SMOOTHED PARTICLE HYDRODYNAMICS

"Four things you may have been told"

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R09: Computational Fluid Dynamics: SPH and Mesh Free Methods (5:00pm - 5:45pm CST)

Interactive,On Demand | Tue, Nov 24, 2020 | 10:00 AM - 10:45 AM

R09:1: SPH simulations of helicopter ditching on calm water and in waves.
Presenter: Guillaume Oger, Ecole Centrale Nantes

R09:2: Graph Neural Network for Lagrangian Fluid Simulation
Presenter: Zijie Li, Carnegie Mellon University

R09:3: Turbulence Modeling in Smoothed Particle Hydrodynamics
Presenter: Francesco Ricci, New Jersey Institute of Technology

R09:4: DualSPHysics: from fluid dynamics to multiphysics problems
Presenter: Angelo Tafuni

R09:5: Coupling the Finite Volume Particle Method with the Finite Element Method for fluid-structure interaction for large deformations
Presenter: Maryrose McLoone, NUI Galway

Welcome to SPHE

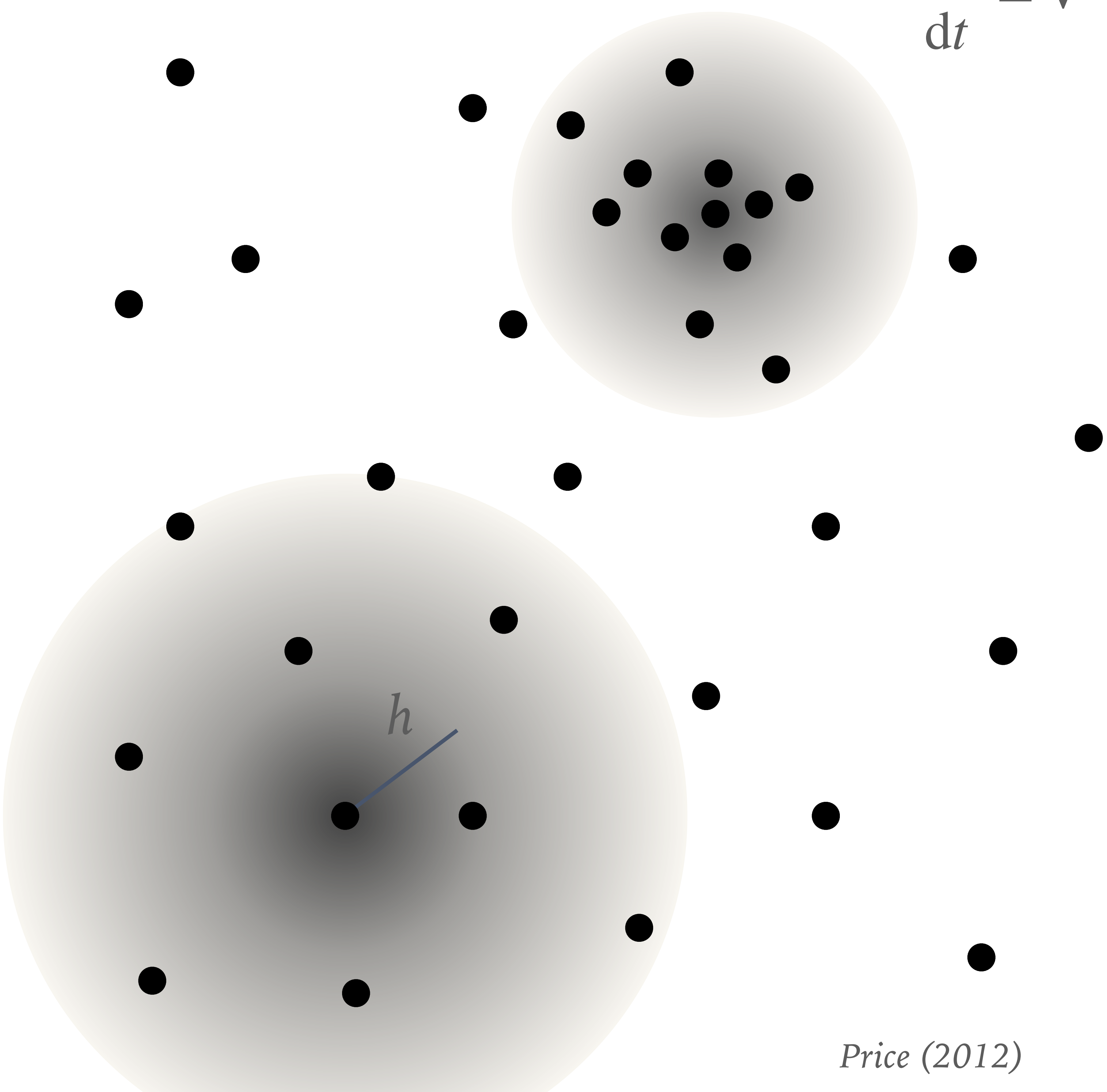
SPHERIC is the interna
researchers and indus
(SPH).

As a purely Lagrangian
distorting fluids and s
mechanics, multi-phase
where Eulerian metho
applications of this mesh

SMOOTHED PARTICLE HYDRODYNAMICS

e.g. Lucy (1977), Gingold & Monaghan (1977), Monaghan (1992)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$$



Price (2012)

- Discretise fluid onto Lagrangian particles
- Kernel-weighted sums to interpolate fluid quantities and derivatives

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h)$$

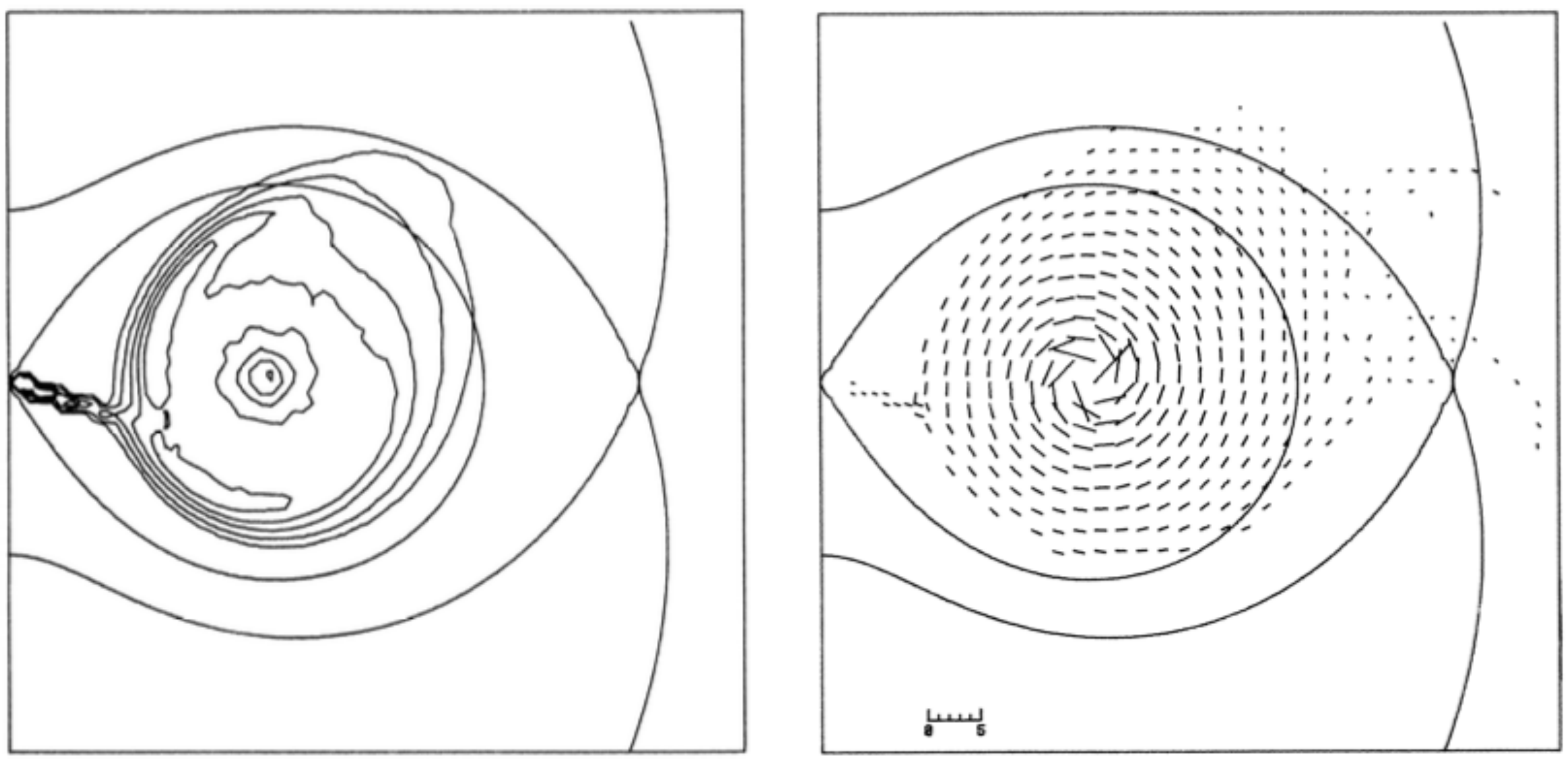
$\rho(x, y)$ m_i, x_i, y_i

THINGS YOU MIGHT HAVE HEARD ABOUT SMOOTHED PARTICLE HYDRODYNAMICS

MYTH 1: SPH DOES NOT SOLVE THE EQUATIONS OF FLUID DYNAMICS

ORIGIN OF THE MYTH: THE STICKY PARTICLE METHOD

We approximate the fluid as being composed of a few thousand individual particles. For most of the time these particles move under Newton's laws as isolated test particles in the potential of the binary system (restricted three-body problem). Every so often each particle is forced to interact instantaneously with its neighbours in a viscous manner. In



Lin & Pringle (1976)

$$\frac{d^2x}{dt^2} = 2\frac{dy}{dt} + x + 1 - \mu + \frac{(1+x)(\mu-1)}{r_2^3} - \frac{\mu x}{r_1^3}$$

$$\frac{d^2y}{dt^2} = -2\frac{dx}{dt} + y\left\{1 + \frac{\mu-1}{r_2^3} - \frac{\mu}{r_1^3}\right\}$$

Shu: I think you would be the first to agree that what you do is not fluid mechanics. It does give some aspects of the role of viscosity, but not all. Furthermore, I would suspect that the results of the calculation are quite sensitive to the value chosen for the parameter l.

