spheres $I(\rho_t(g_1))$ and $I(\rho_t(g_2))$, which are specified by space–like vectors $w_1(t)$ and $w_2(t)$ with norm 1. Then $\langle w_1(t), w_2(t)\rangle$ is a real analytic function of t. Hence, the set of times t where $I(\rho_t(g_1))$ and $I(\rho_t(g_2))$ are tangent is finite. □

The next lemma essentially is a list of ways that Euclidean hemispheres (isometric spheres) can emerge out from other Euclidean hemispheres in a real analytic path.

Lemma 5.4. *In a real analytic path through the space of minimally parabolic, geometrically finite uniformizations of M, the ways in which an isometric sphere may become visible (or invisible) are as follows:*

1. *On the boundary at infinity: two nested isometric spheres become tangent at a point on the boundary at infinity, then the inner one pushes through the outer.*
2. *On the boundary at infinity: two visible isometric spheres meet at a point on the boundary at infinity, a third moves into the point of their intersection, then pushes through.*
3. *Away from the boundary at infinity: two visible isometric spheres meet at an edge of $\mathcal{F}$, a third also meets the length of the edge, then pushes through.*

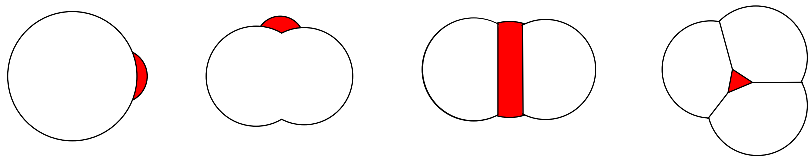

FIGURE 8. The ways in which isometric spheres can become visible.

(4) *Away from the boundary at infinity: three or more visible isometric spheres intersect in a vertex of $\mathcal{F}$, another moves into the vertex and then pushes through.*

It is also possible that multiple new isometric spheres become visible or invisible simultaneously at the same points on the boundary at infinity, or on the same edge or vertex of $\mathcal{F}$.

No isometric sphere may become visible without intersecting any other visible isometric sphere.

The options for single faces becoming visible are illustrated in Figure 8.

Proof. The fact that no isometric sphere may spontaneously arise without intersecting any other isometric sphere follows from Lemma 2.6 and the fact that the path is real analytic and hence continuous: each isometric sphere has positive radius for all time.

We now show that the above four possibilities are the only possibilities. Suppose $I(g)$ is visible for time $t \in (t_0, t_0 + \epsilon)$, but not at time t_0. Then at time t_0, the isometric sphere corresponding to $I(g)$ must have one of the following forms.

(1) It is covered by a single isometric sphere. In this case, it will be tangent to another hemisphere at time t_0, then push through at a point that is visible on the boundary at infinity. This is option (1) above.
(2) It is not covered by a single isometric sphere, but is covered by two visible isometric spheres at time t_0. Then it intersects two hemispheres at their edge of intersection at time t_0, then pushes through. In this case, one of the following options holds.
   (a) The newly visible isometric sphere expands in such a way as to completely cover the old visible edge. This gives option (3) above.
   (b) The new isometric sphere slides in one direction, covering only a portion of the visible edge, and appearing on the boundary at infinity. This gives option (2) above.
   (c) The new isometric sphere slides in one direction, to cover only a portion of the visible edge, but meets a third isometric sphere. Then the new isometric sphere will become visible in a vertex of $\mathcal{F}$. This is option (4) above.
(3) Finally, at time t_0, if the new isometric sphere is not covered by either one or two isometric spheres alone, but is covered by three or more, then in this case the isometric sphere will meet the point where these isometric spheres intersect. As it moves out from under the intersection, we will obtain option (4) above.

