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# Toy stars in one dimension

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**ABSTRACT**

This paper describes a simple model of a star where compressibility is retained but the gravitational force is replaced by a force between pairs which is directed along their line of centres and proportional to their separation. This force was analysed by Newton. It may be considered the simplest many body force because the system reduces to a set of independent particles moving in a harmonic oscillator potential. We call such models 'toy stars'. In this paper we simplify further and focus on the one-dimensional problem. We show that a non-linear solution exists where the velocity is a linear function of the coordinate and the density a quadratic function of the coordinates. We generalize this result to include a magnetic field. Our results provide a very useful benchmark for algorithms designed to simulate gravitating gas and we show that Smoothed Particle Hydrodynamics (SPH) simulations of toy stars with or without magnetic fields give results in good agreement with theory.

**Key words:** gravitation – hydrodynamics – MHD – methods: numerical – stars: general.

## 1 INTRODUCTION

In order to understand a complex astrophysical system it is always useful to study simplified models. The classical ellipsoidal liquid stars are an example of simplified models which have engaged the attention of numerous investigators including Riemann, Dedikind, and Dirichlet (for a description of their results see Lamb 1932). These stars are models of real stars where the compressibility of the fluid is given up but the gravitational force is retained. They are relatively easy to analyse because the gravitational potential of an ellipsoid of constant density is a quadratic function of the coordinates, and the resulting forces are therefore linear functions of the coordinates. A comprehensive account of ellipsoidal figures and their stability is given by Chandrasekhar (1969).

In this paper we consider another class of models where compressibility is retained but the gravitational force is replaced by a force linear in the coordinates. This force is the simplest many-body force. It was discovered by Newton who pointed out that if two particles attract each other with a linear force then they move as if attracted to the centre of mass of the pair (see Chandrasekhar (1995) for a modern interpretation of Newton's Principia and, in particular, Newton's proposition LXIV which discusses this force). If there are $N$ particles attracting each other with a force proportional to the separation, and directed along the line joining pairs of particles, then each particle moves as if independent of the others. The force is a linear force towards the centre of mass of the $N$ particles. In the case of two particles the trajectories are ellipses. A gaseous system with this force has a number of attractive features the most important of which is that a non-linear solution of a pair of ordinary differential equations can be found for polytropic equations of state. We call these models toy stars.

The simplest version of the toy star assumes the pressure $P$ is given in terms of the density $\rho$ by $P = K\rho^2$ where $K$ is a constant. This makes the problem analogous to the problem of shallow water motion in paraboloidal basins. There is an extensive literature on this problem including the early papers of Goldsbrough (1930), the seminal papers of Ball (1963, 1965) and the general analysis by Holm (1991). Some of our results could be extracted from those papers but it is more convenient, and clearer, to cast our results in terms of astrophysical systems from the outset.

Many benchmark calculations for numerical algorithms in astrophysics assume periodic boundaries or infinitely extended systems. A primary aim of this paper is to establish benchmarks for algorithms based on a finite system. Not only do the toy stars have a non-linear solution which can be easily computed, but the modes of oscillation are given in terms of known functions. Furthermore, an easily computed non-linear solution can be found for an MHD version of the toy star.

In this paper we consider the one-dimensional toy star and use it as a benchmark for Smoothed Particle Hydrodynamics (SPH) calculations without and with magnetic fields using a recent Smoothed Particle Magnetohydrodynamics (SPMHD) algorithm (Price & Monaghan 2004a,b).

## 2 THE FORCE LAW

Newton proposed the linear force law in the Principia but for our purposes the modern discussion by Chandrasekhar (1995) is clearer.

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Suppose for example that we have an isolated group of $N$ particles in one dimension interacting with linear forces so that the force on particle $j$ due to particle $k$ is $\nu m_j m_k\,(x_k - x_j)$. The potential energy is

$$\Phi = \frac{1}{4}\nu \sum_j \sum_k m_j m_k (x_j - x_k)^2. \tag{1}$$

The equation of motion of the $j$th particle is then

$$m_j \frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} = -\nu m_j \sum_k m_k (x_j - x_k). \tag{2}$$

However, the centre of mass

$$\frac{\sum_k m_k x_k}{\sum_k m_k} \tag{3}$$

can be chosen as the origin so the equation of motion becomes

$$\frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} = -\nu M x_j, \tag{4}$$

where $M$ is the total mass. The potential can then be written

$$\Phi = \frac{1}{2}\nu M \sum_j m_j x_j^2. \tag{5}$$

The motion of the $N$-body system is therefore identical to the independent motion of each particle in a harmonic potential. In the following we replace $M\nu$ by $\Omega^2$.

