NON-IDEAL SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS

Can non-ideal MHD solve the magnetic braking catastrophe?

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ASTRONUM-2016, 6th-10th June, Monterey, California
The “Magnetic Braking Catastrophe” in Protostellar Disc Formation

Allen et al. (2003), Galli et al. (2006), Price & Bate (2007), Mellon & Li (2008), Hennebelle & Fromang (2008), Commerçon et al. (2010), Krasnopolsky et al. (2010) and many others

- Assumes ideal MHD (not true)
- Our previous work used Euler potentials to solve div \( B = 0 \)
  \[
  B = \nabla \alpha \times \nabla \beta
  \]
  (no outflows)
- No turbulence

See Seifried et al. (2012), Joos et al. (2013) and others

Price & Bate (2007)
SMOOTHED PARTICLE HYDRODYNAMICS

- Lagrangian/Hamiltonian particle method for solving equations of fluid dynamics
- Symmetry-preserving, maintain exact conservation of linear and angular momentum, energy, entropy and circulation in spatial discretisation
- Zero intrinsic dissipation
- Adaptive — resolution follows mass not volume
- No geometry restrictions, easily handle free surfaces
SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS

Euler-Lagrange equations give discrete form of:

\[ L = \int \left( \frac{1}{2} \rho v^2 - \rho u - \frac{1}{2\mu_0} B^2 \right) dV \]

\[ L = \sum_a m_a \left( \frac{1}{2} v_a^2 - u_a - \frac{B_a^2}{2\mu_0 \rho_a} \right) \]

Dissipationless: Must add dissipation terms to handle shocks and discontinuities

Divergence advection test from Dedner et al. (2002)

These equations are equivalent to the 8-wave formulation of Powell et al. 1994
HYPERBOLIC/PARABOLIC DIVERGENCE CLEANING

\[
\frac{\partial B}{\partial t} \equiv = \nabla \psi
\]

\[
\frac{\partial \psi}{\partial t} = -c_h^2 (\nabla \cdot \mathbf{B}) - \frac{\sigma^2 c_h}{c_p^2} \psi
\]

Use dimensionless parameter
Price & Monaghan (2005); Mignone & Tzeferacos (2010)

Critical damping on resolution length
Price & Monaghan (2005); Mignone & Tzeferacos (2010)

\[
\frac{1}{c_h^2} \frac{\partial^2 (\nabla \cdot \mathbf{B})}{\partial t^2} + \nabla^2 (\nabla \cdot \mathbf{B}) + \frac{1}{\lambda c_h} \frac{\partial (\nabla \cdot \mathbf{B})}{\partial t} = 0
\]

Wavelength of critical damping

Hyperbolic term only
WHEN CLEANING ATTACKS

5.3. Static cleaning test: free boundaries

A further variant of the divergence advection test we consider replaces the periodic boundaries by a free boundary, since many applications of SPMHD involve free boundaries (e.g. the merger of two neutron stars [36], or studies of galaxy interactions [15,16]).

5.3.1. Setup

The setup is identical to the divergence advection problem (Section 5.1) with $r_0 = \frac{1}{\sqrt{8}}$, except that the domain is a circular area of fluid with $q = 1$ for $r < 1$ and $q = 0$ (no particles) for $r > 1$, set up using a total of 1976 particles placed on a cubic lattice. The divergence perturbation is introduced at the centre of the circle, and the velocity field is set to zero. Rather than impose an external confining potential, we solve only Eqs. (16) and (17) without the full MHD equations, as in Section 5.2.

5.3.2. Results

Fig. 6 shows the results of purely hyperbolic cleaning ($r = 0$) for this case. As in Fig. 4, the top row shows the unconstrained and non-conservative difference/difference formulation, while the bottom row shows results using the conservative difference/symmetric combination. Similar results are also found in this case, with divergence errors piling up at the free boundary in the non-conservative formulation leading to numerical instability, but our constrained formulation remaining stable.