As for multiple isometric spheres: In each case above it is possible to have more than one hemisphere meeting the point(s) where an isometric sphere is about to emerge. In the case that a hemisphere is covered by another visible hemisphere, it is possible to have multiple hemispheres tangent at the same point, nested within each other, at time t_0. It is feasible that at time $t_0 + \epsilon$, for any sufficiently small $\epsilon > 0$, a smaller hemisphere has pushed out farther than a larger one, and so we obtain two new visible isometric spheres.

Multiple distinct hemispheres may both meet the same edge of intersection of visible isometric spheres, and then push through to form new visible isometric spheres. Similarly, multiple distinct hemispheres may meet the point of intersection of multiple visible isometric spheres, and push through to become visible at the same time. □

We have seen in examples that as the isometric spheres in a Ford domain bump into each other, new isometric spheres become visible. In the next two lemmas, we show that this is the only way new isometric spheres may become visible. First, we set up some notation.

In the arguments below, we will consider a fixed collection of isometric spheres and how they change. Rather than considering the entire Ford domain, we will consider instead whether given isometric spheres are visible with respect to other isometric spheres in the collection.

Definition 5.5. We will say an isometric sphere $I(g)$ is visible with respect to a collection of group elements $\{k_1 \dots, k_n\} \subset \Gamma$ if there is an open subset of $I(g)$ that is not contained in $\Gamma_\infty(\bigcup_{j=1}^n \overline{B(k_j)})$. Recall $B(k_j)$ is the open half space bounded by the isometric sphere $I(k_j)$. Similarly, we say the intersection of two isometric spheres $I(g) \cap I(h)$ is visible with respect to $\{k_1, \dots, k_n\}$ if $I(g) \cap I(h)$ contains an open set which is not contained in $\Gamma_\infty(\bigcup_{i=1}^n B(k_i))$.

Suppose we have a real analytic path, parameterized by time t, through the interior of the space of minimally parabolic geometrically finite uniformizations of M, where M is a hyperbolizable 3–manifold with only one rank 2 cusp. For any time t, we obtain the region $\mathcal{F}(t)$ of Definition 2.7. We may choose vertical fundamental domains in a continuous manner to obtain a path of Ford domains, given by finite polyhedra P_t.

Lemma 5.6. *Suppose that at time t_0, the polyhedron P_{t_0} is cut out by (a vertical fundamental domain and) isometric spheres corresponding to group elements $h_1, \dots, h_n$; and for some $\epsilon > 0$, and all time $t \in [t_0, t_0 + \epsilon)$ the combinatorics of the visible intersections of these isometric spheres do not change. That is, no new visible intersections of these particular faces arise, and no visible intersections of these faces disappear. Then for all $t \in [t_0, t_0 + \epsilon)$, faces corresponding to $h_1, \dots, h_n$ remain exactly those faces that are visible in a Ford spine at time t.*

To summarize, when the combinatorics of the visible intersections of faces is unchanged, no new visible faces may arise.

Proof. The proof is by the Poincaré polyhedron theorem. For any $t \in (t_0, t_0 + \epsilon)$, let Q_t be the polyhedron cut out by isometric spheres corresponding to the group elements $h_1, \dots, h_n$ and the vertical fundamental domain of P_t. Let $\mathcal{G}_t$ be the orbit of Q_t under Γ_∞.

Because there are no new visible intersections, and no visible intersections disappear, for each edge of Q_t arising from intersections of isometric spheres, the faces meeting that edge cycle must be unchanged from that of P_{t_0}, and therefore the monodromy around that edge is unchanged from that at time t_0. Because the monodromy is the identity at time t_0, it must be the identity at time t, all $t \in (t_0, t_0 + \epsilon)$. Moreover, since the dihedral angles about any edge at time t_0 sum to 2π, and since dihedral angles about an edge with monodromy the identity must sum to a multiple of 2π, continuity implies that the dihedral angles sum to 2π for all $t \in (t_0, t_0 + \epsilon)$. Similarly, this is true of translates of edges under Γ_∞, so holds for edges of $\mathcal{G}_t$.