## 3 THE EQUATIONS OF MOTION

The system is one-dimensional with velocity $v$, density $\rho$, and pressure $P$. The acceleration equation is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \Omega^2 x. \tag{6}$$

We assume the equation of state is

$$P = K\rho^2, \tag{7}$$

since that makes our equations identical in form to those for the shallow water equations with density replacing the water depth. The acceleration equation is then

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2K\frac{\partial \rho}{\partial x} - \Omega^2 x. \tag{8}$$

The static model has density

$$\rho = \rho_0 - \frac{\Omega^2 x^2}{4K}. \tag{9}$$

The radius $x_e$ of the static model is therefore

$$x_e^2 = \frac{4K\rho_0}{\Omega^2}. \tag{10}$$

The mass $M$ is

$$M = \int_{-x_e}^{x_e} \rho_0 \left(1 - \frac{x^2}{x_e^2}\right)\,\mathrm{d}x, \tag{11}$$

so that $M = 4\,\rho_0 x_e\,/3$,

The conserved energy of the system is

$$E = \frac{1}{2}\int \rho v^2\,\mathrm{d}x + \int P\,\mathrm{d}x + \frac{1}{2}\Omega \int \rho x^2\,\mathrm{d}x. \tag{12}$$

If $M$, $K$ and $\Omega$ are specified then $\rho_0$ and therefore $x_e$ can be calculated. To simplify the following equations we use $x_e$ as the unit of length, and we use $1/\Omega$ as the unit of time. The acceleration equation then becomes

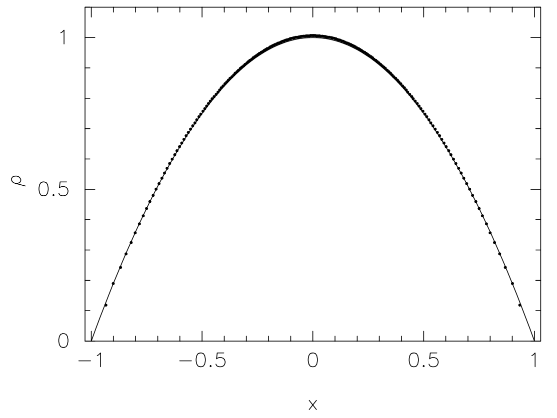


**Figure 1.** Toy star static structure. We set up 200 SPH particles in an initially uniform distribution along the $x$ axis and allow them to evolve under the influence of the linear force. The SPH particles are shown by the solid points after damping to an equilibrium distribution. The agreement with the exact quadratic ($\rho = 1 - x^2$) solution (solid line) is extremely good.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial \rho}{\partial x} - x, \tag{13}$$

and then the static density $\bar{\rho}$ is

$$\bar{\rho} = 1 - x^2, \tag{14}$$

while $P = \rho^2/4$ and $M = 4/3$. The static structure is compared with a numerical solution in Section 5.2 which is illustrated in Fig. 1.

The energy scaled in units $Mx_e^2\,\Omega^2$ is

$$E = \frac{1}{2}\int \rho v^2\,\mathrm{d}x + \int P\,\mathrm{d}x + \frac{1}{2}\int \rho x^2\,\mathrm{d}x. \tag{15}$$

It is worth noting that our equations of motion lead to the result that the centre of mass motion is decoupled from any other motion and, in particular, from the oscillations of the gas. To see this we multiply (6) by $\rho$ and integrate over the toy star. We find

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\Omega^2 R, \tag{16}$$

where $R$ is the centre of mass which therefore oscillates sinusoidally. This result can be deduced from one of Ball's general theorems (Ball 1963). It implies that the oscillations of the gas are decoupled from the centre of mass motion.

## 4 OSCILLATIONS

We now consider small oscillations of our toy star. We assume $v$ is small and we write the density in the form

$$\rho = \bar{\rho} + \eta. \tag{17}$$

If we retain only quantities which are linear in the perturbations the acceleration equation becomes

$$\frac{\partial v}{\partial t} = -\frac{1}{2}\frac{\partial \eta}{\partial x}, \tag{18}$$

and the continuity equation becomes

$$\frac{\partial \eta}{\partial t} = -\frac{\partial(\bar{\rho} v)}{\partial x}. \tag{19}$$

We let the time variation be $e^{i\omega t}$ and, by combining the equations, the equation for $v$ becomes

$$(1 - x^2)\frac{d^2 v}{dx^2} - 4x\frac{dv}{dx} + 2(\omega^2 - 1)v = 0. \quad (20)$$

The solutions of this equation are the Gegenbauer polynomials $G_n(x)$. The solution for $G_n$ requires that

$$2(\omega^2 - 1) = n^2 + 3n, \quad (21)$$

or

$$\omega^2 = \frac{(n+1)(n+2)}{2}. \quad (22)$$

Typical examples of Gegenbauer polynomials are $G_0 = 1$, $G_1 = 3x$, $G_2 = 3(5x^2 - 1)/2$, and $G_3 = 5(7x^3 - 3x)/2$, where higher orders may be calculated via the recurrence relation

$$nG_n(x) = (2n+1)xG_{n-1}(x) - (n+1)G_{n-2}(x). \quad (23)$$

The standard normalization is

$$\int_{-1}^{1} G_n^2(1 - x^2)\,dx = 2\frac{(n+1)(n+2)}{2n+3}. \quad (24)$$

Other properties of $G_n$ can be found in Abramowitz & Stegun (1972).