5.4. 2D Blast wave in a magnetised medium

We now turn to tests that are more representative of the dynamics encountered in typical astrophysical simulations, beginning with a blast wave expanding in a magnetised medium. In this case the initial magnetic field is divergence-free, meaning that the only divergence errors are those created by numerical errors during the course of a simulation – rather than the artificial errors we have induced in the previous tests. Based on the results from the previous tests, in this and subsequent tests we apply cleaning only using constrained, energy-conserving formulations – that is, with conjugate operators $r/C_1$ and $r/w$. We use this problem to examine the effectiveness of the divergence cleaning in the presence of strong shocks, as well as to investigate whether cleaning should be performed using the difference or symmetric $r/C_1$ operator. As with the divergence advection test, a key goal is to find optimal values for the damping parameter $r$.

5.4.1. Setup

The implementation of the blast wave follows that of Londrillo and Del Zanna [18]. The domain is a unit square with periodic boundaries, set up with $512^2$ particles on a hexagonal lattice with $q = 1$. The fluid is at rest with magnetic field $B_x = 10$. The pressure of the fluid is set to $P = 1$, with $c_{\text{iso}} = 1/4$, except a region in the centre of radius $0.125$ has its pressure increased by a factor of 100 by increasing its thermal energy. An adiabatic equation of state is used.

Divergence advection test (Dedner et al. 2002) with 10:1 jump in density
“CONSTRAINED” HYPERBOLIC/PARABOLIC DIVERGENCE CLEANING

Tricco & Price (2012); Tricco, Price & Bate (2016), submitted to JCP

- Define energy associated with cleaning field

\[ E = \int \left[ \frac{1}{2} \frac{B^2}{\mu_0} + \frac{1}{2} \frac{\psi^2}{\mu_0 c_h^2} \right] dV \]

- Enforce energy conservation in hyperbolic terms

\[ \frac{dE}{dt} = \int \left[ \frac{B}{\mu_0} \cdot \left( \frac{dB}{dt} \right) + \frac{\psi}{\mu_0 c_h^2} \frac{d\psi}{dt} - \frac{\psi^2}{2\mu_0 \rho c_h^2} \frac{d\rho}{dt} - \frac{\psi^2}{\mu_0 c_h^3} \frac{dc_h}{dt} \right] dV = 0 \]

- Requires particular choice of operators here

\[ \frac{dB}{dt} = -\nabla \psi \]

\[ \frac{d\psi}{dt} = -c_h^2 (\nabla \cdot B) - \sigma c_h \frac{\rho}{h} \psi - \frac{1}{2} \psi (\nabla \cdot \mathbf{v}) \]

- Can enforce exact energy conservation in SPH discretisation
5.3. Static cleaning test: free boundaries

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Parabolic term is negative definite!
WHAT IF THE CLEANING SPEED VARIES?

Tricco, Price & Bate (2016), submitted to JCP

\[ \frac{dB}{dt} = -\nabla \psi \]

\[ \frac{d}{dt} \left( \frac{\psi}{\varsigma_h} \right) = -\varsigma_h (\nabla \cdot B) - \frac{\psi}{2\varsigma_h} (\nabla \cdot v) - \frac{\sigma \psi}{\varsigma \varsigma_h} \]

Hyperbolic terms conserve energy even with variable wave speed!

Non-conservative method  Conservative method
SHOCK DISSIPATION SWITCHES

- Cullen & Dehnen (2010) switch for shock viscosity

\[ A = \max \left[-\frac{d}{dt}(\nabla \cdot \mathbf{v}), 0\right] \quad \alpha_{loc} = \min \left(\frac{10h^2A}{c_s^2 + h^2A}, 1\right) \]

- Tricco & Price (2013) switch for resistivity

\[ \alpha^B = \min \left(\frac{h|\nabla \mathbf{B}|}{|\mathbf{B}|}, 1\right) \]

- Revised further in Phantom - 2nd order artificial resistivity, vanishes when \( v=\text{const} \)

**Figure 3.** Shocktube test 5A from RJ95 performed in 2D with left state \((\rho, P, v_x, v_y, B_x) = (1, 1, 0, 0, 1)\) and right state \((\rho, P, v_x, v_y, B_x) = (0.125, 0.1, 0, 0, -1)\) with \(B_y = 0.75\) at \(t = 0.1\). Black circles represent the particles and the red line represents the solution obtained with the ATHENA code using \(10^4\) grid cells.
PHANTOM SPMHD CODE

Performed with all dissipation, shock capturing and divergence cleaning turned on

Price et al. (2016) in prep.