Pringle: I think it is a bit strong to say that what we are doing is not fluid mechanics. We treat the mechanics correctly and I would contend that we are dealing with a fluid. The real question is whether or not the equation of state and properties we have bestowed upon our fluid are sufficiently realistic for our present purposes. We feel that in some respects they probably are. Since the size of the para-

TRUTH: DISCRETE HYDRODYNAMICS FROM THE FLUID LAGRANGIAN

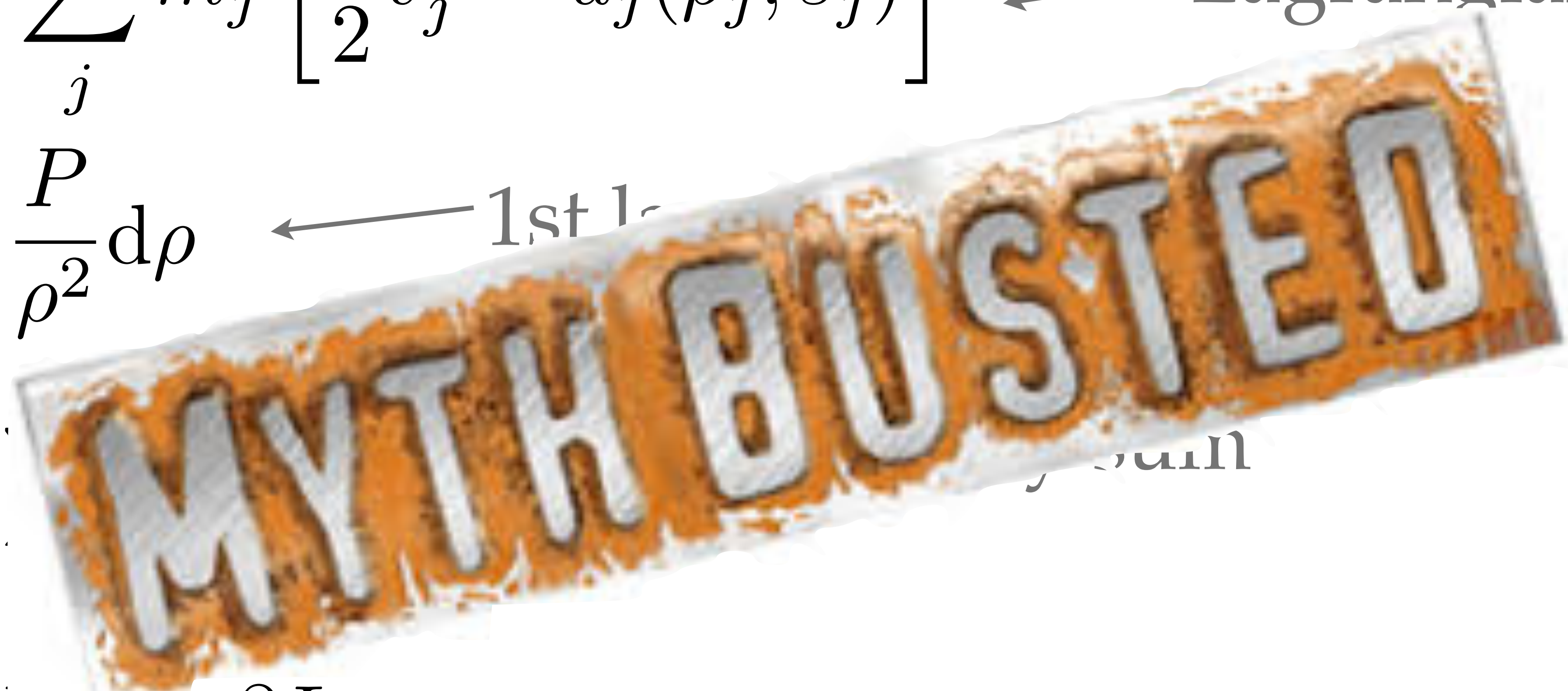
$$L_{sph} = \sum_j m_j \left[\frac{1}{2}v_j^2 - u_j(\rho_j, s_j)\right]$$ ← Lagrangian

$+$

$$\mathrm{d}u = \frac{P}{\rho^2}\mathrm{d}\rho$$ ← 1st l

$+$

$\nabla\rho_i =$



$+$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

$=$

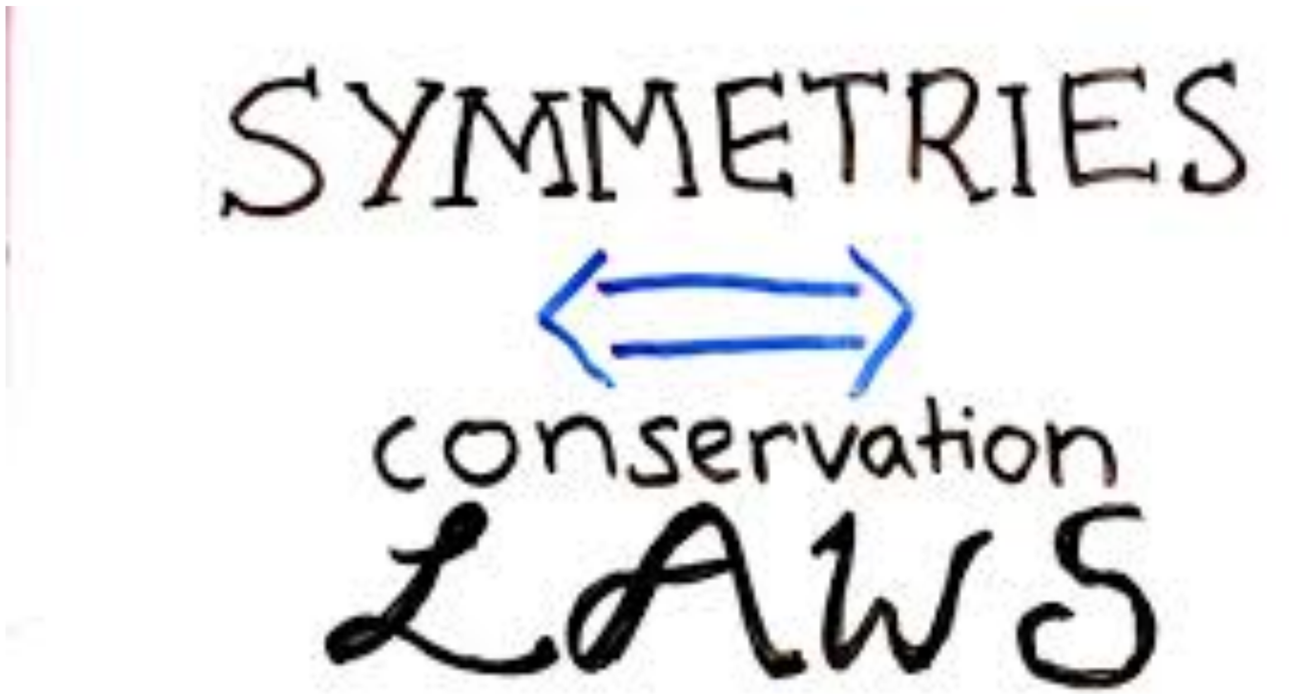
$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)\nabla_i W_{ij}(h)$$

← equations of motion!

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho}\right)$$

WHAT THE LAGRANGIAN GIVES US

Noether's theorem: "the most beautiful idea in physics"



Emmy Noether 1882-1935

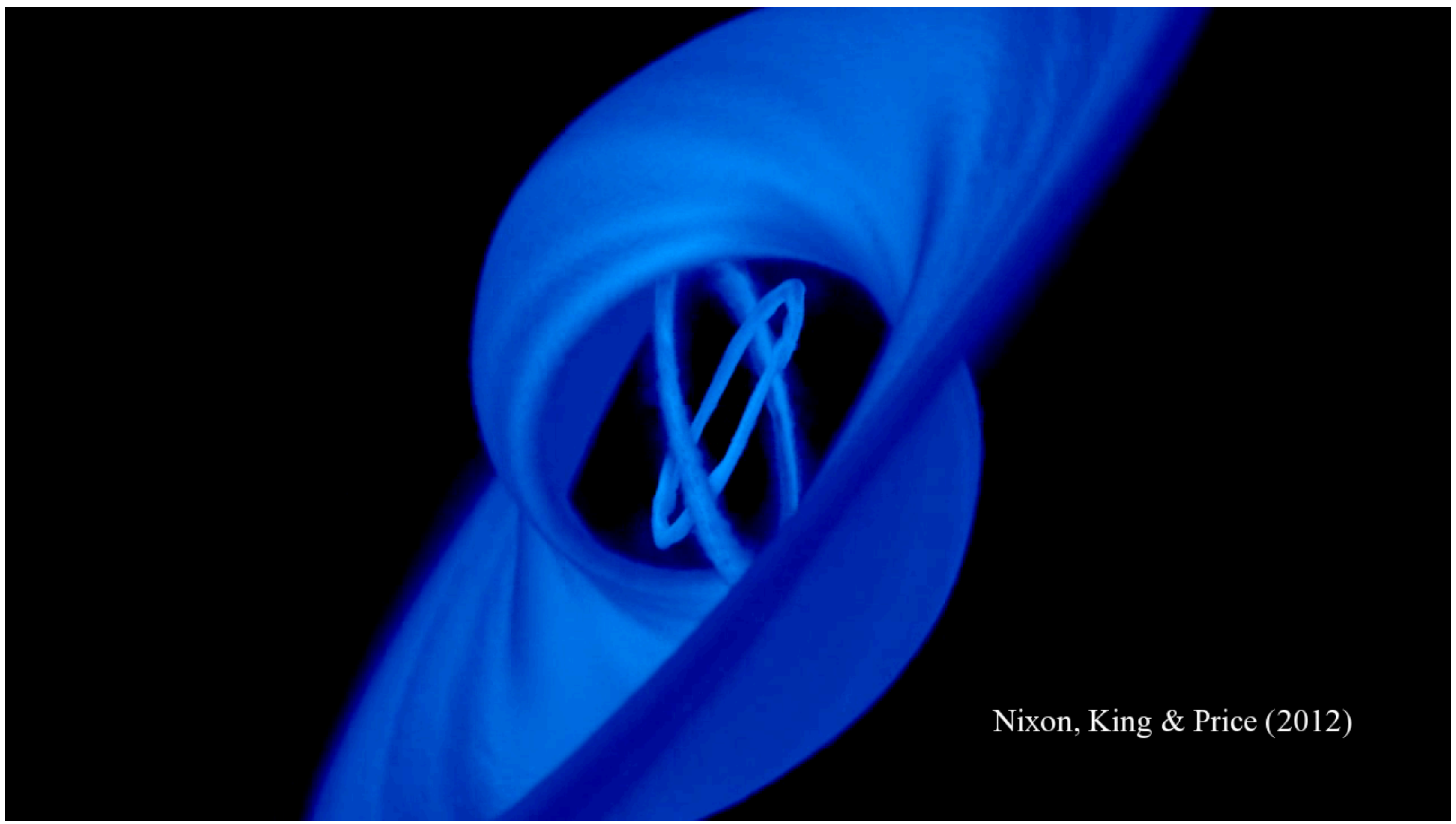
- Conservation of both linear and angular momentum to machine precision (translational and rotational symmetry)
- Conservation of energy in spatial discretisation (time symmetry)

$$\sum_a m_a \frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = 0$$

$$\sum_a m_a \left(\mathbf{r}_a \times \frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} \right) = 0$$

$$\sum_a m_a \frac{\mathrm{d}e_a}{\mathrm{d}t} = 0$$

EXAMPLE: CONSERVATION OF ANGULAR MOMENTUM



Warping of an accretion disc by a spinning, supermassive black hole
Nixon, King & Price (2012), ApJL 757, L24

Orbits are accurate… even when motions not aligned with any symmetry axis.

EXAMPLE: GENERAL RELATIVISTIC HYDRODYNAMICS

Monaghan & Price (2001)
Rosswog (2010)
Liptai & Price (2019)

$$L_{grsph} = -\sum_j m_j (1 + u_j) \sqrt{-g_{\mu\nu} v_j^\mu v_j^\nu}$$ ← Lagrangian

$+$

$$\mathrm{d}u = \frac{P}{\rho^2} \mathrm{d}\rho$$ ← 1st law of thermodynamics

$+$

$$\nabla \rho_i = \sum_j m_j \nabla W_{ij}(h)$$ ← density sum

$+$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ ← Euler-Lagrange equations