Additionally, all isometric sphere faces of $\mathcal{G}_t$ are glued isometrically by continuity: They are glued isometrically at time t_0, when $\mathcal{G}_0$ is the equivariant Ford domain $\mathcal{F}$, and by Lemma 2.17

their intersections with other isometric spheres continue to be glued isometrically. Therefore, visible regions continue to be glued isometrically.

By Theorem 2.25, gluing faces of $\mathcal{G}_t$ yields a hyperbolic manifold with fundamental group generated by the face pairings $h_1, \dots, h_n$, equivariant with respect to Γ_∞. Therefore when we quotient by Γ_∞, we get a manifold whose fundamental group is isomorphic to that of the original manifold. Then Lemma 2.26 implies that $\mathcal{G}_t$ must equal the equivariant Ford domain at time t. Hence only the faces $h_1, \dots, h_n$ are visible at time t. □

Lemma 5.7. *Suppose that at time t_0, the equivariant Ford domain $\mathcal{F}_{t_0}$ is cut out by isometric spheres corresponding to group elements $h_1, \dots, h_n$ and their translates under Γ_∞; and for some $\epsilon > 0$ and all time $t \in [t_0, t_0 + \epsilon)$, there are no new visible intersections of faces corresponding to the h_j or their translates, although some visible intersections may disappear. Then no new visible faces arise in this time interval.*

Proof. Again let $\mathcal{G}_t$ be the polyhedron cut out by isometric spheres corresponding to $h_1, \dots, h_n$ at time t and their translates under Γ_∞, so that $\mathcal{G}_{t_0} = \mathcal{F}_{t_0}$.

If the combinatorics of intersections of isometric spheres remains as it was at time t_0, then the previous lemma implies there are no new visible faces. So suppose the combinatorics changes. By hypothesis, no visible intersections of faces corresponding to $h_1, \dots, h_n$ arise. Hence some intersection visible at time t_0 must disappear. Without loss of generality, suppose faces corresponding to h_1 and h_2 intersect visibly at time t_0, but not at time t.

If a visible edge disappears, it must do so in one of the ways of Lemma 5.4. Note that each of the ways (1), (2), and (4) in this lemma involve the Euclidean length of the edge shrinking to zero. Only possibility (3) does not. However, in that case, an edge disappears by sliding into another edge which was not initially visible. Because it was not initially visible, the two isometric spheres meeting in this edge did not initially intersect visibly. Thus in case (3), two isometric spheres that did not intersect visibly at time t_0 must intersect visibly thereafter, contradicting hypothesis. Therefore, this option of Lemma 5.4 does not happen.

Thus the Euclidean length of the visible intersection between faces corresponding to h_1 and h_2 must decrease to zero. Lemma 2.17 implies that the Euclidean length of the image of the visible intersection under isometries corresponding to h_1 and h_2 must also decrease to zero (as the visible edge is mapped isometrically). Applying the result to all edges in this edge class, we see that the edge class must vanish from the Ford domain entirely. That is, all faces which meet the edge corresponding to the visible intersection of h_1 and h_2 at time t_0 will cease to intersect in pairs by time t and the edge will be removed.

Now consider an edge class that remains visible with respect to faces corresponding to $h_1, \dots, h_n$ and their translates under Γ_∞. By the above argument, the edge cannot meet fewer faces than it meets at time t_0, for then the entire edge would disappear. Since there are no additional visible intersections of the h_i and its translates, no additional face corresponding to $h_1, \dots, h_n$ and their translates may meet the edge. Hence a visible edge with respect to the h_i and their translates at time t corresponds to a visible edge at time t_0, and has the same monodromy, and therefore the monodromy is the identity. Since this is true for all $t \in (t_0, t_0 + \epsilon)$, continuity implies the dihedral angles about the edge sum to 2π.