The equation for the density perturbation is

$$(1 - x^2)\frac{d^2\eta}{dx^2} - 2x\frac{d\eta}{dx} + 2\omega^2\eta = 0. \quad (25)$$

The solution to this equation are Legendre polynomials $P_m$. We note that

$$\frac{dP_{m+1}}{dx} = G_m. \quad (26)$$

We compare the perturbation solution with a numerical solution of toy star oscillations below.

## 5 STATIC AND LINEAR TEST CASES

As we remarked earlier, the toy star provides a good benchmark for both ideal gas and MHD codes. We illustrate this in the following and in Section 9 for a recent Smoothed Particle Magnetohydrodynamics (SPMHD) code (Price & Monaghan 2004a,b).

### 5.1 SPH implementation

For details of the SPH method in general we refer the reader to the review by Monaghan (1992). The specific implementation used here is described in detail in Price & Monaghan (2004a). The SPH equations are implemented using the summation over particles to calculate the density and the usual momentum equation with the linear force subtracted. The equation of state is specified by using $P = K\rho^{\gamma}$, where for the cases shown we set $K = 0.25$. The smoothing length is allowed to vary with the particle density, where we take simple averages of kernel quantities in the SPH equations in order to conserve momentum. In some cases we apply a small amount of artificial viscosity to the simulation. The formulation of the dissipative terms and the details of the viscosity switch are described in Price & Monaghan (2004a). Where viscosity is applied the minimum artificial viscosity coefficient $\alpha_{\min}$ is set to 0.1. The viscosity coefficient for each particle is then evolved using a switch that responds to the divergence of the velocity. The net result is minimal excess damping of the simulation.

### 5.2 Static structure

The simplest test with the toy star is to verify the static structure. We setup 200 SPH particles equally spaced along the $x$ axis with $x = [-1, 1]$ with zero initial velocity and a total mass $M = 4/3$. The particles are then allowed to evolve under the influence of the linear force, with the velocities damped each time-step according to

$$\frac{dv_x}{dt} = \frac{dv_x}{dt} - 0.02v_x. \quad (27)$$

Artificial viscosity is also applied to further increase the damping. The particle distribution at equilibrium is shown in Fig. 1 and shows extremely good agreement with the exact solution (equation 14).

### 5.3 Oscillations

The one-dimensional toy star is initially set up using 400 particles distributed along the $x$ axis. Although in principle we could use the particle distribution obtained in the previous test as the initial conditions, it is simpler just to set up the particles according to the $1 - x^2$ density profile. We consider both the case where the particle masses are initially varied and the case where equal mass particles are used with a variable initial separation. The results in both cases are similar, although slightly better in the former case since the resolution is improved in the outer regions.

For higher order oscillations a small amount of artificial viscosity is necessary to maintain order in the particle distribution. Using the switch discussed in Section 5.1, this results in some dissipation at the outer edges (where the velocity divergence is highest).

In the linear regime, oscillations of the toy star may be compared with the solution given in Section 4. We set up the toy star oscillations with initial velocity $v = 0.05\ C_s Gn(x)$ [with perturbation solution $v = 0.05\ C_s G_n(x)\cos\omega t$] and density $\rho = \bar{\rho} + \eta$ [where $\eta = 0.1\ C_s\omega\ P_{n+1}(x)\sin(\omega t)$]. Note that we can consider all the oscillation modes because although the centre of mass moves in the even modes, this motion is decoupled from the oscillations (cf. Section 3).

The sound speed $C_s = 1/\sqrt{2}$. In each case we apply a small amount of artificial viscosity using the switch discussed in Price & Monaghan (2004a). The results of the $n = 3$ calculation are shown in Fig. 2 after ten oscillation periods. In the top panel, 400 equal mass particles have been used with $\gamma = 2$ and no magnetic fields. There is good agreement with the perturbation solution (solid line) except for a slight discrepancy between the central densities. It was found that the numerical solution tended to oscillate around the perturbation solution slightly, in a quasi-periodic manner. This could be observed as an amplitude modulation in the kinetic energy. The error can be attributed to the lack of total energy conservation in the SPH code when the smoothing length is allowed to vary with density (total energy conservation is not explicitly enforced in this simulation since the pressure is determined from the density via 7). A formulation of SPH that self-consistently accounts for a changing smoothing length is given in Price & Monaghan (2004b). The results using this formulation are shown in the central panel of Fig. 2 and give a much smaller discrepancy in central densities (and correspondingly a much smaller modulation of the kinetic energy amplitude) than in the previous case. A simulation of this mode using particles with different masses and a constant initial separation is shown in the bottom panel. The oscillatory error is very small (total energy conserved to 1 part in $10^{-5}$), since the variation in the smoothing length over the course of the simulation is not significant. For the calculation of the higher order modes we therefore set up the simulation using particles of varying mass. This gives a higher resolution to the outer edges of the toy star (which is particularly important