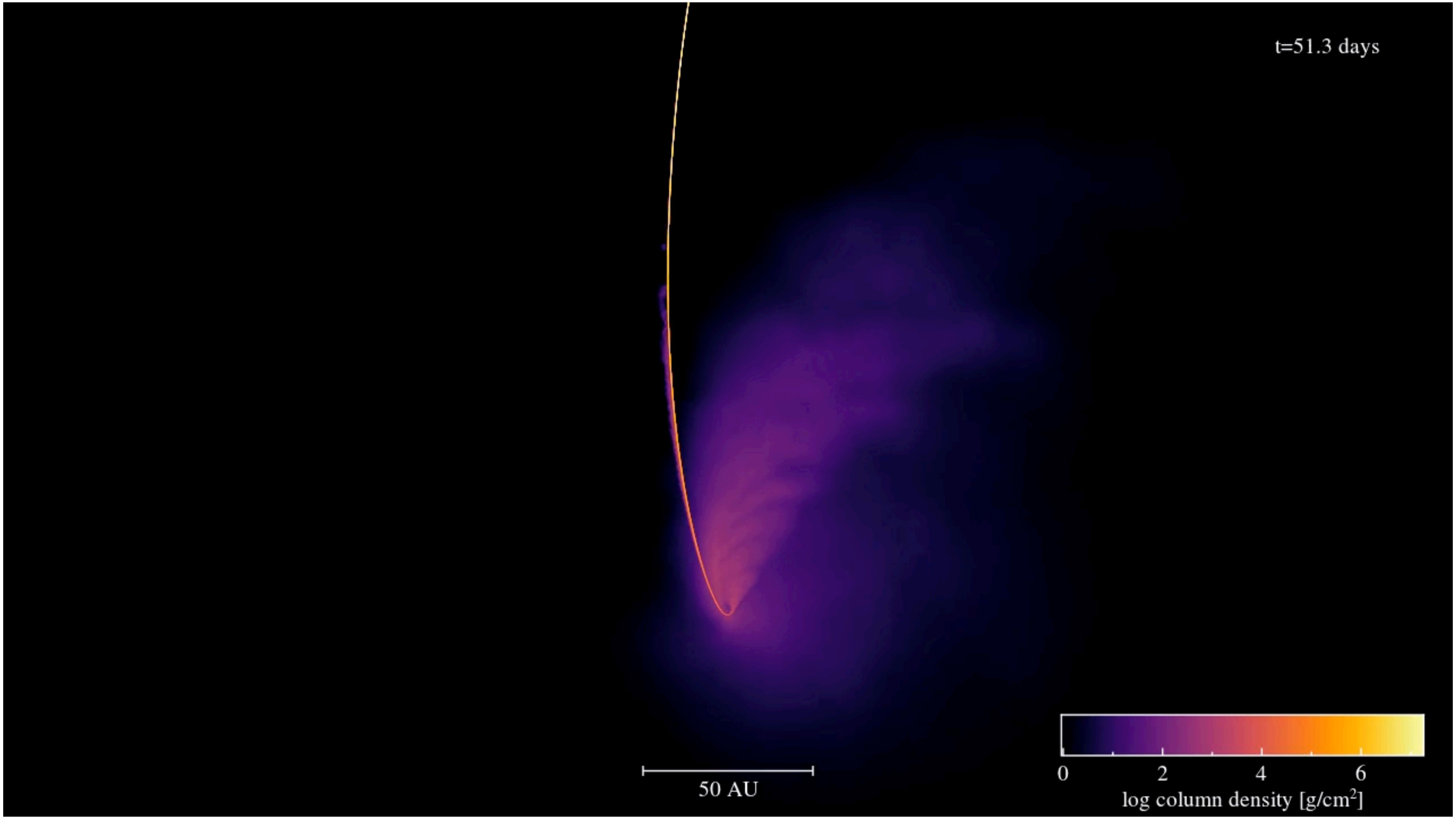
$=$

$$\frac{\mathrm{d}\mathbf{p_i}}{\mathrm{d}t} = -\sum_j m_j \left(\frac{\sqrt{-g_i} P_i}{\rho_i^{*2}} + \frac{\sqrt{-g_j} P_j}{\rho_j^{*2}} \right) \nabla_i W_{ij}(h)$$

← equations of motion!

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\nabla(\sqrt{-g} P)}{\rho *} \right)$$

TIDAL DISRUPTION OF STARS BY SUPERMASSIVE BLACK HOLES



- High speed flow, huge range of timescales

Liptai et al. (2020)

- Most of domain is empty, immensely challenging problem!

MYTH 2: SPH CAN'T CAPTURE SHOCKS

ORIGIN OF THE MYTH: "HIGH RESOLUTION SHOCK CAPTURING" METHODS

An efficient shock-capturing central-type scheme for multidimensional relativistic flows

I. Hydrodynamics

L. Del Zanna and N. Bucciantini

Numerical Relativistic Hydrodynamics: HRSC Methods
Luciano Rezzolla
Olindo Zanotti
DOI:10.1093/acprof:oso/9780198528906.003.0009

This chapter is devoted to the analysis of those numerical methods based on the conservative formulation of the equations, as is the case of the relativistic-hydrodynamics equation.

Eulerian conservation form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P\boldsymbol{I}) = 0$$

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot [(\rho e + P)\boldsymbol{v}] = 0$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \left[\mathbf{F}(\mathbf{U})\right] = 0 \qquad \mathbf{U} = \left[\rho, \rho\mathbf{v}, \rho e\right]^T$$

Lagrangian conservation form:

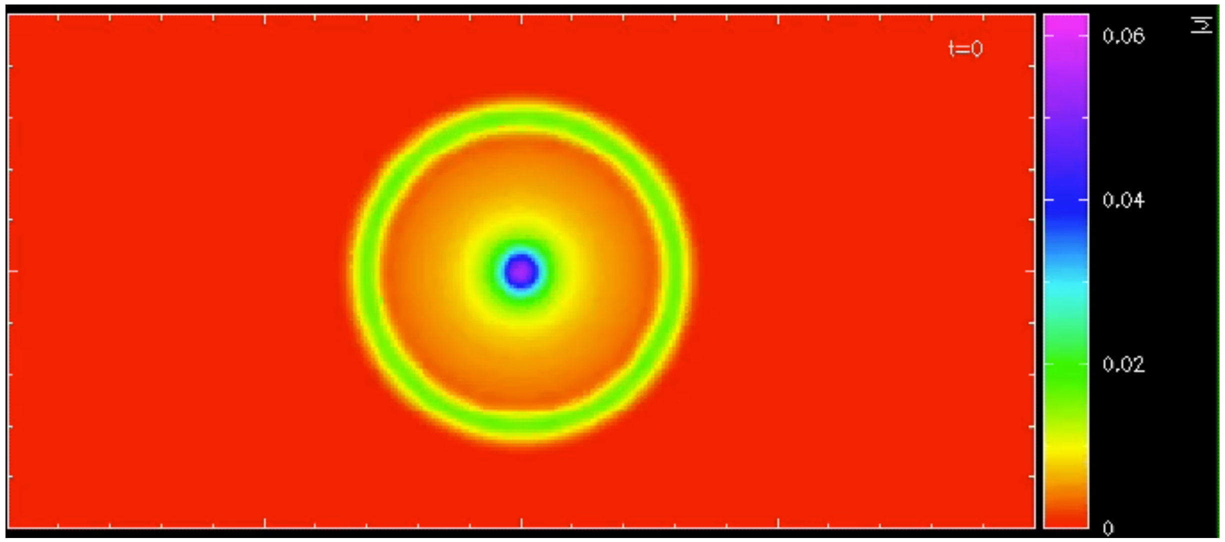
$$\frac{\mathrm{d}m}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla \cdot (P\boldsymbol{I})}{\rho}$$

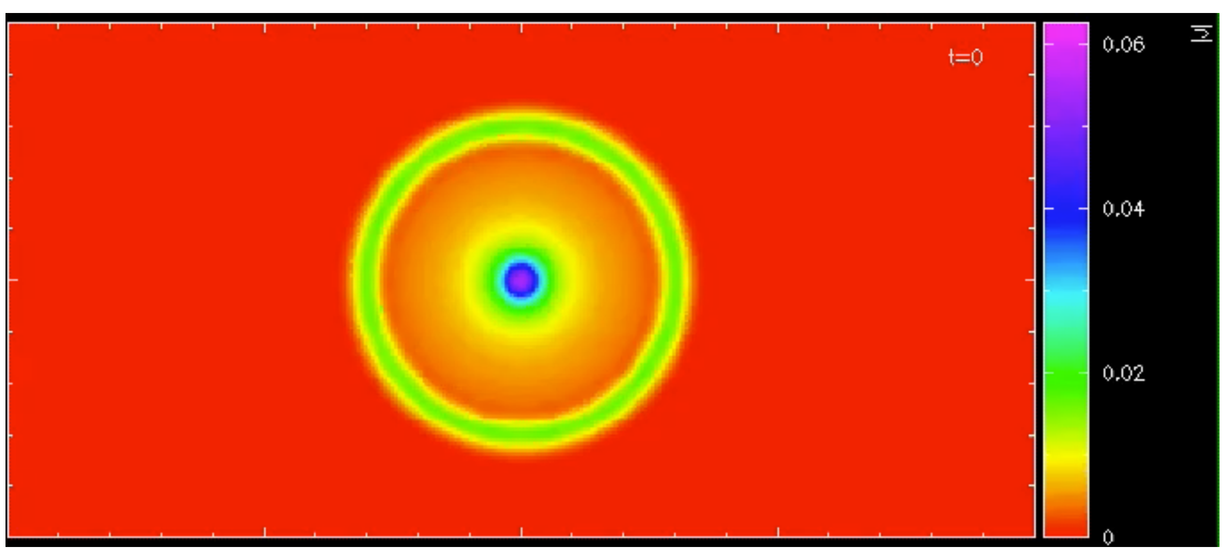
$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\nabla \cdot (P\boldsymbol{v})}{\rho}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{\nabla \cdot \mathbf{F}(\mathbf{u})}{\rho} \qquad \mathbf{u} = [\mathbf{v}, e]^T$$

TRUTH: ADVECTION IS PERFECT IN LAGRANGIAN SCHEMES



first 25 crossings



1000 crossings (Rosswog & Price 2010)

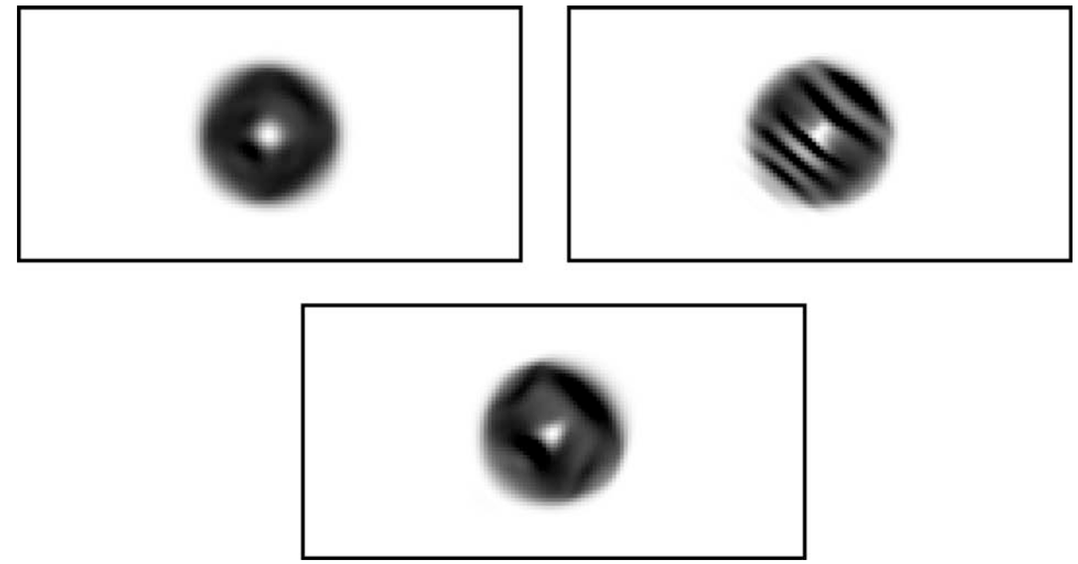


Fig. 3. Gray-scale images of the magnetic pressure ($B_x^2 + B_y^2$) at $t = 2$ for an advected field loop ($v_0 = \sqrt{5}$) using the $\mathcal{E}_z^\alpha$ (top left), $\mathcal{E}_z^\circ$ (top right) and $\mathcal{E}_z^c$ (bottom) CT algorithm.

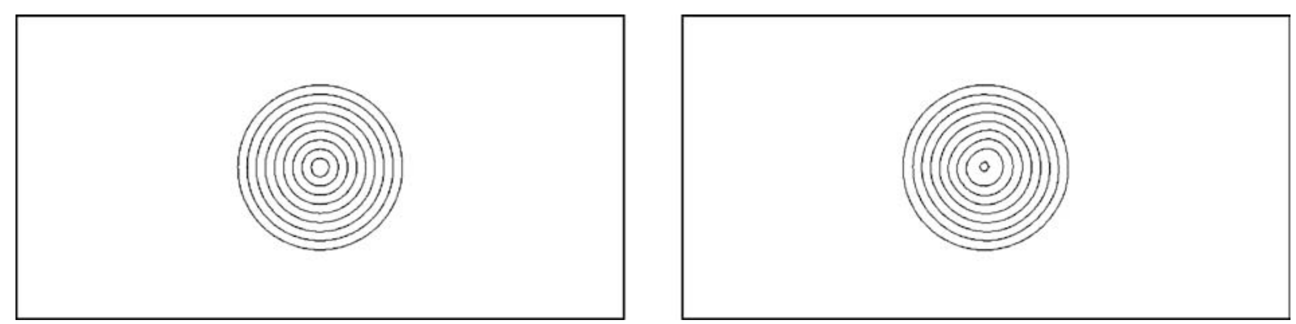


Fig. 8. Magnetic field lines at $t = 0$ (left) and $t = 2$ (right) using the CTU + CT integration algorithm.