Next we show that faces corresponding to the h_i are still glued isometrically. Lemma 2.17 implies that their intersections map to other intersections isometrically. It could happen that one of the faces corresponding to $h_1, \dots, h_n$ is no longer visible with respect to the h_i at time t. Then we ignore that face. For other faces, the argument of Lemma 2.15 implies that if some portion of h_j (or a translate) is visible with respect to the other $h_k's$, then so must be a portion of h_j^{-1}. Continuity implies visible faces glue isometrically.

By the above work, when we glue via face pairings, the result must be a manifold by the Poincaré polyhedron theorem, Theorem 2.25. Because one of the faces h_i may no longer be visible, it could happen that the group generated by the pairings of visible faces (and the quotient by Γ_∞) no longer generates $\pi_1(M)$, and so these isometric spheres do not give the full equivariant Ford domain. However, if all the h_i remain visible, then Lemma 2.26 implies that $\mathcal{G}_t$ is the equivariant Ford domain of our manifold, and we are finished in this case.

So now suppose some h_i becomes invisible. In this case, there must be some initial time at which a face h_i is no longer visible, say all the h_i are visible for $t \in (t_0, t_1)$, but h_j is not visible at time t_1. Up until this time, the above argument implies that the visible isometric spheres corresponding to the h_i and their translates under Γ_∞ cut out the equivariant Ford domain of our manifold.

Suppose that at time t_1, the remaining visible isometric spheres no longer cut out the equivariant Ford domain. This means that at time t_1, some other isometric sphere, say corresponding to k, must be visible. Lemma 5.2 implies that there is some $\epsilon > 0$ such that the isometric sphere corresponding to k is visible for $t \in (t_1 - \epsilon, t_1 + \epsilon)$. However, for $t \in (t_1 - \epsilon, t_1)$, the equivariant Ford domain is not cut out by an isometric sphere corresponding to k. This is a contradiction.

Thus in all cases, we have the setup of Lemma 2.26. So $\mathcal{G}_t$ is the equivariant Ford domain, and hence there are no new visible isometric spheres. □

In a real analytic path of minimally parabolic geometrically finite uniformizations of M, the dual structure to the Ford domain will be changing. It follows from Lemma 5.2 that a dual edge will exist for an open set of time. The dual structure changes smoothly during the path, except at a discrete set of points corresponding to the addition or removal of a cell of the dual structure.

In Example 4.1, a new edge and a new 2–cell in the dual structure are created when two visible isometric spheres meet across portions of their boundaries on $\mathbb{C}$. In Example 4.2, a new edge, two new 2–cells, and a single 3–cell are created when two visible isometric spheres slide into each other along a third visible isometric sphere. In this case the boundaries of the isometric spheres on $\mathbb{C}$ initially meet at a point where two other boundaries of visible isometric spheres intersect.

Definition 5.8. If in a real analytic path of minimally parabolic geometrically finite uniformizations of M, two visible isometric spheres move to intersect across portions of their boundaries on $\mathbb{C}$, we will refer to the move as *bumping* at the boundary. The reverse of this move, where two isometric spheres pull apart at the boundary, we will refer to as *reverse bumping*. This is the move of Example 4.1.

If an isometric sphere slides into the visible intersection of two other isometric spheres at a point where the intersection meets the boundary $\mathbb{C}$, we call the move *sliding* at the boundary. Its reverse we will call *reverse sliding*. This is the move of Example 4.2.

Finally, isometric spheres may also shift and change intersections internally, without affecting the combinatorics of the boundary of the dual structure. We refer to these intersections as *internal moves*.