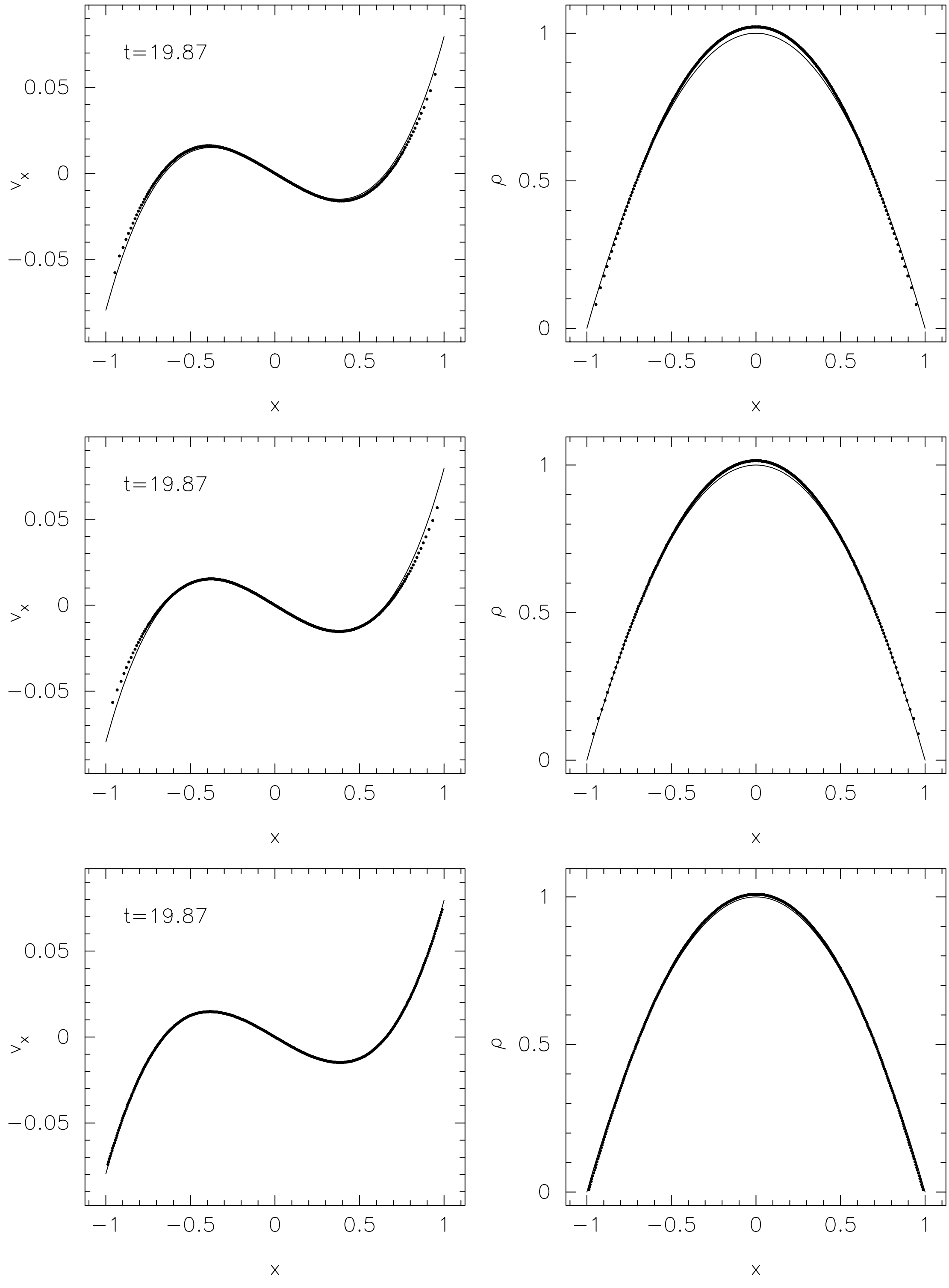


**Figure 2.** Results of the linear toy star simulation with initial velocity perturbation $v = 0.05C_sG_3(x)$, where $C_s$ is the sound speed and $G_3(x)$ is the third Gegenbauer polynomial. The velocity and density profiles are shown after ten oscillation periods in the case of equal-mass particles (top), equal-mass particles with a consistent treatment of smoothing length terms (middle) and using particles of different masses (bottom). 400 SPH particles are used in each case, denoted by the solid points, whilst the solid line denotes the perturbation solution (which after each period is the same as the initial conditions). The slight discrepancy between the central densities in the top figure highlights small errors in the energy conservation properties of the SPH code when the smoothing length is allowed to vary with density. A consistent treatment of smoothing length terms (middle) improves the situation substantially whilst the error is not detectable in the variable particle mass case (bottom) since the particles have a constant initial separation (resulting in little variation in particle smoothing lengths.