2 crossings (Gardiner & Stone 2005)

Test problem: Advection of a magnetic current loop in a uniform flow

HIGH RESOLUTION SHOCK CAPTURING METHODS FOR SPH

Monaghan (1997), Chow & Monaghan (1997)

- Finite volume method

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \left[\mathbf{F}(\mathbf{U})\right] = 0 \quad \Rightarrow \quad \frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} = -\left[\frac{\mathbf{F}^*_{i+\frac{1}{2}} - \mathbf{F}^*_{i-\frac{1}{2}}}{\Delta x}\right]$$

$$\mathbf{U} = \left[\rho, \rho\mathbf{v}, \rho e\right]^T$$

$$\mathbf{F}^* = \frac{1}{2}\left[\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R)\right] - \frac{v_{\rm sig}}{2}(\mathbf{U}_R - \mathbf{U}_L)$$

Godunov-type solution

- SPH

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{\nabla \cdot \mathbf{F}(\mathbf{u})}{\rho} \quad \Rightarrow \quad \frac{\mathrm{d}\mathbf{u}_a}{\mathrm{d}t} = -\sum_b m_b \left[\frac{\mathbf{F}_a}{\Omega_a \rho_a^2} \cdot \nabla_a W_{ab}(h_a) + \frac{\mathbf{F}_b}{\Omega_a \rho_a^2} \cdot \nabla_a W_{ab}(h_b)\right]$$

$$\mathbf{u} = [\mathbf{v}, e]^T \qquad -\sum_b \frac{m_b}{\bar{\rho}_{ab}} \bar{v}_{\rm sig}(\mathbf{u}_a - \mathbf{u}_b)\hat{\mathbf{r}}_{ab} \cdot \overline{\nabla_a W_{ab}},$$

Similar for SPH, but dissipation does NOT affect advection terms

c.f. Chow & Monaghan (1997), Inutsuka (2002), Cha & Whitworth (2003), Price (2008)

THE KEY IS A GOOD SWITCH

- Use shock detector to turn off shock dissipation where there are no shocks

$$A = \xi \max\left[-\frac{\mathrm{d}}{\mathrm{d}t}(\nabla \cdot \boldsymbol{v}), 0\right]$$

- Nearly undamped linear waves

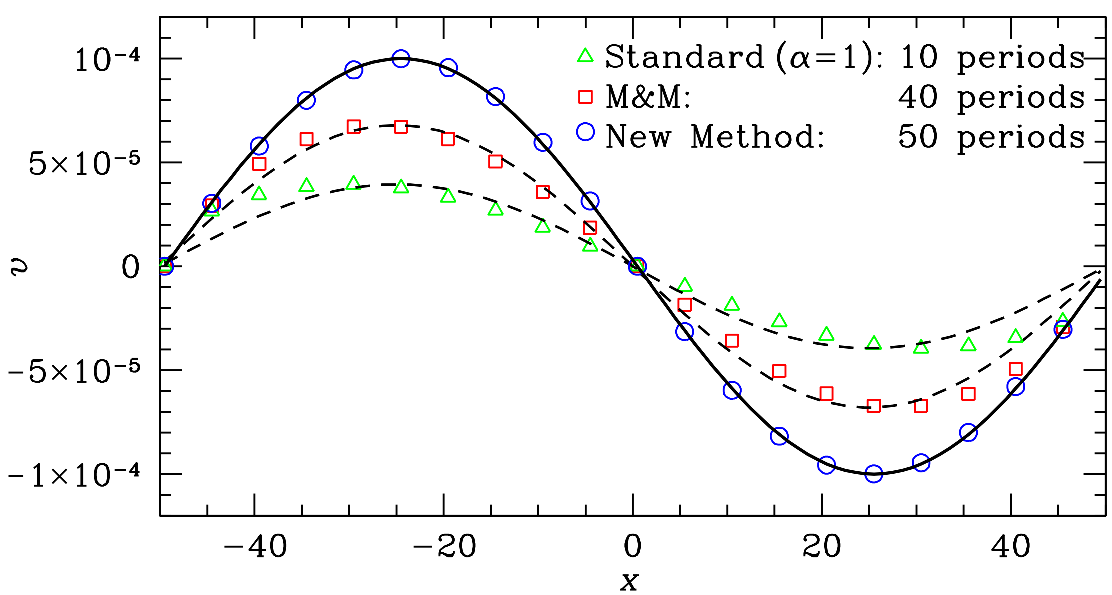


Figure 2. As Fig. 1, but for SPH with standard ($\alpha = 1$) or Morris & Monaghan (1997) artificial viscosity, as well as our new method (only every fifth particle is plotted). Also shown are the undamped wave (*solid*) and lower-amplitude sinusoidals (*dashed*). Only with our method the wave propagates undamped, very much like SPH without any viscosity, as in Fig. 1.

6 *Lee Cullen & Walter Dehnen*

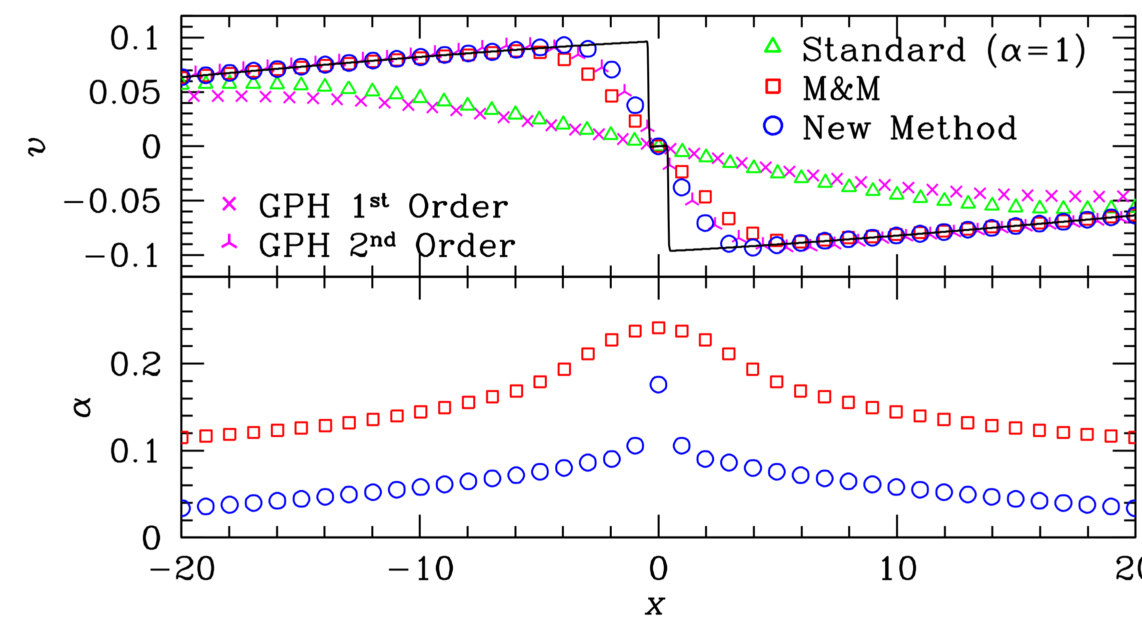


Figure 6. Steepening of a 1D sound wave: velocity and viscosity parameter vs. position for standard SPH, the M&M method, our new scheme, and Godunov particle hydrodynamics of first and second order (GPH, Cha & Whitworth 2003), each using 100 particles per wavelength. The solid curve in the top panel is the solution obtained with a high-resolution grid code.

c.f. Cullen & Dehnen (2010), Price et al. (2018)

MACH 10, SUPERSONIC TURBULENCE: SPH VS GRID

Price & Federrath (2010)
Tricco, Price & Federrath (2016)

SPH

phantomsph.bitbucket.io

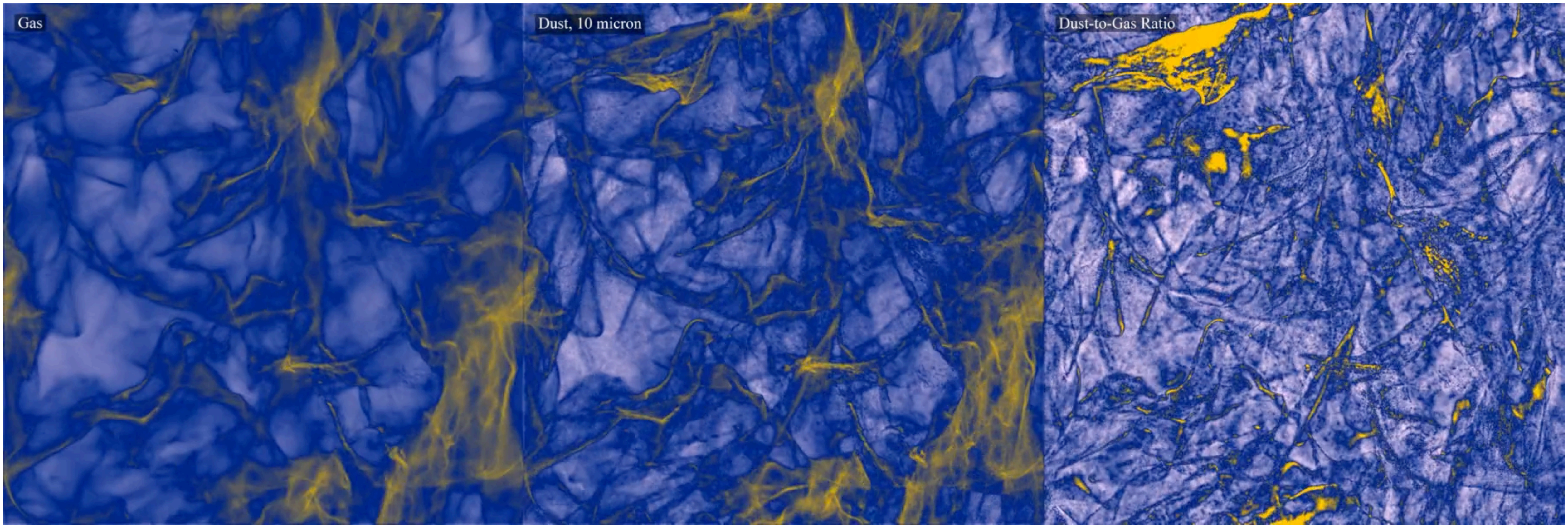
FLASH

flash.uchicago.edu

Main advantage of SPH: resolution follows mass

PARTICLE-LADEN SUPERSONIC TURBULENCE AT MACH 10

Tricco, Price & Laibe (2017), MNRAS 471, L52



Using "one fluid" model for dust-gas mixtures

Laibe & Price (2014a,b,c) MNRAS 440, 2136

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho(\nabla \cdot \boldsymbol{v}),$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P_{\mathrm{g}}}{\rho},$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{1}{\rho}\nabla \cdot (\epsilon t_{\mathrm{s}} \nabla P_{\mathrm{g}}),$$

$$\rho = \rho_{\mathrm{g}} + \rho_{\mathrm{d}}$$

$$\mathbf{v} = \frac{\rho_{\mathrm{g}}\mathbf{v}_{\mathrm{g}} + \rho_{\mathrm{d}}\mathbf{v}_{\mathrm{d}}}{\rho}$$

