DENSITY SPECTRA FOR KNOTS

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Abstract. We recently discovered a relationship between the volume density spectrum and the determinant density spectrum for infinite sequences of hyperbolic knots. Here, we extend this study to new quantum density spectra associated to quantum invariants, such as Jones polynomials, Kashaev invariants and knot homology. We also propose related conjectures motivated by geometrically and diagrammatically maximal sequences of knots.

In celebration of Józef Przytycki's 60th birthday

1. Volume and determinant density spectra

In [6], we studied the asymptotic behavior of two basic quantities, one geometric and one diagrammatic, associated to an alternating hyperbolic link K: The *volume density* of K is defined as $\mathrm{vol}(K)/c(K)$, and the *determinant density* of K is defined as $2\pi \log \det(K)/c(K)$.

For any diagram of a hyperbolic link K, an upper bound for the hyperbolic volume $\mathrm{vol}(K)$ was given by D. Thurston by decomposing $S^3 - K$ into octahedra at crossings of K. Any hyperbolic octahedron has volume bounded above by the volume of the regular ideal octahedron, $v_{\mathrm{oct}} \approx 3.66386$. So if $c(K)$ is the crossing number of K, then

$$\frac{\mathrm{vol}(K)}{c(K)} \leq v_{\mathrm{oct}}. \tag{1}$$

The following conjectured upper bound for the determinant density is equivalent to a conjecture of Kenyon [11] for planar graphs. We have verified this conjecture for all knots up to 16 crossings.

Conjecture 1.1 ([6])**.** *If K is any knot or link,* $\dfrac{2\pi \log \det(K)}{c(K)} \leq v_{\mathrm{oct}}$.

This motivates a more general study of the spectra for volume and determinant density.

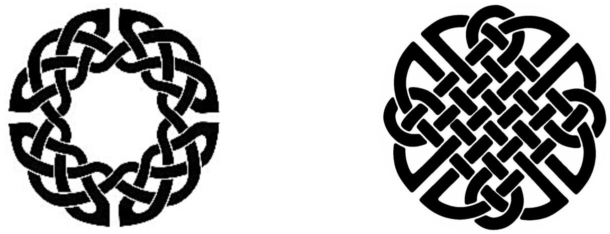
Definition 1.2. Let $\mathcal{C}_{\mathrm{vol}} = \{\mathrm{vol}(K)/c(K)\}$ and $\mathcal{C}_{\det} = \{2\pi \log \det(K)/c(K)\}$ be the sets of respective densities for all hyperbolic links K. We define $\mathrm{Spec}_{\mathrm{vol}} = \mathcal{C}'_{\mathrm{vol}}$ and $\mathrm{Spec}_{\det} = \mathcal{C}'_{\det}$ as their derived sets (set of all limit points).

Equation (1) and Conjecture 1.1 imply

$$\mathrm{Spec}_{\mathrm{vol}},\ \mathrm{Spec}_{\det} \subset [0, v_{\mathrm{oct}}]$$

Twisting on two strands of an alternating link gives 0 as a limit point of both densities: $0 \in \mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\det}$. Moreover, by the upper volume bound established in [1], v_{oct} cannot occur as a volume density of any finite link; i.e., $v_{\mathrm{oct}} \notin \mathcal{C}_{\mathrm{vol}}$. However, v_{oct} is the volume density of the *infinite weave* $\mathcal{W}$, the infinite alternating link with the infinite square grid projection graph (see [6]).

To study $\mathrm{Spec}_{\mathrm{vol}}$ and $\mathrm{Spec}_{\det}$, we consider sequences of knots and links. We say that a sequence of links K_n with $c(K_n) \to \infty$ is *geometrically maximal* if $\displaystyle\lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)} = v_{\mathrm{oct}}$.



(a) (b)

FIGURE 1. (a) A Celtic knot diagram that has a cycle of tangles. (b) A Celtic knot diagram with no cycle of tangles, which could be in a sequence that satisfies conditions of Theorem 1.4.