Advection of current loop (Gardiner & Stone 2005, 2008)

Convergence on circularly polarised Alfvén wave with ALL dissipation turned on
JETS FROM THE FIRST CORE

*Price, Tricco & Bate (2012); see also Machida et al. (2008)*
PROTOTERRAL JETS: SECOND COLLAPSE

Bate, Tricco & Price (2014)

Performed with radiation magnetohydrodynamics (grey FLD: Whitehouse & Bate 2004a,b; Whitehouse, Bate & Monaghan 2006)
MAGNETICALLY LAUNCHED OUTFLOWS

First core (100 x 100 au)

Second (protostellar) core (10 x 10 au)

Bate, Tricco and Price (2013)
SMALL SCALE DYNAMO: FLASH VS PHANTOM

Tricco, Price & Federrath (2016)

**Flash**

<table>
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**Phantom**

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Figure 3. z-column integrated \( \langle B \rangle \) and defined \( \langle \langle B \rangle \rangle = \frac{\int B \, dz}{\int dz} \) for Flash (top) and Phantom (bottom) at resolutions of \( 256 \times 256 \) for \( t/t_{c} = 2, 4, 6, 8 \). The density field has similar structure in both codes at early times, but diverge at late times due to the non-deterministic behaviour of the turbulence. The magnetic field is strongest in the densest regions, while the mean magnetic field strength throughout the domain increases with time.

Analytic studies of the exponential growth rate of the small-scale dynamo have shown that for \( P_{m} \sim 1 \), the growth rate scales with \( R_{m}^{1/2} \), while for \( P_{m} \ll 1 \), it scales with \( R_{e}^{1/2} \) (Schober et al. 2012a; Bovino et al. 2013). Theoretical predictions of the growth rate for \( P_{m} \ll 1 \), which is the Prandtl number regime for numerical codes in the absence of explicit dissipation terms, are more uncertain. The growth rate in the transition region between \( 0 < P_{m} < 10 \) was probed by Federrath et al. (2014) using Flash simulations with explicit viscous and resistive dissipation. They found that the magnetic energy growth rate for \( P_{m} \ll 1 \) exhibited a steep dependence on \( P_{m} \) and only agreed qualitatively with the analytical expectations of Schober et al. (2012a) and Bovino et al. (2013). Conversely, the growth rate for \( P_{m} \gg 1 \) quantitatively agreed with analytical expectations, with, by comparison, relatively little variation with respect to \( P_{m} \).
Strong coupling approximation: $\rho \approx \rho_n$, $\rho_i \ll \rho$

$$\frac{dB}{dt} = -B(\nabla \cdot v) + (B \cdot \nabla)v$$

$$-\nabla \times \left[ \eta_O J + \eta_H J \times \hat{B} - \eta_A (J \times \hat{B}) \times \hat{B} \right]$$

- Spatial discretisation exactly conserves energy
- Guaranteed positive definite contribution to entropy

**Whistler/Ion-cyclotron modes**

**C-shock**

**Damped Alfvén wave**

**Standing C-shock**

Figure C1. Dispersion relation for the left- and right-circularly polarised wave, corresponding to $\gamma_{AD} < 0$ and $\gamma > 0$, respectively. The solid circles are the numerically calculated phase velocities.

Tests:
- Mac-Low et al. (1995)
- Choi et al. (2009)
- Falle (2003)

Figure C2. The analytical (solid line) and numerical (crosses) results for the isothermal standing shock. The initial conditions are given in the text. At any given position, the analytical and numerical solutions agree within 3 per cent.
CONDUCTIVITIES


- Solve cosmic ray ionisation/recombination chemical network for grains, ions and neutrals
- Currently assume single grain species 0.1µm
- Gives number density of electrons, ions and grains
- Compute Ohmic, Ambipolar and Hall coefficients at given density, temperature

NICIL Code: Wurster (2016), submitted to PASA
Can non-ideal MHD save discs?