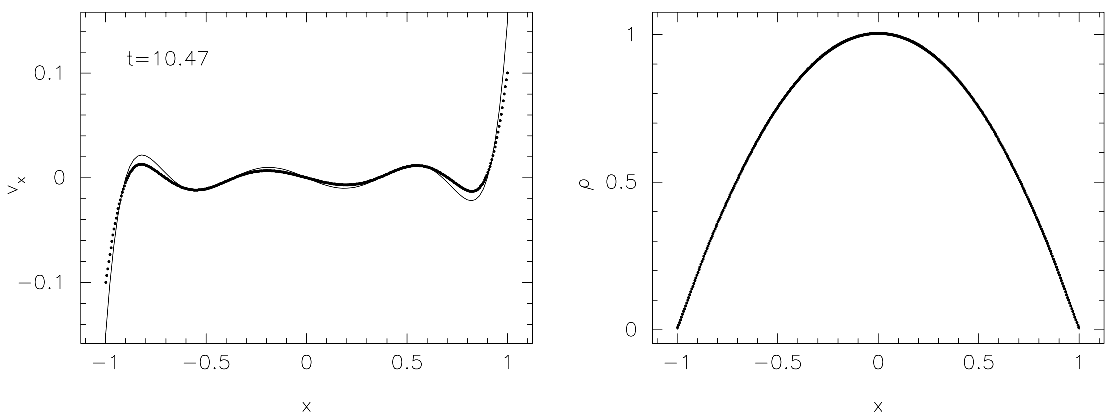


**Figure 3.** Results of the linear toy star simulation with initial perturbation $= 0.05\, C_s G_7(x)$where $C_s$ is the sound speed and $G_7(x)$ is the seventh Gegenbauer polynomial. Velocity and density profiles are shown after ten oscillation periods. 400 particles are used with a constant initial separation and therefore variable particle masses. SPH particles are given by the solid points, whilst the solid line denotes the perturbation solution (which after each period is the same as the initial conditions).

in the high order linear modes, since the Gegenbauer polynomials turn sharply near the outer edges). The result is that the higher order modes last much longer before damping to the fundamental ($n = 1$) than in the equal particle mass case. For example, in the $n = 7$ mode the equal mass simulation damps to the fundamental in ∼8 periods, whilst the variable mass case damps after ∼80 periods, with the damping time correspondingly shorter for higher order modes.

Results for the $n = 7$ mode are shown in Fig. 3 using 400 particles with constant initial separation and variable particle masses. The velocity and density profiles are shown after ten oscillation periods. The agreement with the perturbation solution (solid line) is very good.

We measure the frequency of oscillation of the SPH solution by the spacing of maxima in the kinetic energy, taking the average period over the first five periods. The frequency of oscillation computed in this manner for each mode is compared with the exact frequency (equation 22) in Fig. 4. 400 particles have been used in each case, using variable particle masses and a constant initial separation. The comparison with theory is excellent (<1 per cent) up to at least $n = 20$ for the first 5 oscillation periods. Using equal mass particles the results are similar up to around $n = 9$, but thereafter the higher order modes tend to damp to the $n = 1$ mode. A similar effect is observed when the variable mass runs are followed for a longer time. Note for the higher order modes that whilst the frequencies correspond very well to the perturbation solution the amplitudes are significantly damped.

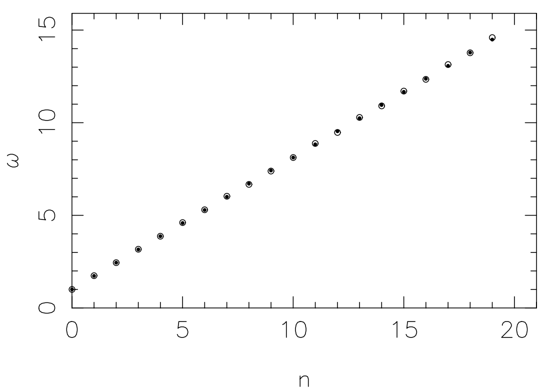


**Figure 4.** Toy star oscillation spectrum. The exact oscillation frequencies are given by the filled circles for the various modes, whilst open circles denote the SPH solution. Initial perturbations are given by $v = 0.05\, C_s G_n(x)$, where modes $n = 1$ to $n = 20$ are shown. 400 particles are used with a constant initial separation and variable particle masses.

## 6 AN EXACT NON-LINEAR SOLUTION

Let us return to the exact equations and seek a solution in the form

$$v = A(t)x, \tag{28}$$

with

$$\rho = H(t) - C(t)x^2, \tag{29}$$

so that the time-dependent radius of the toy star is $\sqrt{H/C}$, and the mass $M$ is given by

$$M = 2\int_0^{\sqrt{H/C}} \rho\, \mathrm{d}x = \frac{4}{3}\left(\frac{H^3}{C}\right)^{1/2}. \tag{30}$$

Substitution of the expressions for $v$ and $\rho$ into the acceleration equation gives

$$\dot{A} + A^2 = C - 1, \tag{31}$$

and from equating coefficients of powers of $x$ the continuity equation gives

$$\dot{H} = -AH, \quad \text{and} \quad \dot{C} = -3AC. \tag{32}$$

We deduce from the last two equations that $H^3 \propto C$, which from (30) guarantees the conservation of mass. The solution of the original equations of motion therefore reduce to the solution of the two first-order autonomous equations

$$\dot{A} = C - 1 - A^2, \tag{33}$$

$$\dot{C} = -3AC, \tag{34}$$

after which $H$ can be found from $C$. The equations for A and C can be solved for given initial conditions on $v$ and $\rho$. From the previous two equations we deduce that