For an example of an internal move, suppose two isometric spheres $I(g)$ and $I(h)$ form a visible edge, and two additional isometric spheres $I(k)$ and $I(\ell)$ slide together over that edge, such that at some instant $t = t_0$ all four isometric spheres meet in a single point. At this instant, neither the intersection of $I(g)$ and $I(h)$ is visible, nor is the intersection of $I(k)$ and $I(\ell)$. However, for some $\epsilon > 0$, the intersection of $I(k)$ and $I(\ell)$ will be visible for time $(t_0, t_0 + \epsilon)$, and the intersection of $I(g)$ and $I(h)$ will be visible for time $(t_0 - \epsilon, t_0)$. This gives

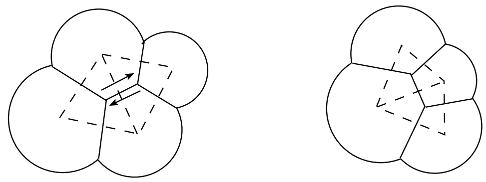

FIGURE 9. A retriangulation of the dual structure.

a "retriangulation" of the existing dual structure, in which faces in the interior are removed and replaced by other faces, and interior edges of the dual structure appear or disappear. An example of this phenomenon is a 2–3 Pachner move of a triangulation, or its reverse, a 3–2 move. See Figure 9.

5.2. **Paths and geodesic core tunnels.** We now present results that give evidence for Conjecture 1.1. We will be considering the $(1;2)$–compression body C once more.

Fix the following notation. As before, let α, β, and γ generate $\pi_1(C)$, with α and β generating $\pi_1(\partial_- C) \cong (\mathbb{Z}\times\mathbb{Z})$. Suppose $\rho_t\colon\ \pi_1(C) \to PSL(2,\mathbb{C})$ is a real analytic path of minimally parabolic geometrically finite uniformizations of C. We will assume that $\rho_t(\pi_1(\partial_- C)) = \Gamma_\infty$ fixes the point at infinity of $\mathbb{H}^3$.

The following lemma will guarantee that all structures on a particular path through the space of minimally parabolic geometrically finite uniformizations of C have geodesic core tunnel.

Lemma 5.9. *Suppose $\rho_t\colon\ \pi_1(C) \to PSL(2,\mathbb{C})$ is a real analytic path of minimally parabolic geometrically finite uniformizations of C such that at time $t = 0$, $M_0 = \mathbb{H}^3/\rho_0(\pi_1(C))$ admits a Ford spine such that*

(a) *the isometric sphere corresponding to $\rho_0(\gamma)$ is visible, and*
(b) *the core tunnel is isotopic to the geometric dual of this face of the Ford spine.*

Suppose that for $t \in (0, t_0)$, the isometric sphere corresponding to $\rho_t(\gamma)$ remains visible. Then the core tunnel is geodesic for all $t \in (0, t_0)$.

Proof. Consider the dual structure. For each $t \in [0, t_0)$, since $\rho_t(\gamma)$ is visible, there is an edge dual to it, which is a geodesic. The path ρ_t gives a (real analytic) one–parameter family of embedded edges dual to $\rho_t(\gamma)$. For any $t_1 \in (0, t_0)$, this restricts to an ambient isotopy of the edge dual to $\rho_0(\gamma)$ to the edge dual to $\rho_{t_1}(\gamma)$. Since the edge dual to $\rho_0(\gamma)$ is isotopic to the core tunnel, the edge dual to $\rho_{t_1}(\gamma)$ is also isotopic to the core tunnel, and so the core tunnel is geodesic. □

Now, we present a result that guarantees the core tunnel is geodesic for many paths of uniformizations of C. In the proof, for $g \in \pi_1(C)$, we will sometimes denote $\rho_t(g)$ by g_t, or when ρ_t is clear, we will simply write g to simplify notation.

Theorem 5.10. *Suppose $\rho_t\colon\ \pi_1(C) \to PSL(2,\mathbb{C})$ is a real analytic path of minimally parabolic geometrically finite uniformizations of C such that $M_0 = \mathbb{H}^3/\rho_0(\pi_1(C))$ admits a Ford spine with just one face. Suppose for all $t > 0$, there is a compression disk D_t properly embedded in C, which does not meet any faces of the Ford spine of $M_t = \mathbb{H}^3/\rho_t(\pi_1(C))$. Then for any $t > 0$, the core tunnel is geodesic, isotopic to an edge dual to the Ford spine.*