Figure 2. Face-on gas column density using ideal MHD. The initial rotation is counterclockwise and the initial magnetic field is directed out of the page (i.e. $B_0 \cdot \Omega_0 > 0$). Each model is initialised with $\sim 3 \times 10^5$ particles within the sphere. From left to right, the columns represent snapshots at a given time (in units of the free-fall time, $t_{ff} = 2.4 \times 10^4$ yr). The rows represent models with different initial magnetic field strengths given in terms of $\mu_0$ (i.e. the initial mass-to-flux ratio normalised to the critical mass-to-flux ratio). The top row has no initial magnetic field and the bottom row has the strongest magnetic field (i.e. increasing magnetic field strength corresponds to a decreasing value of $\mu_0$). The white circles represent the sink particle with the radius of the circle representing the accretion radius of the sink particle. Each frame is 300 AU. The discs grow in size and mass with time. At any given time, the models with stronger magnetic fields have smaller and less massive discs than the models with the weaker initial magnetic field. The hydrodynamic model yields the largest and most massive disc in our entire suite of simulations.

In agreement with (e.g.) Allen et al. (2003), PB07, Mellon & Li (2008), and Hennebelle & Fromang (2008), this demonstrates the magnetic braking catastrophe.

4.1.1 Resolution

Fig. 4 shows a comparison of the discs formed at resolutions of $\sim 3 \times 10^5$ particles in the collapsing sphere (top row) and $\sim 10^6$ particles (bottom row) using $\mu_0 = 7.5$. This magnetic field strength was used so that disc characteristics could be compared. The $\sim 10^6$ particle model took $\sim 3.5$ times longer to run, which is reasonable given the increase in resolution.

The two resolutions follow the same general trend, with large discs forming. For $1.1 \leq t/t_{ff} \leq 1.21$, the star + disc mass, disc mass and disc radius typically differ by less than 20 per cent. Thus, these results are relatively robust to the resolution increase presented here.

Wurster, Price & Bate (2016)
NON-IDEAL MHD: ALIGNED INITIAL FIELD

\[
\mu_0 = 10 \\
B_0 \cdot \Omega_0 > 0 \\
1.01t_{ff} \\
1.06t_{ff} \\
1.12t_{ff} \\
1.16t_{ff} \\
1.21t_{ff} \\
\]

\[
\mu_0 = 7.5 \\
B_0 \cdot \Omega_0 < 0 \\
1.01t_{ff} \\
1.06t_{ff} \\
1.12t_{ff} \\
1.16t_{ff} \\
1.21t_{ff} \\
\]

\[
\mu_0 = 5 \\
B_0 \cdot \Omega_0 < 0 \\
1.01t_{ff} \\
1.06t_{ff} \\
1.12t_{ff} \\
1.16t_{ff} \\
1.21t_{ff} \\
\]

Figure 6. Face-on gas column density, as in Fig. 2 but for non-ideal MHD including the effect of Ohmic resistivity, the Hall effect and ambipolar diffusion. The top plot has the magnetic field initialised with \( B_0 \cdot \Omega_0 > 0 \), and the bottom plot with \( B_0 \cdot \Omega_0 < 0 \). Compare to ideal MHD, discs sizes are smaller for \( B_0 \cdot \Omega_0 > 0 \), but larger for \( B_0 \cdot \Omega_0 < 0 \). This indicates that the Hall effect is the most important non-ideal MHD term for disc formation.

Thus, to save computational costs of the models with weaker magnetic fields, the bottom panel in Fig. 6 shows the lower resolution models. For consistency, we thus present all the models in Sections 4.1 and 4.3 at the lower resolution. The remainder of this study is performed using the \( \sim 10^6 \) particle models, with the exception of our discussion of the cosmic ionisation rate. Note that the non-ideal MHD model with \( \mu_0 = 5 \) and \( B_0 \cdot \Omega_0 < 0 \) has only evolved to \( t \approx 1.18 t_{ff} \).
NON-IDEAL MHD: ANTI-ALIGNED INITIAL FIELD

Wurster, Price & Bate (2016)

\[ B_0 \cdot \Omega_0 < 0 \quad 1.01t_{ff} \]

\[ \mu_0 = 10 \]

\[ B_0 \cdot \Omega_0 > 0 \quad 1.06t_{ff} \]

\[ \mu_0 = 7.5 \]

\[ \mu_0 = 5 \]

\[ 1.12t_{ff} \]

\[ 1.16t_{ff} \]

\[ 1.21t_{ff} \]

100 AU

see also Tsukamoto et al. (2015)
Can non-ideal MHD save discs?