Similarly, it is *diagrammatically maximal* if $\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_{\rm oct}$. In [6], we found many families of geometrically and diagrammatically maximal knots and links that are related to the infinite weave $\mathcal{W}$.

Definition 1.3. Let G be any possibly infinite graph. For any finite subgraph H, the set ∂H is the set of vertices of H that share an edge with a vertex not in H. We let $|\cdot|$ denote the number of vertices in a graph. An exhaustive nested sequence of connected subgraphs, $\{H_n \subset G: \ H_n \subset H_{n+1}, \ \cup_n H_n = G\}$, is a *Følner sequence* for G if

$$\lim_{n\to\infty} \frac{|\partial H_n|}{|H_n|} = 0.$$

For any link diagram K, let $G(K)$ be the projection graph of the diagram. Let $G(\mathcal{W})$ be the projection graph of $\mathcal{W}$, which is the infinite square grid. We will need a particular diagrammatic condition called a *cycle of tangles*, which is defined in [6]. For an example, see Figure 1.

Theorem 1.4 ([6])**.** *Let K_n be any sequence of hyperbolic alternating link diagrams that contain no cycle of tangles, such that*

(1) *there are subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(\mathcal{W})$, and*
(2) $\lim_{n\to\infty} |G_n|/c(K_n) = 1$.

Then K_n is geometrically maximal: $\lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)} = v_{\rm oct}$.

Theorem 1.5 ([6])**.** *Let K_n be any sequence of alternating link diagrams such that*

(1) *there are subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(\mathcal{W})$, and*
(2) $\lim_{n\to\infty} |G_n|/c(K_n) = 1$.

Then K_n is diagrammatically maximal: $\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_{\rm oct}$.

Many families of knots and links are both geometrically and diagrammatically maximal. For example, weaving knots are alternating knots with the same projection as torus knots, and are both geometrically and diagrammatically maximal [7, 6]. These results attest to the non-triviality of $\mathrm{Spec}_{\rm vol} \cap \mathrm{Spec}_{\rm det}$:

Corollary 1.6. $\{0, v_{\rm oct}\} \subset \mathrm{Spec}_{\rm vol} \cap \mathrm{Spec}_{\rm det}$.

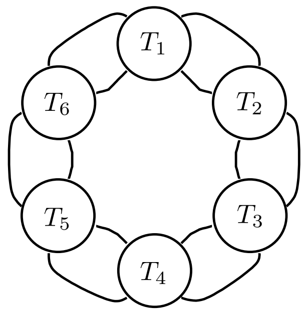


FIGURE 2. A 6–cycle of 2–tangles.

After we presented these results and conjectures, Stephan Burton [3] recently proved the following:

Theorem 1.7 ([3]).

$$\mathrm{Spec}_{\mathrm{vol}} = [0, v_{\mathrm{oct}}], \text{ and } [0, v_{\mathrm{oct}}] \subset \mathrm{Spec}_{\mathrm{det}}, \text{ hence } \mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}} = [0, v_{\mathrm{oct}}].$$

Burton's non-constructive proof leaves open how to explicitly realize a particular volume or determinant density by a sequence of links. Below, we prove how to explicitly realize many elements in $\mathrm{Spec}_{\mathrm{vol}}$ and $\mathrm{Spec}_{\mathrm{det}}$. In Section 2, we extend these ideas to spectra related to quantum invariants of knots and links.

Additionally, although $\mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}} = [0, v_{\mathrm{oct}}]$, the following remains an interesting open question:

Question 1.8. For which $\alpha \in [0, v_{\mathrm{oct}}]$, does there exist a sequence of links K_n such that

$$\lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \alpha \ ?$$

At present, the only values in this spectrum that we can prove are $\alpha \in \{0, \frac{10}{3} v_{\mathrm{tet}}, v_{\mathrm{oct}}\}$, where $v_{\mathrm{tet}} \approx 1.01494$ is the hyperbolic volume of the regular ideal tetrahedron.