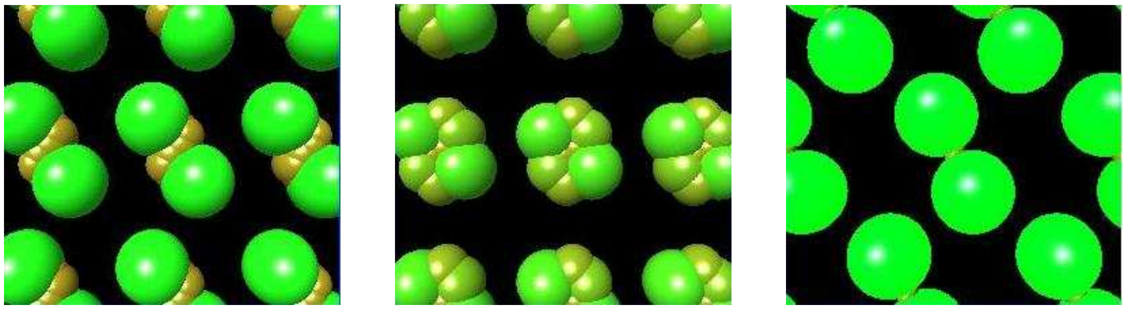

FIGURE 10. Shown are examples of structures to which Theorem 5.10 applies.

Proof. Suppose the isometric sphere corresponding to γ has remained visible for all time t in $(0, t_0)$. We will show it is still visible at time t_0. Because isometric spheres are visible for an open set of time, it will follow from Lemma 5.9 that the core tunnel is geodesic at time t_0.

Consider a lift of the disk D_{t_0} to $\mathbb{H}^3$, which we will continue to write as D_{t_0}, abusing notation slightly.

Without loss of generality, we may assume ∂D_{t_0} encircles a single connected component of the isometric spheres of $\mathcal{F}$, for if not, we may replace D_{t_0} with a disk which has this property, as follows. If ∂D_{t_0} encircles more than one connected component, then there is an arc α in $\mathbb{C}$ from ∂D_{t_0} to itself which meets no isometric spheres of $\mathcal{F}$. Then there is a disk in $\mathbb{H}^3$ with boundary on $\alpha \subset \mathbb{C}$ and on D_{t_0} which is disjoint from the isometric spheres of $\mathcal{F}$ and with interior disjoint from D_{t_0}. Replace D_{t_0} with a portion of D_{t_0} and this new disk with boundary α, reducing the number of components encircled by D_{t_0}. Repeat, as necessary, to obtain D_{t_0} whose boundary encircles a single connected component of the isometric spheres of $\mathcal{F}$.

Without loss of generality, we may assume ∂D_{t_0} encircles $I(\gamma)$ at time t_0. Then note that $I(\gamma)$ cannot meet $p(I(\gamma))$ for any $p \in \Gamma_\infty \setminus \{1\}$, or else the faces $p^n(I(\gamma))$, $n \in \mathbb{Z}$ would form an infinite strip of isometric spheres, and ∂D_{t_0} would have to intersect this strip, contradicting assumption. So we may assume $I(\gamma)$ (and hence $I(\gamma^{-1})$) meets none of its translates under $\Gamma_\infty = \Gamma_\infty(t_0)$.

Change generators, if necessary, so that the isometric sphere $I(\gamma)$ is at least as close to $I(\gamma^{-1})$ as to any of the translates of $I(\gamma^{-1})$ under $\rho_{t_0}(\Gamma_\infty)$ at time t_0.

Suppose first that $I(\gamma)$ and $I(\gamma^{-1})$ are disjoint (or only meet at a single point on the boundary at infinity). Then in this case, as in the proof of Lemma 2.27, the Poincaré polyhedron theorem implies that the object obtained by gluing isometric spheres corresponding only to $I(\gamma)$ and $I(\gamma^{-1})$ and their translates under Γ_∞, quotiented out by Γ_∞, must be a manifold with fundamental group isomorphic to $\pi_1(C)$. Then Lemma 2.26 implies that the equivariant Ford domain in this case consists only of faces $I(\gamma)$ and $I(\gamma^{-1})$ (and their translates under Γ_∞). Thus M_{t_0} must have a simple Ford spine consisting of one face, so by Proposition 4.7, the core tunnel is geodesic.