$$\frac{\mathrm{d}A}{\mathrm{d}C} = \frac{A^2 - C + 1}{3AC}, \tag{35}$$

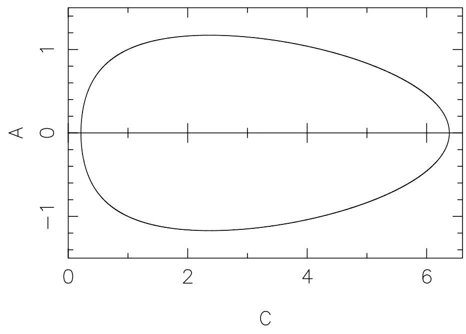


**Figure 5.** Closed curve connecting exact solution parameters $A$ and $C$ demonstrating that the motion of the toy star is periodic. The curve shown is for the case where initially $A = 1$ and $C = 1$.

which can be integrated to give

$$A^2 = -1 - 2C + kC^{\frac{2}{3}}, \tag{36}$$

where $k$ is a constant.

The equation connecting $A$ and $C$ is a closed curved in the $A$, $C$ plane showing that the motion is periodic. In Fig. 5 we show this curve for the case where initially $A = 1$, $C = 1$ and therefore $k = 4$.

## 7 THE MAGNETIC TOY STAR

The Lagrangian rate of change of the magnetic field for ideal MHD (Price & Monaghan 2004a) is given by

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - \boldsymbol{B}\nabla \cdot \boldsymbol{v}. \tag{37}$$

For our one-dimensional system this shows that the $x$ component of $\boldsymbol{B}$ is constant. For the present analysis we assume it is zero. The rate of change of the $y$ component $B^y$ is given by

$$\frac{\mathrm{d}B^y}{\mathrm{d}t} = -B^y \frac{\partial v}{\partial x}. \tag{38}$$

This last equation shows that $B^y \propto \rho$. The coefficient of proportionality is constant following the motion and could be taken as a different constant for each element of fluid or equivalently each SPH particle. However, we wish to determine the non-linear solution in the product form. We therefore assume the constant is the same for each particle and write $B = \sigma\rho$ with the understanding that $B = B^y$. The acceleration equation (6) then becomes

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(P + \frac{B^2}{2\mu_0}\right) - \Omega^2 x, \tag{39}$$

where $\mu_0$ is the permeability of free space.

The dynamics of the MHD system is then the same as for the ideal gas except that the constant $K$ is replaced by

$$K' = K + \frac{\sigma^2}{2\mu_0}. \tag{40}$$

The scaled form of the equations is the same as before with an effective pressure $P = \rho^2/4$ including both the gas pressure and the magnetic pressure. The non-linear solution in terms of two ordinary differential equations carries over to the MHD case.

## 8 ARBITRARY POLYTROPIC EQUATIONS OF STATE

The previous analysis applies to the case where the equation of state is $P = K\rho^2$ which maps directly to the shallow water equations. If the equation of state is $P = K\rho^\gamma$ then the analysis for the case with no magnetic field is similar. In particular, it is straightforward to show that the equations have a solution with $v = A(t)x$ and

$$\rho^{\gamma-1} = H(t) - C(t)x^2. \tag{41}$$

From the coefficient of $x$ in the acceleration equation we find

$$\dot{A} + A^2 = \frac{2K\gamma}{\gamma - 1}C - 1. \tag{42}$$

The coefficients of powers of $x$ in the equation of continuity give

$$\dot{H} = -AH(\gamma - 1) \tag{43}$$

and

$$\dot{C} = -AC(1 + \gamma). \tag{44}$$

From the last two equations we deduce that $H^{\gamma+1} \propto C^{\gamma-1}$, which guarantees the conservation of mass. It is possible to integrate the equations for the rate of change of $A$ and $C$ to deduce that

$$A^2 = -1 - \frac{2\sigma C}{\gamma - 1} + kC^{2/(\gamma+1)}, \tag{45}$$

where $\sigma = 2K\gamma/(\gamma - 1)$, and $k$ is an arbitrary constant. For physical problems $C > 0$ and the curve in the $A$, $C$ plane is closed.