$$\epsilon = \frac{\rho_{\mathrm{d}}}{\rho}$$

c.f. special sessions on particle-laden flows

MYTH 3:
SPH CAN'T SIMULATE KELVIN-HELMHOLTZ INSTABILITIES

ORIGIN OF THE MYTH

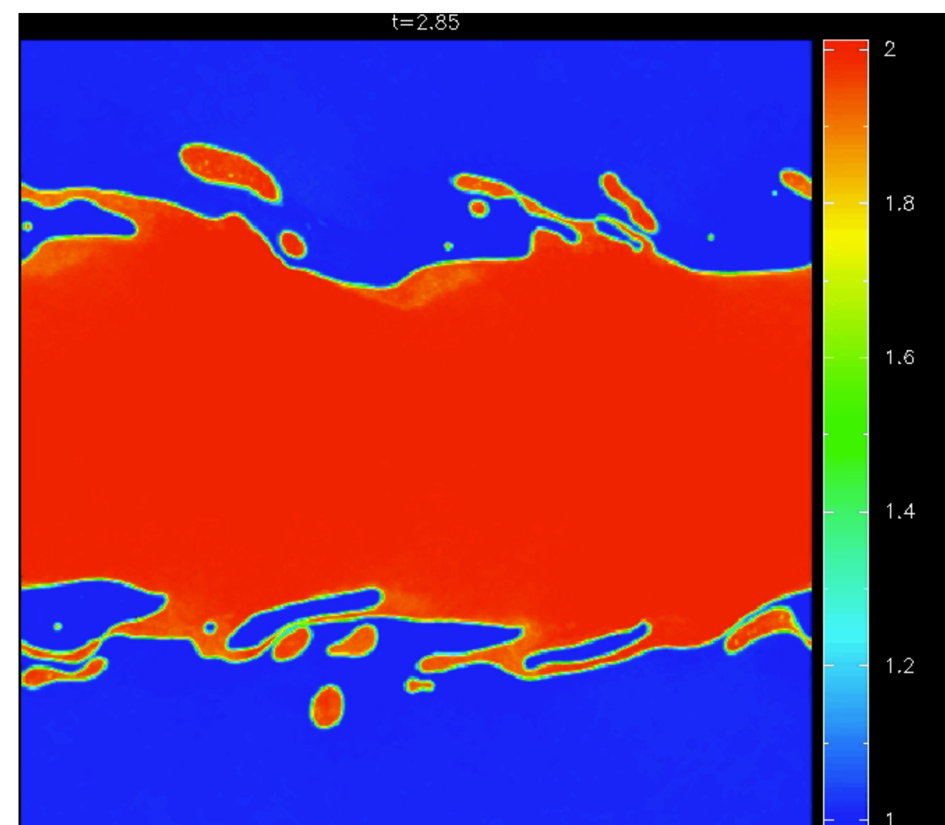
Mon. Not. R. Astron. Soc. 380, 963–978 (2007) doi:10.1111/j.1365-2966.2007.12183.x

Fundamental differences between SPH and grid methods

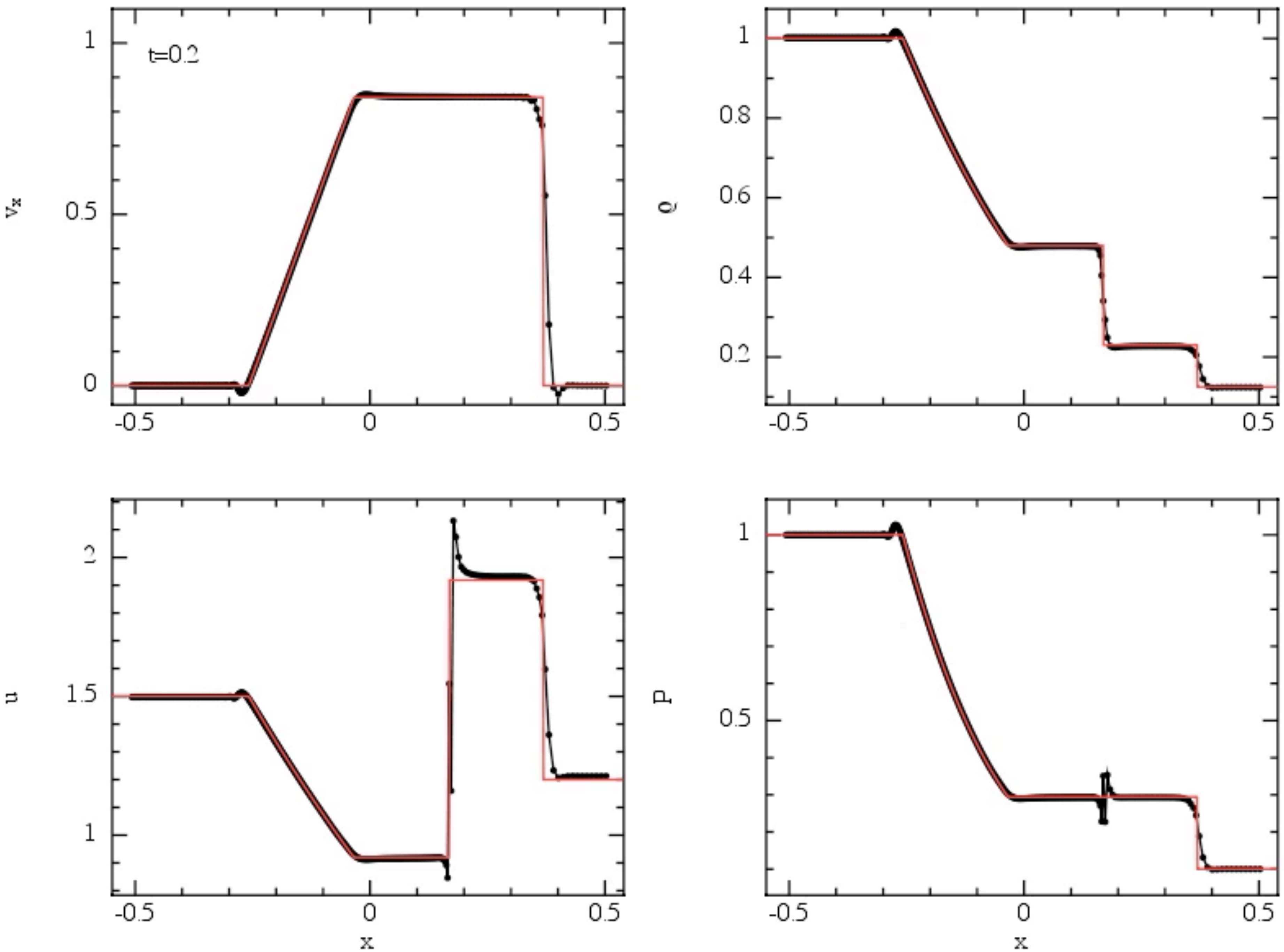
Oscar Agertz,[1★] Ben Moore,[1] Joachim Stadel,[1] Doug Potter,[1] Francesco Miniati,[2] Justin Read,[1] Lucio Mayer,[2] Artur Gawryszczak,[3] Andrey Kravtsov,[4] Åke Nordlund,[5] Frazer Pearce,[6] Vicent Quilis,[7] Douglas Rudd,[4] Volker Springel,[8] James Stone,[9] Elizabeth Tasker,[10] Romain Teyssier,[11] James Wadsley[12] and Rolf Walder[13]

[1]*Institute for Theoretical Physics, University of Zürich, CH-8057 Zürich, Switzerland*
[2]*Department of Physics, Institute für Astronomie, ETH Zürich, CH-8093 Zürich, Switzerland*
[3]*Nicolaus Copernicus Astronomical Centre, Bartycka 18, Warsaw PL-00-716, Poland*
[4]*Department of Astronomy & Astrophysics, The University of Chicago, Chicago, IL 60637, USA*

- Apparent problems with K-H instability in SPH when simulations performed with 2:1 density contrast
- Manifests as numerical "surface tension"



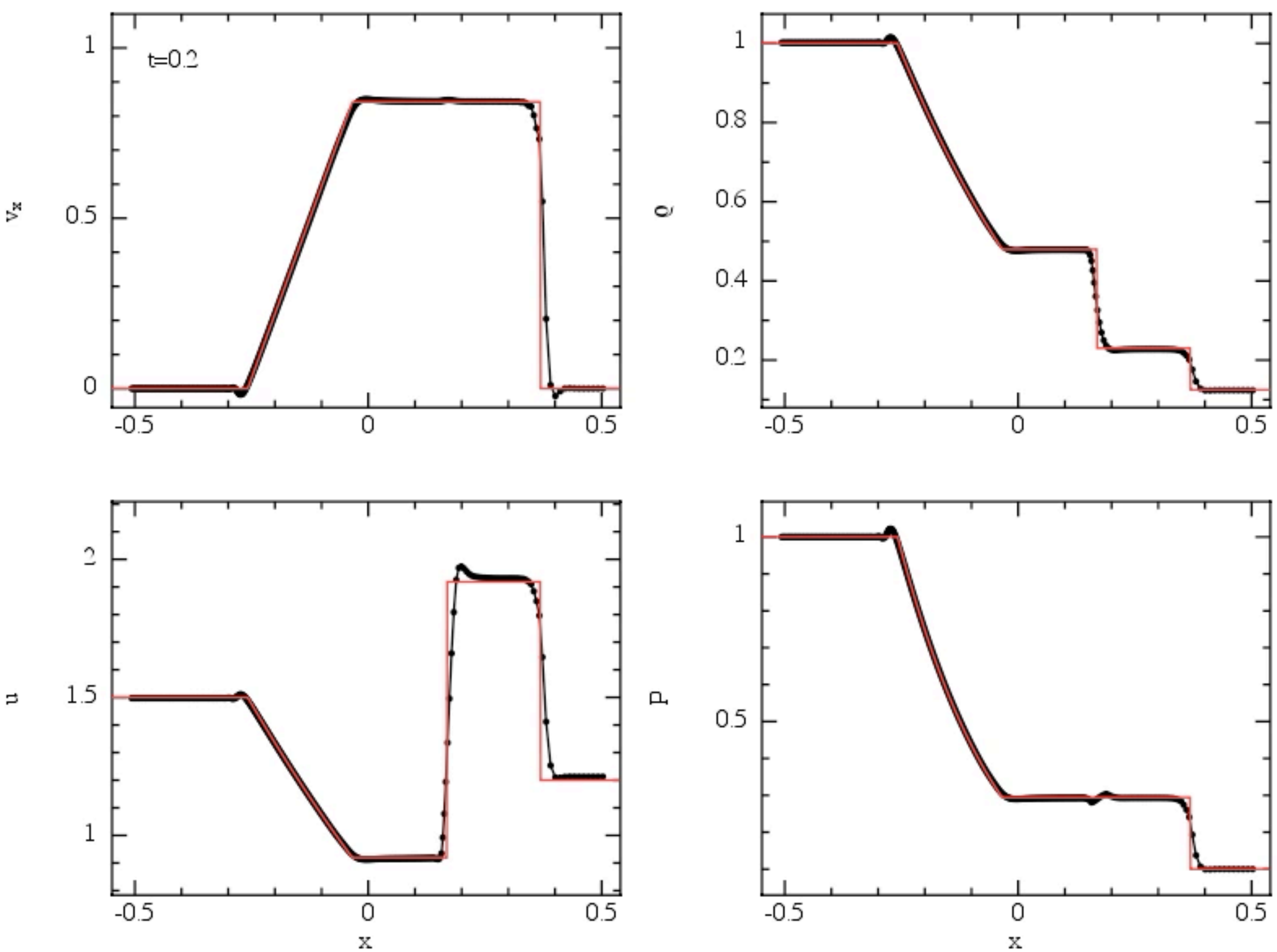
THE RIGHT WAY TO THINK ABOUT IT



1D Sod shock tube with artificial viscosity

- Shock capturing dissipation terms required at discontinuities
- Artificial viscosity applied at shock
- What about the contact discontinuity?