Next suppose $I(\gamma)$ and $I(\gamma^{-1})$ intersect. Then they (i.e. their boundaries) are contained within the region of $\mathbb{C}$ bounded by ∂D_{t_0}. Let $I(g)$ and $I(h)$ be any isometric spheres within this region. Then note that for any nontrivial parabolic $p \in \Gamma_\infty \setminus \{1\}$, $p(I(g))$ cannot meet $I(h)$, for $p(I(g))$ must lie outside the region bounded by ∂D_{t_0}.

We claim that in this case, all visible isometric spheres in the region bounded by ∂D_{t_0} are of the form $I(g)$ for g an element of the cyclic group $\langle\gamma\rangle$. Again this will follow from

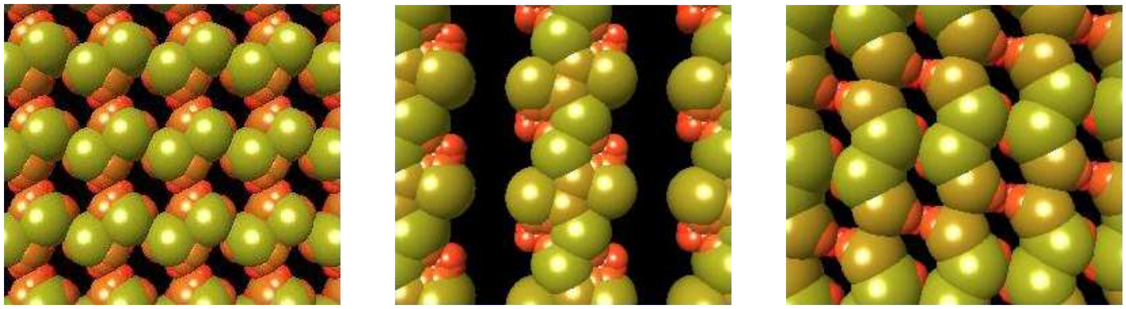

Figure 11. Snapshots along paths. In the figures shown, the core tunnel is geodesic because $I(\gamma)$ remains visible.

Lemma 2.26, as follows. Consider the isometric spheres corresponding to the cyclic group $\langle\gamma\rangle$. Ford domains of cyclic groups have been studied by Jørgensen [16] and Drumm and Poritz [11]. In particular, it is known that $\langle\gamma\rangle$ is geometrically finite, so a finite number of isometric spheres corresponding to this group will be visible with respect to the other isometric spheres of the group. Moreover, they will glue to give a manifold, namely a layered solid torus. Additionally, the Ford domain for $\langle\gamma\rangle$ is connected. Hence it lies entirely within ∂D_{t_0}, and thus it is disjoint from all its translates under Γ_∞. Therefore when we consider all translates under Γ_∞ of visible isometric spheres corresponding to the cyclic group $\langle\gamma\rangle$, the result is a domain in $\mathbb{H}^3$ cut out by isometric spheres, which glue to give a manifold. If we further take the quotient by Γ_∞ then we obtain a manifold homeomorphic to the $(1;2)$–compression body. The fundamental group of this quotient manifold clearly contains Γ_∞; it also contains γ because it contains all of $\langle\gamma\rangle$. Hence the fundamental group of this manifold is $\rho_t(\pi_1(C))$. Lemma 2.26 implies that we have found the entire (equivariant) Ford domain.