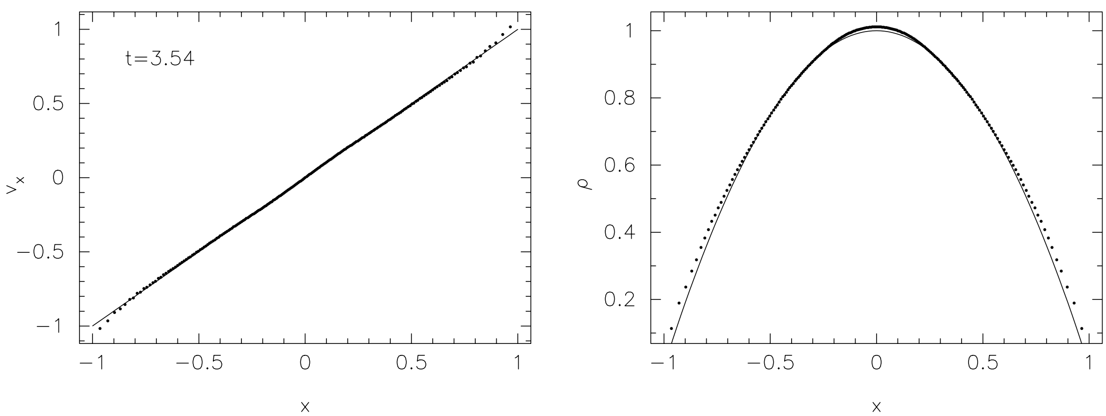


**Figure 6.** Results of the SPH non-linear toy star simulation with $\gamma = 2$ and initial conditions $v = x$, $\rho = 1 - x^2$ (ie. $A = C = H = 1$). Velocity and density profiles are shown after approximately one oscillation period, with the SPH particles indicated by the solid points and the exact solution by the solid line in each case. Equal mass particles are used with a variable initial separation.

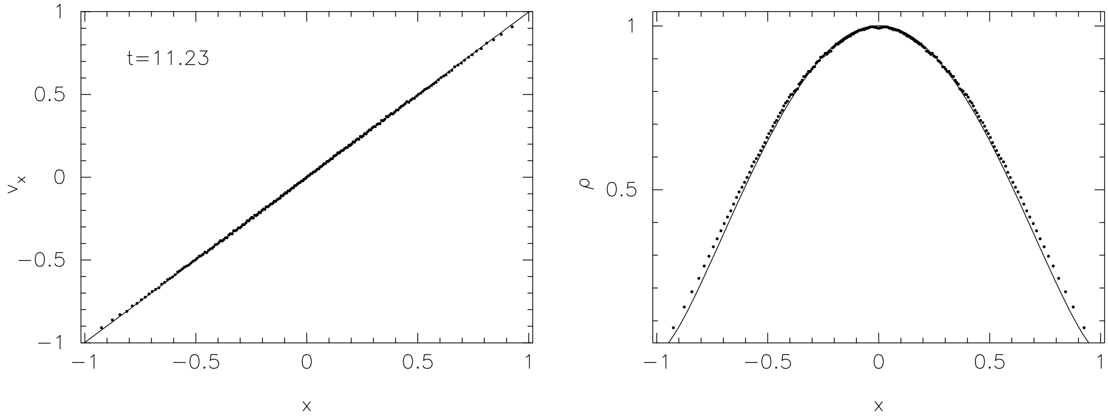


**Figure 7.** Results of the SPH non-linear toy star simulation with $\gamma = 5/3$ and initial conditions $v = x$, $\rho = (1 - x^2)^{3/2}$ (ie. $A = C = H = 1$ with $\gamma = 5/3$). Velocity and density profiles are shown after approximately three oscillation periods and the exact solution is given by the solid line.