1.1. **Volume and determinant density spectra from alternating links.** For any reduced alternating diagram D of a hyperbolic alternating link K, Adams [2] recently defined the following notion of a *generalized augmented link* J. Take an unknotted component B that intersects the projection sphere of D in exactly one point in each of two non-adjacent regions of D. Then $J = K \cup B$. In [2, Theorem 2.1], Adams proved that any such generalized augmented link is hyperbolic.

Theorem 1.9. *For any hyperbolic alternating link K,*

(a) *if $K \cup B$ is any generalized augmented alternating link,* $\mathrm{vol}(K \cup B)/c(K) \in \mathrm{Spec}_{\mathrm{vol}}$,

(b) $2\pi \log \det(K)/c(K) \in \mathrm{Spec}_{\mathrm{det}}$.

Proof of part (a). View K as a knot in the solid torus $S^3 - B$. Cut along the disk bounded by B (cutting K each time K intersects the disk bounded by B), obtaining an m–tangle T. Let K^n denote the n–periodic reduced alternating link with quotient K, formed by taking n copies of T joined in an n–cycle of tangles as in Figure 2. Thus, $K^1 = K$ and $c(K^n) = n \cdot c(K)$.

Let B also denote the central axis of rotational symmetry of K^n. Then [9, Theorem 3.1], using results of [8], implies that

$$n\left(1-\frac{2\sqrt{2}\,\pi^2}{n^2}\right)^{3/2}\mathrm{vol}(K\cup B)\ \le\ \mathrm{vol}(K^n)\ \le\ n\,\mathrm{vol}(K\cup B).$$

Therefore,

$$\lim_{n\to\infty}\frac{\mathrm{vol}(K^n)}{c(K^n)}=\lim_{n\to\infty}\frac{n\cdot\mathrm{vol}(K\cup B)}{n\cdot c(K)}=\frac{\mathrm{vol}(K\cup B)}{c(K)}.$$

This completes the proof of part (a). □

For the proof of part (b), we recall some notation. Any alternating link K is determined up to mirror image by its Tait graph G_K, the planar checkerboard graph for which a vertex is assigned to every shaded region and an edge to every crossing of K. Thus, $e(G)=c(K)$. Let $\tau(G)$ denote the number of spanning trees of G. For any alternating link, $\tau(G)=\det(K)$, which is the determinant of K [15].

We will need the following special case of [12, Corollary 3.8]. Let $V(G)$ denote the set of vertices of G, and let $|G|$ denote the number of vertices.

Proposition 1.10. *Given $d>0$, let G_n be any sequence of finite connected graphs with degree at most d such that $\lim_{n\to\infty}\frac{\log\tau(G_n)}{|G_n|}=h$. If G'_n is a sequence of connected subgraphs of G_n such that*

$$\lim_{n\to\infty}\frac{\#\{x\in V(G'_n):\ \deg_{G'_n}(x)=\deg_{G_n}(x)\}}{|G_n|}=1,$$

then $\lim_{n\to\infty}\frac{\log\tau(G'_n)}{|G'_n|}=h.$

Proof of Theorem 1.9 part (b). Proceed as in the proof of part (a), but now view K as a closure of a 2–tangle T. Let K^n denote the n–periodic link formed by an n–cycle of tangles T as in Figure 2. Let $L^n=K\#\cdots\#K$ denote the connect sum of n copies of K, which has a reduced alternating diagram as the closure of n copies of T joined in a row. Note that $c(K^n)=c(L^n)=n\cdot c(K)$, and $\det(L^n)=(\det(L))^n$.

In terms of Tait graphs, G_{K^n} is obtained from G_{L^n} by identifying one pair of vertices, so that G_{L^n} is a subgraph of $G_{K^{n+1}}$, and $|G_{L^n}|=|G_{K^n}|+1$. Hence, by Proposition 1.10,

$$\lim_{n\to\infty}\frac{\log\tau(G_{K^n})}{|G_{K^n}|}=\lim_{n\to\infty}\frac{\log\tau(G_{L^n})}{|G_{L^n}|}.$$

Therefore,

$$\lim_{n\to\infty}\frac{2\pi\log\det(K^n)}{c(K^n)}=\lim_{n\to\infty}\frac{2\pi\log\det(L^n)}{c(L^n)}=\lim_{n\to\infty}\frac{n\cdot 2\pi\log\det(K)}{n\cdot c(K)}=\frac{2\pi\log\det(K)}{c(K)}.$$

This completes the proof of part (b). □

Note that part (a) of Theorem 1.9 generalizes [7, Corollary 3.7], where B was the braid axis.