Figure 5. Edge-on gas column density using ideal MHD and zoomed out to $(3000 \text{ AU})^2$ and using a density range shifted down by a factor of ten to visualize the full extent of the outflows launched shortly after the collapse ($t \approx 1.02 t_{\text{ff}}$) in the magnetic models. The models with strongermagnetic fields have faster and more collimated outflows.

Increasing the resolution for the non-ideal MHD models has a minimal effect on the disc over the time of analysis ($t \leq 1.15 t_{\text{ff}}$; i.e., the life of the disc in the $\sim 3 \times 10^5$ model).

By increasing the resolution, the mass of the star+disc system decreases only by $\sim 5$ per cent. The high-resolution disc is more massive.
OUTFLOWS: NON-IDEAL MHD / ALIGNED INITIAL FIELD

Can non-ideal MHD save discs?

The top plot has the magnetic field initialised with $B_0 \cdot \Omega_0 > 0$, and the bottom plot with $B_0 \cdot \Omega_0 < 0$. Outflows for the calculations that form small discs are shown in Fig. 9.

The most interesting aspect is that outflows appear to anticorrelate with the presence of discs. That is, outflows carry away angular momentum, which hinders the formation of discs. This is counterintuitive since one would normally expect outflows to be launched from a disc. Here, as in Price et al. (2012), the outflows are powered by a rotating, sub-Keplerian flow, and carry away sufficient angular momentum to prevent the formation of a Keplerian disc. Non-ideal MHD, in general, appears to suppress the formation of outflows. This is quantified further in Section 4.6, where we discuss the influence of individual non-ideal MHD terms.

Wurster, Price & Bate (2016)
Outflows are anti-correlated with disc formation!
WHICH NON-IDEAL EFFECTS ARE IMPORTANT?

<table>
<thead>
<tr>
<th>Ideal</th>
<th>1.01t_{ff}</th>
<th>1.06t_{ff}</th>
<th>1.12t_{ff}</th>
<th>1.16t_{ff}</th>
<th>(\mu_0=5)</th>
<th>1.12t_{ff}</th>
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<td>Hall (B_0,\Omega_0 &lt; 0)</td>
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<td>Ambipolar</td>
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- Hall effect is dominant during disc formation
- Produces counter-rotating envelope when B and rotation are misaligned
- Maybe why half of all stars have planets?
WHICH NON–IDEAL EFFECTS ARE IMPORTANT?

Figure 12. Shortly after the formation of the disc, the Hall effect is included in the five models in this figure, a sink particle is formed at the limited information from their respective processes diffuse the magnetic field, and the maximum plasma beta of the sink particle, the maximum plasma beta of the Hall-only model has a maximal magnetic field strength and index that the diffusion terms are important in the disc. Thus, at this magnetic field strength, there are no discs with a chosen definition of ‘disc’. Thus, at this magnetic field strength, the disc is distinct from the other models. Figure 13 shows the masses and sizes of these discs (along with the percentage mass than the Hall-only and non-ideal MHD models, respectively, at which non-ideal effects are important). Given the numerical difficulty associated with the Hall effect, it was included in the Hall-only model, the maximum mass and plasma beta occurs in the non-ideal MHD model. When the Hall effect is included in the ideal MHD model, and the Hall-only and non-ideal MHD models are compared, there is a clear difference in the magnetic field strength and index that the diffusion terms are important in the disc. Thus, at this magnetic field strength, there are no discs with a chosen definition of ‘disc’. Thus, at this magnetic field strength, the disc is distinct from the other models.
CONCLUSIONS

➤ New “constrained” hyperbolic/parabolic divergence cleaning

➤ Can now perform realistic ideal and non-ideal Smoothed Particle Magnetohydrodynamics simulations

➤ Phantom SPMHD code available on request (public soon)

➤ Non-ideal MHD, in particular the Hall effect, plays a crucial role in the formation of protostellar discs