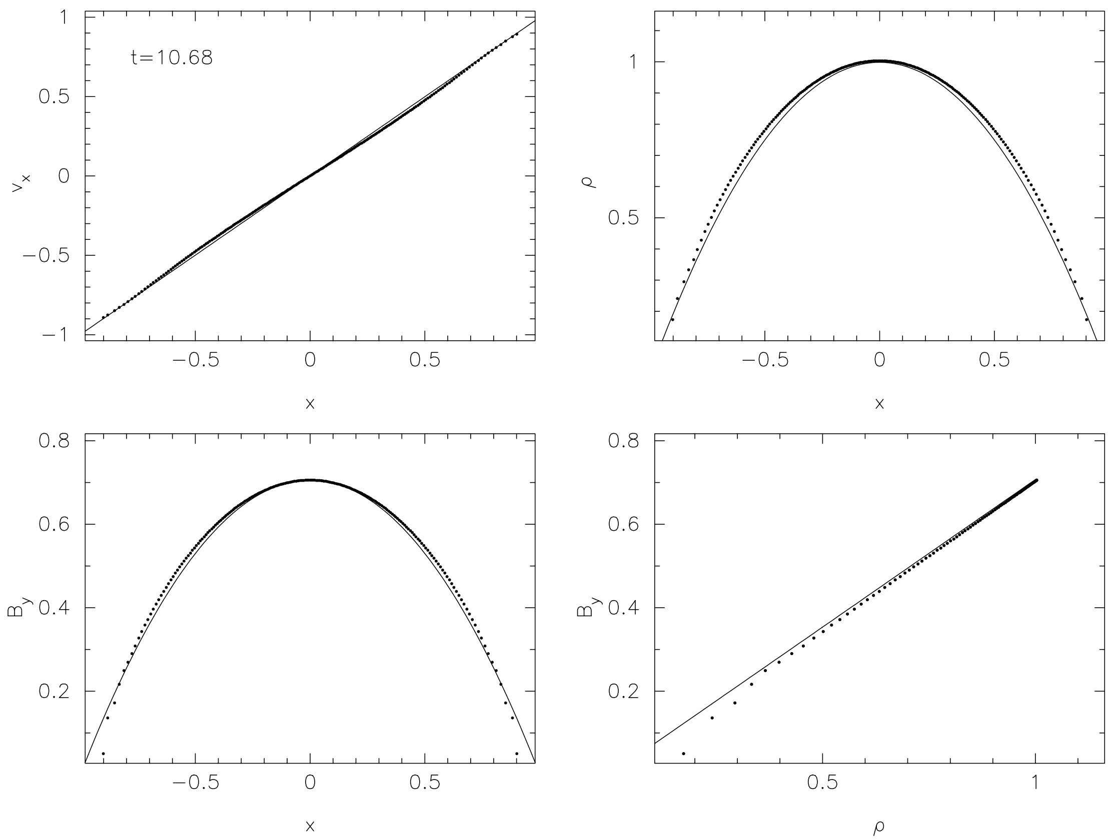


**Figure 8.** Results of the non-linear, magnetic toy star simulation with initial conditions $v = x$, $\rho = (1 - x^2)$, $B^y = \rho/\sqrt{2}$ (ie. $A = C = H = 1, \sigma = 1/\sqrt{2}$ and $\gamma = 2$), shown after approximately three oscillation periods. Equal-mass particles are used with a variable initial separation, whilst the magnetic field is chosen such that gas pressure and magnetic pressure are equal in magnitude.

## 9 NON-LINEAR AND MAGNETIC TEST CASES

The SPH toy star is set up as described in Section 5.3. For the non-linear tests we use 200 equal mass particles in one dimension with variable initial separation and smoothing lengths.

The exact (non-linear) solution is obtained by numerical integration of equations (42)–(44) using a simple improved Euler method. We use the condition (45) as a check on the quality of this integration by evaluating the constant $k$, which should remain close to its initial value.

Results for the case where initially $A = C = H = 1$ (and therefore $k = 4$) are shown in Fig. 6 at $t = 3.54$ (corresponding to approximately one oscillation period) alongside the exact solution shown by the solid lines. No artificial viscosity is applied in this case. The agreement with the exact solution is excellent. Note that the sound speed in this case is $C_s = 1/\sqrt{2}$ such that using the parameter $A = 1$ results in supersonic velocities at the edges of the star (the solution is therefore highly non-linear).

Fig. 7 shows the SPH results for a simulation with $\gamma = 5/3$ (cf. Section 8) and the same initial parameters as Fig. 6. Velocity and density profiles are shown at time $t = 11.23$ corresponding to approximately three oscillation periods. No artificial viscosity is used. The agreement with the exact solution (solid lines) is again extremely good.

In the magnetic case (Section 7) the magnetic field is evolved using the SPH form of equation (37) with the magnetic field and velocity allowed to vary in two dimensions whilst the particles are constrained to move along the $x$-axis. We set $\gamma = 2$ and choose the magnetic field strength such that the ratio of gas to magnetic pressure, $\beta = 1$, i.e. $\boldsymbol{B} = (0, \rho/\sqrt{2}, 0)$. For this simulation we apply a small amount of artificial dissipation, as discussed in Section 5.1.

Results are shown in Fig. 8 at $t = 10.68$, corresponding to approximately three oscillation periods in this case and again show very good agreement with the exact solution.

## 10 DISCUSSION AND CONCLUSIONS

Our aim in this paper has been to establish the properties of a simple dynamical system which has some of the properties of actual stars, but in a sufficiently simple form that the dynamics can be calculated very accurately and used as benchmarks for computer codes. The system is based on a many body attractive force which is proportional to the distance between the masses. This force pulls the matter into an object with a steady-state density profile which is a quadratic function of the coordinates.

For a one-dimensional toy star we have determined the frequencies and modes for linear oscillations and found a solution for a non-linear oscillation in terms of two ordinary differential equations. It is trivial to include a magnetic field, and we have shown that the main results can be extended to an arbitrary polytropic index.

In the case of both linear and non-linear oscillations, with and without magnetic fields, we have compared results from our SPH code with the theoretical results of this paper and found very good agreement.

Two- and three-dimensional toy stars can be analysed in a similar way though the analysis is more complicated. The perturbations to the static model can be written in terms of known functions and the frequencies determined. Non-linear solutions can be found, in some cases exactly, and in other cases from a small number (six in the case of two dimensions) of ordinary differential equations. In addition rotation can be included together with a variety of magnetic field structures. These solutions will be described elsewhere.

While the many body force law we introduced does not occur in nature, the motion of gas in an oscillator potential (to which our force reduces) can occur e.g. gas with negligible mass moving about an equilibrium position due to forces from much greater masses. In some cases rotation may be a complicating feature, as in gas near the stable Lagrangian points of a binary system. However, it is our view that the main value of our toy stars is in providing demanding benchmarks for numerical codes designed to simulate astrophysical systems.

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Calculations of toy star models were first run with the aim of studying the oscillations. We noticed that the toy star stayed in the fundamental mode regardless of the amplitude. This was remarked on to Darryl Holm who suggested that maybe the fundamental was an exact solution which we then worked out. Darryl Holm also introduced us to the fluid dynamical literature of tidal oscillations. Donald Lynden-Bell pointed out to one of us that the oscillator potential was more interesting than we thought because it was really a many-body force that Newton had used.

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