ANALOGY WITH GODUNOV-TYPE SOLVERS

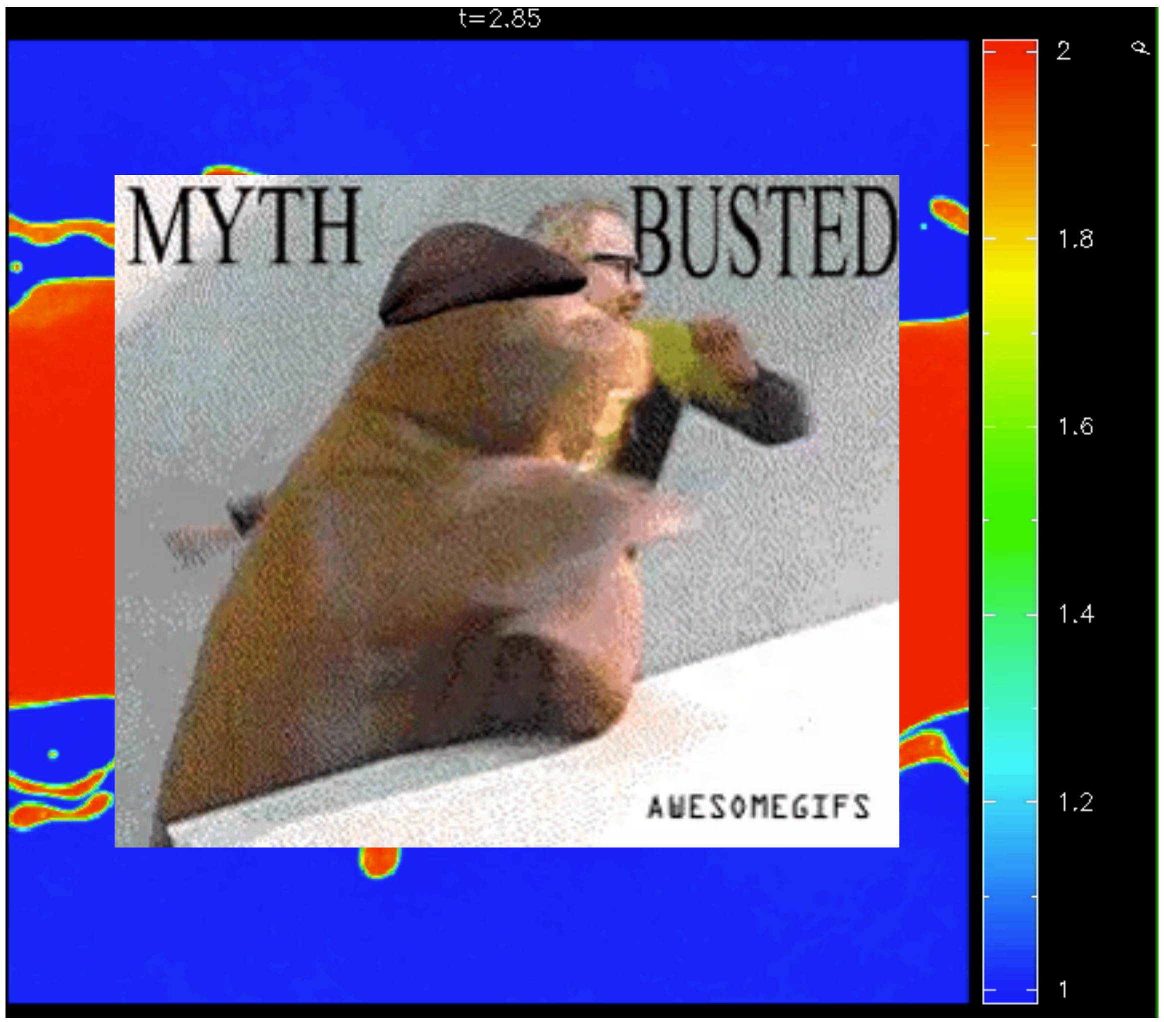


1D Sod shock tube with artificial conductivity

- Godunov-type solvers imply conductivity at the contact discontinuity (Monaghan 1997)
- Use analogous dissipation terms to ensure smooth pressure across discontinuous jumps in density and temperature (Chow & Monaghan 1997, Price 2008)

MUST TREAT DISCONTINUITIES PROPERLY

Price (2008)



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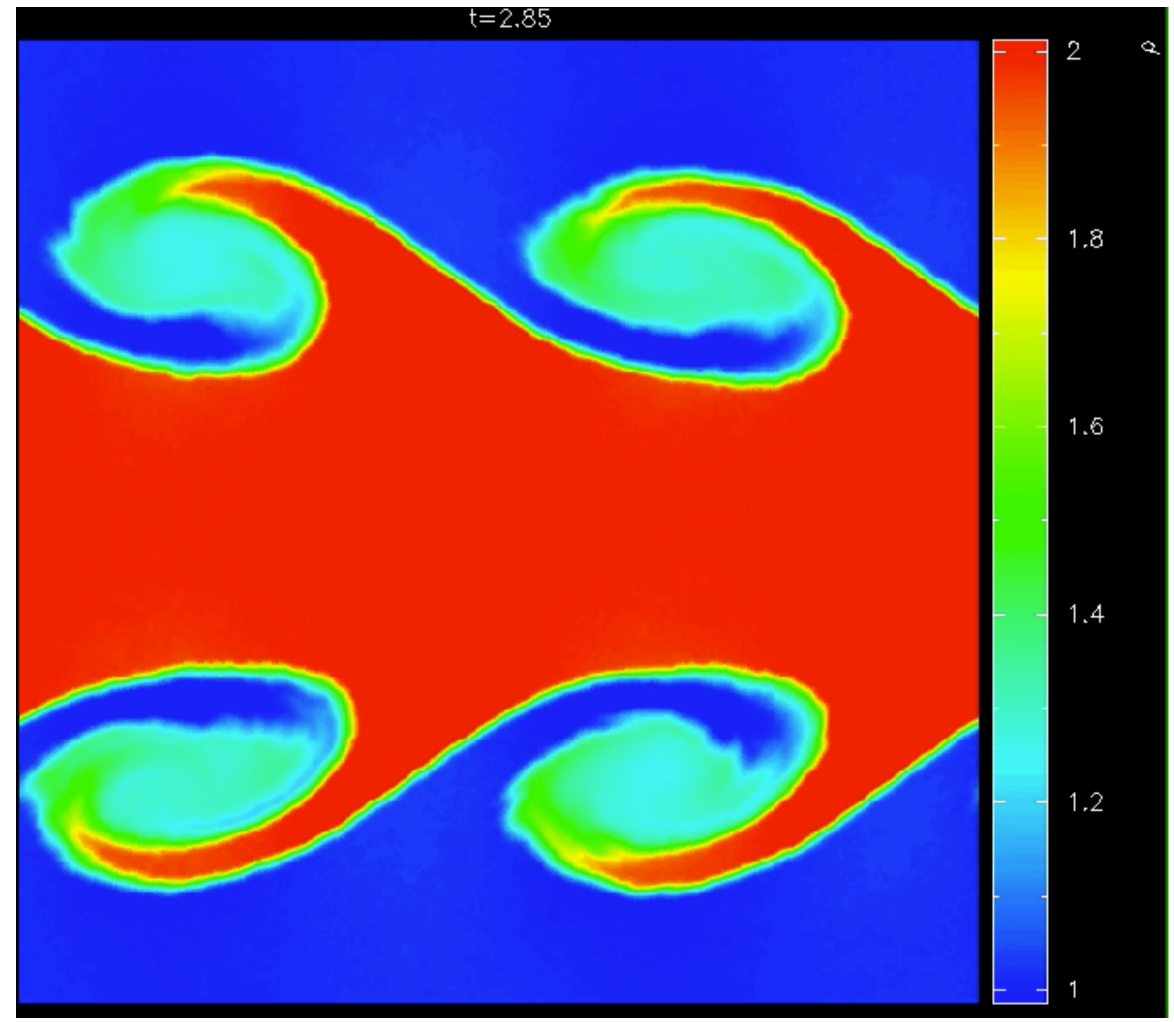
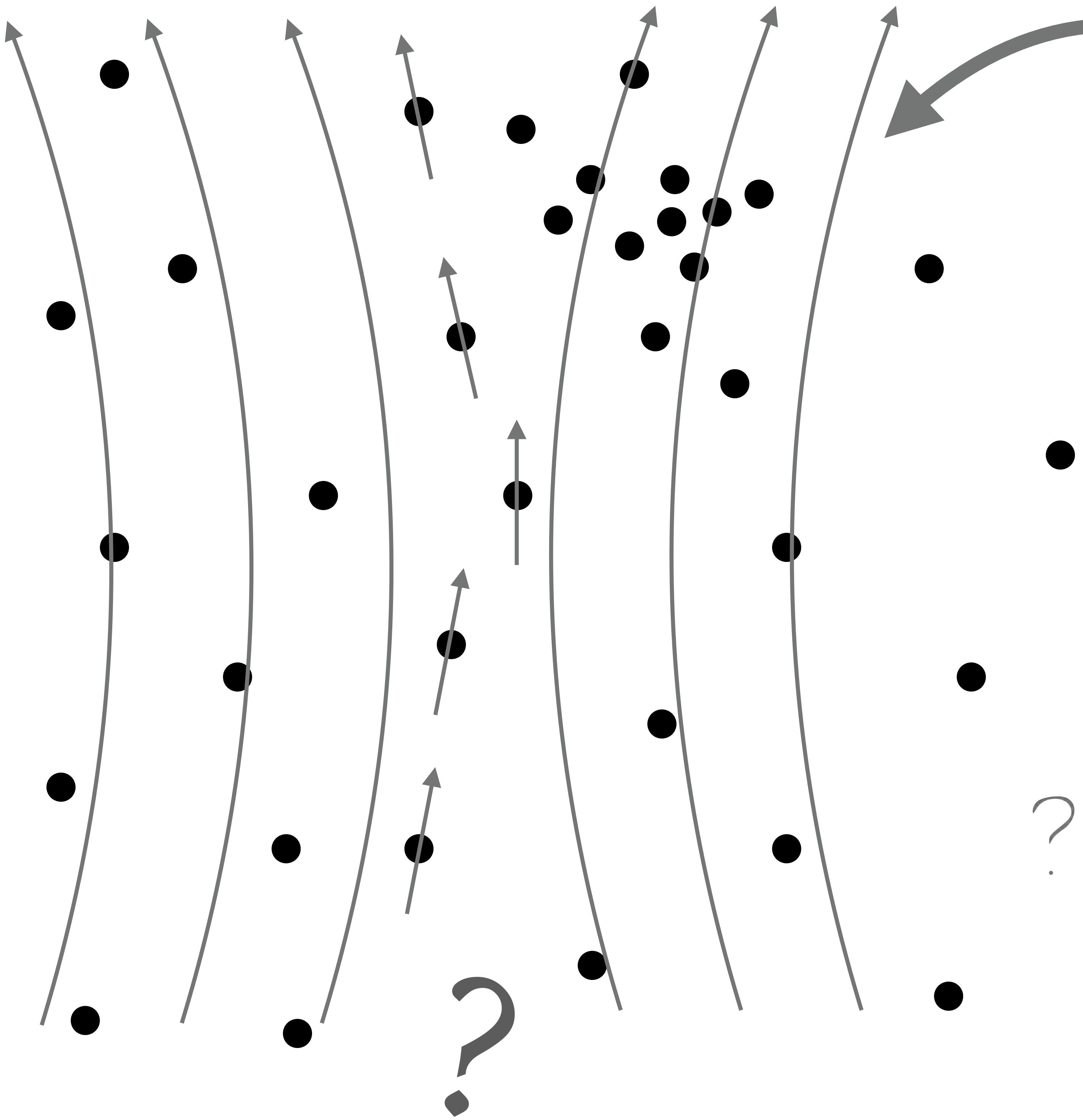


Fixed

This issue has nothing to do with the Kelvin-Helmholtz instability!

MYTH 4: SPH CAN'T DO MAGNETIC FIELDS

ORIGIN OF THE MYTH I: HOW DO YOU EVEN DO THAT?