Work of Jørgensen [15] and Drumm and Poritz [11] implies that the face $I(\gamma)$ is visible in the Ford domain of $\langle\gamma\rangle$. Therefore in our case, $I(\gamma)$ must remain visible at time $t = t_0$ (this is contained in [11, Theorem 7.9], see also the two paragraphs before the statement of that theorem). Then our result follows from Lemma 5.9. □

By Lemma 5.9, in a real analytic path of minimally parabolic geometrically finite uniformizations of C which begins with a simple Ford spine, if the isometric spheres corresponding to γ and γ^{-1} remain visible throughout, then the core tunnel remains visible. We found no topological obstruction to the isometric sphere of γ being covered. However, in practice, we were unable to find examples of paths in which this occurred. All such examples led to indiscrete groups.

Figure 11 shows examples of Ford domains obtained by our computer program which are not guaranteed to have a geodesic core tunnel by Theorem 5.10. However, each of these can be shown to have geodesic core tunnel by observation. In particular, the face $I(\gamma)$ is visible always for each of these examples. Thus by Lemma 5.9, the core tunnel is geodesic for each of these structures. Moreover, it is actually dual to a face of the Ford spine.

This leads us to the following strengthening of Conjecture 1.1.

Conjecture 5.11. *In any geometrically finite hyperbolic structure on a $(1;2)$–compression body, the core tunnel is isotopic to a geodesic dual to a face of the Ford domain.*

The analogue of Conjecture 5.11 is false for finite volume manifolds. Sakuma and Weeks conjectured in [19] that core tunnels in knot complements were isotopic to edges of the

Epstein–Penner canonical polyhedral decomposition of the knot complement [12], which is dual to the Ford domain. Sakuma and Weeks' conjecture was shown to be false by Heath and Song in [14]. They showed that the knot $P(-2,3,7)$ has four distinct core tunnels, but only three edges of the canonical polyhedral decomposition.

Nevertheless, we believe Conjecture 5.11 will be true for compression bodies.

5.3. **Relation to Computer Procedure.** Conjecture 5.11 is intricately related to Procedure 3.2.

Suppose Conjecture 5.11 is false. Then for some geometrically finite hyperbolic structure, the faces corresponding to γ, γ^{-1} are not visible. In this case, it is not clear whether Procedure 3.2 will actually find and draw all the faces of the Ford domain. Additionally, since at least initially the procedure is drawing isometric spheres that will not be faces of the Ford domain, it no longer follows that the procedure is drawing faces of a convex fundamental domain for the group Γ. Hence work of Bowditch will not apply, and the procedure may never terminate.

Similarly, suppose a face F of the Ford domain initially arose as the intersection of two visible faces in a path through Ford domains, but that later in the path, those visible faces or their edge of intersection becomes covered by some other face. Then it could possibly be the case that Procedure 3.2 never draws face F. However, again based on experimental evidence, this seems unlikely.

How might a face become invisible? If it is covered by other faces. In the interior, such a move would occur as the dual of a 3–2 move of tetrahedra, or some similar move.

Question 5.12. *For any geometrically finite hyperbolic structure on a $(1;2)$–compression body, does there exist a smooth path of Ford domains from the simple structure to this particular structure for which there are no internal moves on the Ford domain?*

Note that an affirmative answer to Question 5.12 would imply Conjecture 5.11, as the faces corresponding to γ, γ^{-1} cannot disappear as other faces slide together and apart, meeting only on the boundary $\mathbb{C} \cap \mathcal{F}$.

There is some evidence for a positive answer to Question 5.12. Our proof of Theorem 5.10 shows that 2–3 moves are impossible in the core of the Ford domain when the Ford domain has the form of that theorem. Interestingly, in the case of once–punctured tori, Jørgensen also found that internal moves in the Ford domain are impossible [16]. However, his proof was similar to our proof of Theorem 5.10, and will not answer Question 5.12.

Using our computer program, we have found that 2–3 moves do occur in the $(1;2)$–compression body setting. However, in all examples encountered, there was a straightforward way to reparameterize the path of structures to avoid these moves.

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