Remark 1.11. Motivated by Conjecture 1.1, it is interesting to find proven upper bounds for the determinant density. In terms of graph theory, since every spanning tree is a subset of the edge set, $\tau(G)\le 2^{e(G)}$ for any graph G, so that $\frac{2\pi\log\tau(G)}{e(G)}\le 2\pi\log(2)\approx 4.3552$. We thank Jun Ge for informing us that Stoimenow has improved on this bound: Let $\delta\approx 1.8393$

be the real positive root of $x^3 - x^2 - x - 1 = 0$. Then [14, Theorem 2.1] implies that $\frac{2\pi \log \det(K)}{c(K)} \leq 2\pi \log(\delta) \approx 3.82885$. Note that planarity is required to prove Conjecture 1.1 because Kenyon has informed us that $\frac{2\pi \log \tau(G)}{e(G)} > v_{\mathrm{oct}}$ does occur for some non-planar graphs.

1.2. **Volume and determinant for alternating links.** For alternating knots, the ranks of their reduced Khovanov homology and their knot Floer homology both equal the determinant. Every non-alternating link can be viewed as a modification of a diagram of an alternating link with the same projection, by changing crossings. In [6], we proved that this modification affects the determinant as follows. Let K be a reduced alternating link diagram, and K' be obtained by changing any proper subset of crossings of K. Then

$$\det(K') < \det(K).$$

Motivated by this fact, the first two authors previously conjectured that alternating diagrams similarly maximize hyperbolic volume in a given projection. With K and K' as before, they have verified that for all alternating knots up to 18 crossings (≈ 10.7 million knots),

$$\mathrm{vol}(K') < \mathrm{vol}(K).$$

We conjecture in [6] that the same result holds if K' is obtained by changing any proper subset of crossings of K. Note that by Thurston's Dehn surgery theorem, the volume converges from below when twisting two strands of a knot, so $\mathrm{vol}(K) - \mathrm{vol}(K')$ can be an arbitrarily small positive number.

We have verified the following conjecture for all alternating knots up to 16 crossings, and weaving knots and links for $3 \leq p \leq 50$ and $2 \leq q \leq 50$.

Conjecture 1.12 ([6])**.** *For any alternating hyperbolic link K,*

$$\mathrm{vol}(K) < 2\pi \log \det(K).$$

Conjectures 1.1 and 1.12 would imply that any geometrically maximal sequence of knots is diagrammatically maximal. In contrast, we can obtain K_n by twisting on two strands, such that $\mathrm{vol}(K_n)$ is bounded but $\det(K_n) \to \infty$. We also showed in [6] that the inequality in Conjecture 1.12 is sharp, in the sense that if $\alpha < 2\pi$, then there exist alternating hyperbolic knots K such that $\alpha \log \det(K) < \mathrm{vol}(K)$.

Applying the same arguments as in the proof of Theorem 1.9, Conjecture 1.12 implies the following conjecture, which would be a new upper bound for how much the volume can change after drilling out an augmented unknot:

Conjecture 1.13. *For any hyperbolic alternating link K with an augmented unknot B around any two parallel strands of K,*

$$\mathrm{vol}(K) < \mathrm{vol}(K \cup B) \leq 2\pi \log \det(K).$$

2. Quantum density spectra

2.1. **Jones polynomial density spectrum.** Let $V_K(t) = \sum_i a_i t^i$ denote the Jones polynomial, with $d = \mathrm{span}(V_K(t))$, which is the difference between the highest and lowest degrees of terms in $V_K(t)$. Let $\mu(K)$ denote the average of the absolute values of coefficients of $V_K(t)$, i.e.

$$\mu(K) = \frac{1}{d+1} \sum |a_i|.$$

For sequences of alternating diagrammatically maximal knots, we have:

Proposition 2.1. *If K_n is any sequence of alternating diagrammatically maximal links,*

$$\lim_{n\to\infty} \frac{2\pi \log \mu(K_n)}{c(K_n)} = v_{\rm oct}.$$

Proof. If, as above, G is the Tait graph of K, and $\tau(G)$ is the number of spanning trees, then $\tau(G) = \det(K)$ and $e(G) = c(K)$. It follows from the spanning tree expansion for $V_K(t)$ in [15] that if K is an alternating link,

$$\mu(K) = \frac{\det(K)}{c(K)+1}.$$

Thus, $\dfrac{\log \mu(K)}{c(K)} = \dfrac{\log \det(K) - \log(c(K)+1)}{c(K)}$, and the result follows since K_n are diagrammatically maximal links. □

We conjecture that the alternating condition in Proposition 2.1 can be dropped.

Conjecture 2.2. *If K is any knot or link,*

$$\frac{2\pi \log \mu(K)}{c(K)} \leq v_{\rm oct}.$$

Proposition 2.3. *Conjecture 1.1 implies Conjecture 2.2.*

Proof. By the proof of Proposition 2.1, Conjecture 1.1 would immediately imply that Conjecture 2.2 holds for all alternating links K. By the spanning tree expansion for $V_K(t)$, $\Sigma|a_i| \leq \tau(G(K))$, with equality if and only if K is alternating. Hence, if K is not alternating, then there exists an alternating link with the same crossing number and strictly greater coefficient sum $\Sigma|a_i|$. Therefore, Conjecture 1.1 would still imply Conjecture 2.2 in the non-alternating case. □

Definition 2.4. Let $\mathcal{C}_{\rm JP} = \{2\pi \log \mu(K)/c(K)\}$ be the set of Jones polynomial densities for all links K. We define ${\rm Spec}_{\rm JP} = \mathcal{C}'_{\rm JP}$ as its derived set (set of all limit points).

Conjecture 2.2 is that ${\rm Spec}_{\rm JP} \subset [0, v_{\rm oct}]$.

Corollary 2.5. $[0, v_{\rm oct}] \subset {\rm Spec}_{\rm JP}$.

Proof. The result follows from Theorem 1.7 and the proof of Proposition 2.1. □

For example, twisting on two strands of an alternating link gives 0 as a common limit point. For links K_n that satisfy Theorem 1.4, their asymptotic volume density equals their asymptotic determinant density, so in this case,

$$\lim_{n\to\infty} \frac{{\rm vol}(K_n)}{c(K_n)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \lim_{n\to\infty} \frac{2\pi \log \mu(K_n)}{c(K_n)} = v_{\rm oct}.$$

2.2. **Knot homology density spectrum.** A natural extension of Conjecture 1.12 to any hyperbolic knot is to replace the determinant with the rank of the reduced Khovanov homology $\widetilde{H}^{*,*}(K)$. We have verified the following conjecture for all non-alternating knots with up to 15 crossings.

Conjecture 2.6 ([6])**.** *For any hyperbolic knot K,*

$${\rm vol}(K) < 2\pi \log {\rm rank}(\widetilde{H}^{*,*}(K)).$$

Note that Conjecture 1.12 is a special case of Conjecture 2.6.

Question 2.7. Is Conjecture 2.6 true for knot Floer homology; i.e., is it true that $\mathrm{vol}(K) < 2\pi \log \mathrm{rank}(HFK(K))$?

Definition 2.8. Let $\mathcal{C}_{\mathrm{KH}} = \{2\pi \log \mathrm{rank}(\widetilde{H}^{*,*}(K))/c(K)\}$ be the set of Khovanov homology densities for all links K. We define $\mathrm{Spec}_{\mathrm{KH}} = \mathcal{C}'_{\mathrm{KH}}$ as its derived set (set of all limit points).

Proposition 2.9. *If* $\mathrm{Spec}_{\det} \subset [0, v_{\mathrm{oct}}]$ *then* $\mathrm{Spec}_{\mathrm{KH}} \subset [0, v_{\mathrm{oct}}]$.