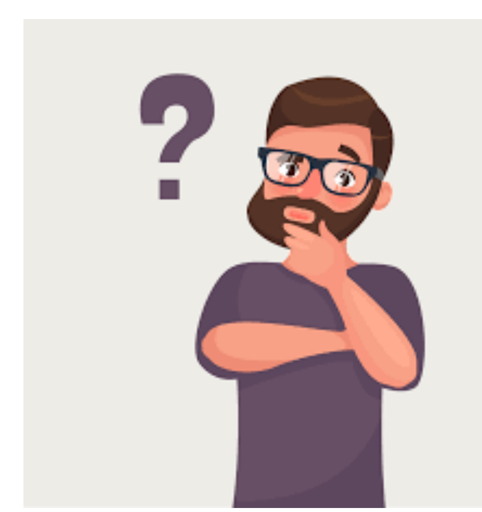
$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho = -\rho(\nabla\cdot\boldsymbol{v})$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho}$$

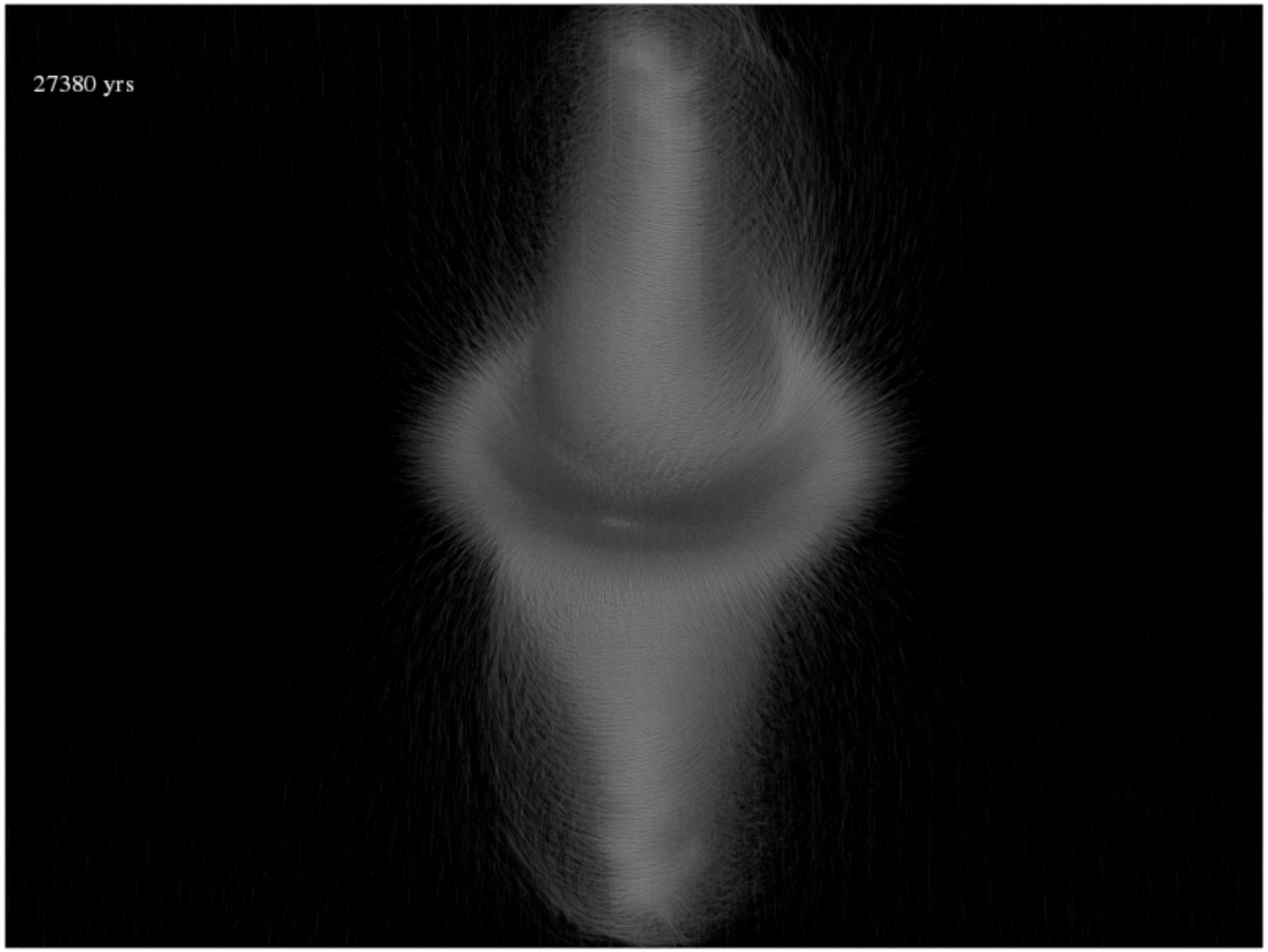
$$\frac{\partial u}{\partial t} + (\boldsymbol{v}\cdot\nabla)u = -\frac{P}{\rho}(\nabla\cdot\boldsymbol{v})$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

$$\nabla\cdot\mathbf{B} = 0$$



TRUTH: SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS



Magnetic jet launched from gravitational collapse of a rotating, magnetised cloud
Price, Tricco & Bate (2012)

- Use the Lagrangian!
- Obtain discretised MHD equations
- Better to think in terms of partial differential equations, not particles

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\nabla\right)\boldsymbol{v}$$

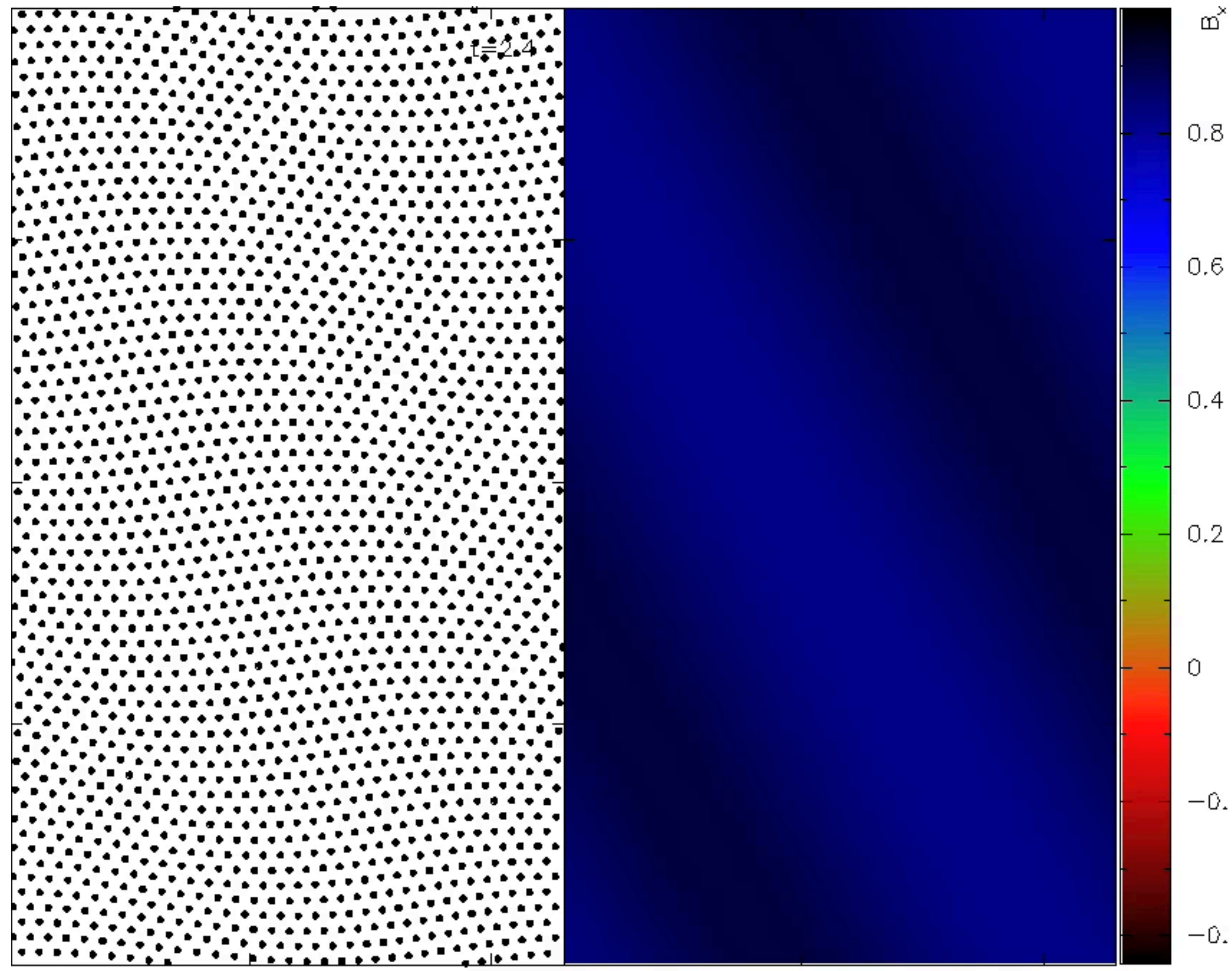
$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\sum_b m_b \left[\left(\frac{S^{ij}}{\rho^2}\right)_a + \left(\frac{S^{ij}}{\rho^2}\right)_b\right]\nabla_a^j W_{ab}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}_a}{\rho_a}\right) = -\sum_b m_b(\boldsymbol{v}_a - \boldsymbol{v}_b)\frac{\boldsymbol{B}_a}{\rho_a^2}\cdot\nabla_a W_{ab}$$

ORIGIN OF THE MYTH II: THE TENSILE INSTABILITY IN SPMHD

Phillips & Monaghan (1985), Børve, Omang & Trulsen (2001), Price & Monaghan (2004a,b), Price (2012)

- Particles attract each other along magnetic field lines when stress tensor is negative (tension forces)
- Fixed by subtracting spurious $\mathbf{B}(\nabla\cdot\mathbf{B})$ term from the numerical force (Børve et al. 2001)



2D circularly polarised Alfvén wave

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} + \frac{\mathbf{J}\times\mathbf{B}}{\rho} + \cancel{\frac{\mathbf{B}(\nabla\cdot\mathbf{B})}{\mu_0\rho}}$$

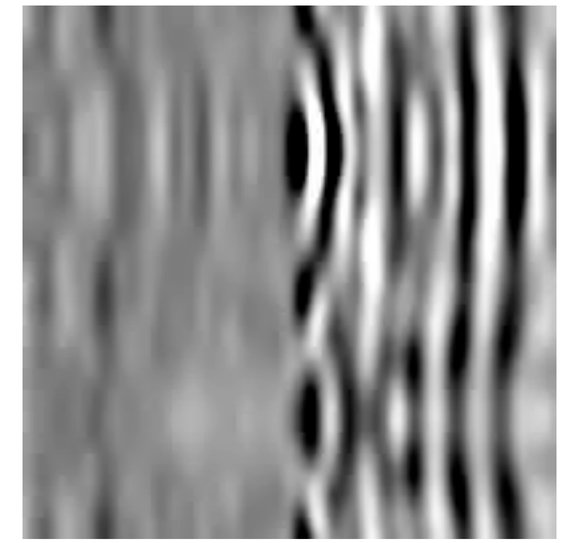
$\nabla \cdot \mathbf{B} = 0$ IN SPMHD

Price & Monaghan (2005), Tricco & Price (2012), Tricco, Price & Bate (2016)

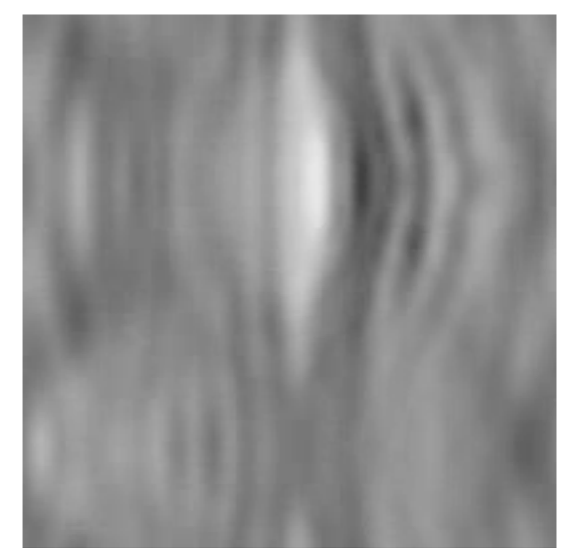
- Constrained hyperbolic/parabolic divergence cleaning based on original scheme by Dedner et al. (2002)
- Formulated so that change in magnetic energy is negative definite

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{clean} = -\nabla\psi$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -c^2(\nabla\cdot\mathbf{B}) - \frac{\psi}{\tau}$$