Proof. For alternating knots, $\mathrm{rank}(\widetilde{H}^{*,*}(K)) = \det(K)$. Let K be an alternating hyperbolic knot, and K' be obtained by changing any proper subset of crossing of K. It follows from results in [5] that $\det(K') \leq \mathrm{rank}(\widetilde{H}^{*,*}(K')) \leq \det(K)$. □

Question 2.10. Does $\mathrm{Spec}_{\mathrm{KH}} = \mathrm{Spec}_{\det}$?

2.3. **Kashaev invariant density spectrum.** The Volume Conjecture (see, e.g. [4] and references therein) is an important mathematical program to bridge the gap between quantum and geometric topology. One interesting consequence of our discussion above is a *maximal volume conjecture* for a sequence of links that is geometrically and diagrammatically maximal.

The Volume Conjecture involves the Kashaev invariant

$$\langle K \rangle_N := \frac{J_N(K; \exp(2\pi i/N))}{J_N(\bigcirc; \exp(2\pi i/N))},$$

and is the following limit:

$$\lim_{N\to\infty} 2\pi \log |\langle K \rangle_N|^{\frac{1}{N}} = \mathrm{vol}(K).$$

For any knot K, Garoufalidis and Le [10] proved

$$\limsup_{N\to\infty} \frac{2\pi \log |\langle K \rangle_N|^{\frac{1}{N}}}{c(K)} \leq v_{\mathrm{oct}}$$

Now, since the limits in Theorems 1.4 and 1.5 are both equal to v_{oct}, we can make the maximal volume conjecture as follows.

Conjecture 2.11 (Maximal volume conjecture)**.** *For any sequence of links K_n that is both geometrically and diagrammatically maximal, there exists an increasing integer-valued function $N = N(n)$ such that*

$$\lim_{n\to\infty} \frac{2\pi \log |\langle K_n \rangle_N|^{\frac{1}{N}}}{c(K_n)} = v_{\mathrm{oct}} = \lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)}.$$

To prove Conjecture 2.11 it suffices to prove

$$\lim_{n\to\infty} \frac{2\pi \log |\langle K_n \rangle_N|^{\frac{1}{N}}}{c(K_n)} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_{\mathrm{oct}},$$

which relates only diagrammatic invariants.

These ideas naturally suggest an interesting quantum density spectrum:

Definition 2.12. Let $\mathcal{C}_q = \{2\pi \log |\langle K \rangle_N|^{\frac{1}{N}}/c(K),\ N \geq 2\}$ be the set of quantum densities for all links K and all $N \geq 2$. We define $\mathrm{Spec}_q = \mathcal{C}'_q$ as its derived set (set of all limit points).

Conjecture 2.11 would imply that $v_{\mathrm{oct}} \in \mathrm{Spec}_q$. The Volume Conjecture would imply that $\mathrm{Spec}_{\mathrm{vol}} \subset \mathrm{Spec}_q$.

Remark 2.13. For every link K for which the Volume Conjecture holds, $\dfrac{\mathrm{vol}(K)}{c(K)} \in \mathrm{Spec}_q$. In particular, since the Volume Conjecture has been proved for torus knots, the figure-eight knot, Whitehead link and Borromean link (see [13]), we know that certain rational multiples of volumes of the regular ideal tetrahedron and octahedron are in Spec_q; namely,

$$\{0,\ \frac{1}{2}v_{\mathrm{tet}},\ \frac{1}{5}v_{\mathrm{oct}},\ \frac{1}{3}v_{\mathrm{oct}}\} \subset \mathrm{Spec}_q.$$

If $N = 2$, then $|\langle K\rangle_N| = \det(K)$, so $\frac{1}{2}\mathrm{Spec}_{\det} \subset \mathrm{Spec}_q$.

Together with Theorem 1.7, the results above suggest the following general conjecture:

Conjecture 2.14.

$$\mathrm{Spec}_{\mathrm{vol}} = \mathrm{Spec}_{\det} = \mathrm{Spec}_q = [0, v_{\mathrm{oct}}].$$

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