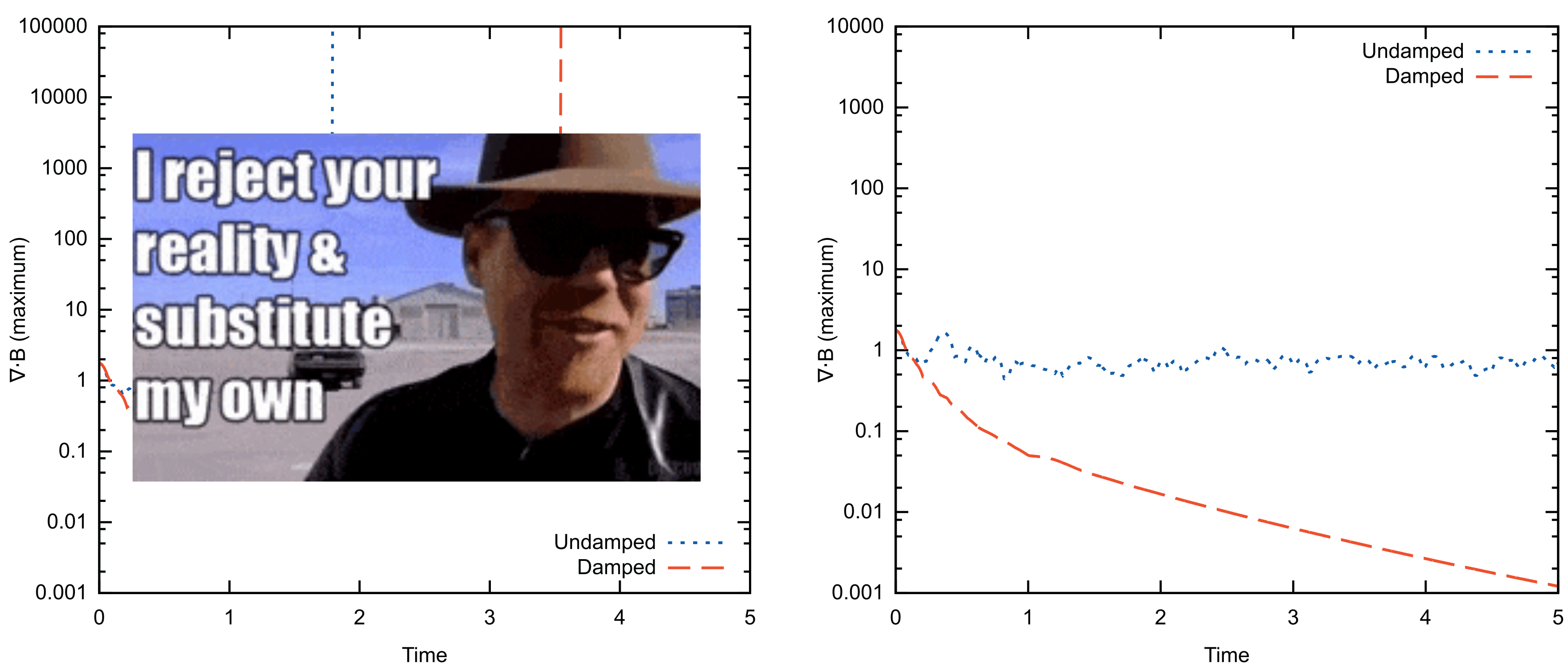


Unconstrained div B cleaning



Constrained div B cleaning

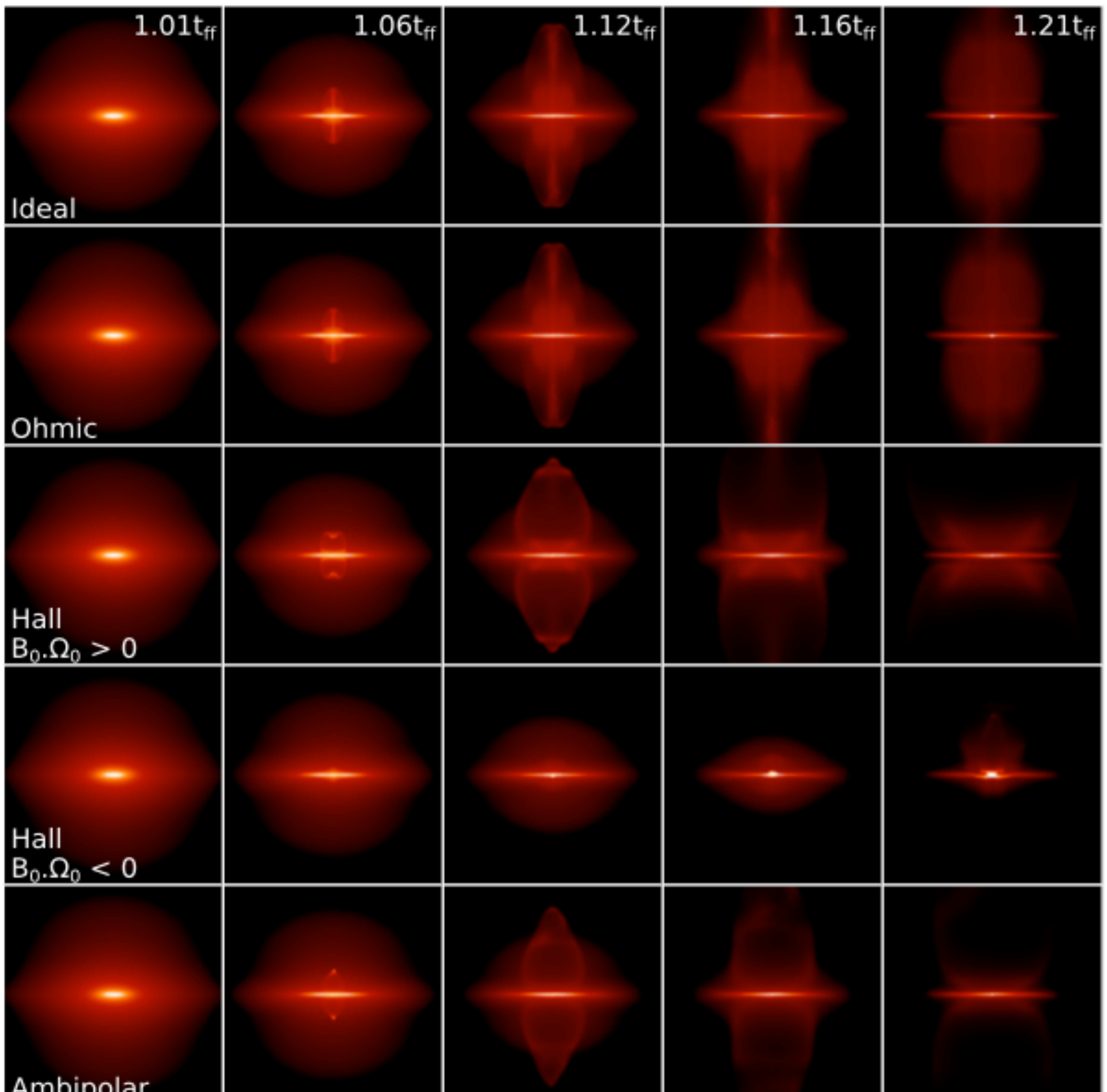
7224 T.S. Tricco, D.J. Price / Journal of Computational Physics 231 (2012) 7214–7236



NON-IDEAL MAGNETOHYDRODYNAMICS

Wurster, Price & Ayliffe (2014),
Wurster, Price & Bate (2016)
Wurster et al. (2017,2018,2019)

- Partially ionised plasmas (ions, electrons, neutrals)



Ohmic resistivity · *Hall effect* · *Ion-neutral drift*

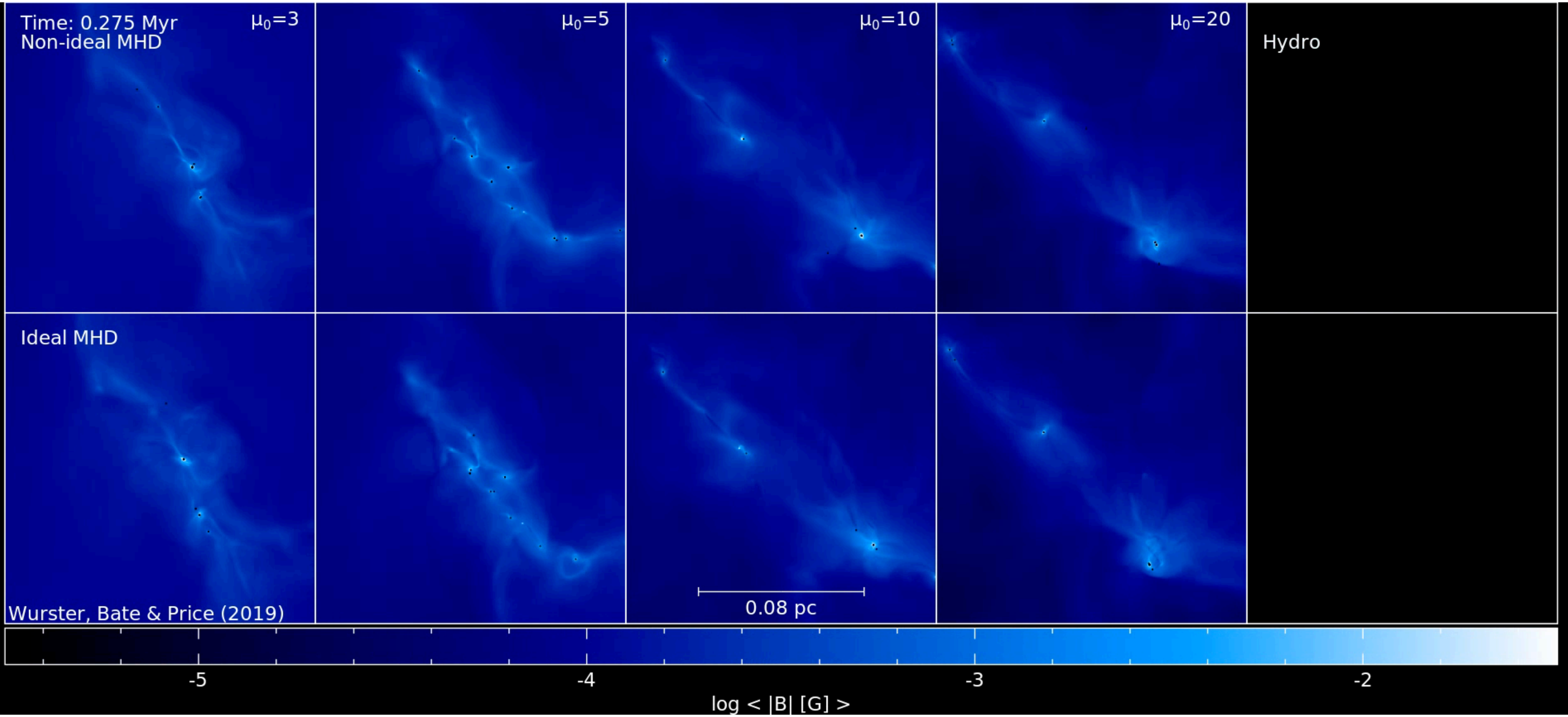
$$\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_{\mathrm{NI}} = -\nabla\times\underbrace{\left[\frac{\boldsymbol{J}}{\sigma} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{en_e} - \frac{(\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}}{\gamma_{\mathrm{AD}}\rho_i}\right]}_{\boldsymbol{D}}$$

$$\boldsymbol{J}_a = \frac{1}{\Omega_a\rho_a}\sum_b m_b(\boldsymbol{B}_a - \boldsymbol{B}_b)\times\nabla_a W_{ab}(h_a)$$

$$\left(\frac{\mathrm{d}\boldsymbol{B}_a}{\mathrm{d}t}\right)_{\mathrm{NI}} = \rho_a\sum_b m_b\left[\frac{\boldsymbol{D}_a}{\Omega_a\rho_a^2}\times\nabla_a W_{ab}(h_a) + \frac{\boldsymbol{D}_b}{\Omega_b\rho_b^2}\times\nabla_a W_{ab}(h_b)\right]$$

STAR CLUSTER FORMATION WITH NON-IDEAL MHD

Price & Bate (2008, 2009), Wurster, Price & Bate (2019)



- Solves issue of how to form circumstellar discs and binary stars despite interstellar magnetic fields
- That is, turbulence + non-ideal MHD solves the "magnetic braking catastrophe"

SUMMARY: THINGS YOU MIGHT HAVE HEARD ABOUT SMOOTHED PARTICLE HYDRODYNAMICS

SUMMARY

- SPH offers powerful solutions to problems that are difficult/ impossible with other methods
- Main strength is in simulating flow with no preferred geometry or with large change in density
- Just needs thought sometimes