SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS

The state of the art

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Weighted sum over neighbours

\[ \rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h) \]

e.g. \[ W = \frac{\sigma}{h^3} e^{-r^2/h^2} \]
RESOLUTION Follows MASS
FROM DENSITY TO HYDRODYNAMICS

\[ L_{sph} = \sum_j m_j \left[ \frac{1}{2} v_j^2 - u_j(\rho_j, s_j) \right] \]

Lagrangian

\[ du = \frac{P}{\rho^2} d\rho \]

1st law of thermodynamics

\[ \nabla \rho_i = \sum_j m_j \nabla W_{ij}(h) \]

density sum

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial r} = 0 \]

Euler-Lagrange equations

\[ \frac{d\mathbf{v}_i}{dt} = - \sum_j m_j \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}(h) \]

\[ \left( \frac{d\mathbf{v}}{dt} = - \frac{\nabla P}{\rho} \right) \]
equations of motion!
WHAT THIS GIVES US: ADVANTAGES OF SPH

➤ Exact solution to the mass continuity equation
➤ Resolution follows mass
➤ Zero numerical dissipation
➤ Advection done perfectly
➤ Exact and simultaneous conservation of mass, momentum, angular momentum, energy and entropy
➤ A guaranteed minimum energy state
Existence of minimum energy state guarantees local ordering of particle distribution

BUT: requires positive pressure
**SMOOTHED PARTICLE MAGNETOHYDRODYNAMICS**

First, we define the Lagrangian as follows:

$$L = \int \left( \frac{1}{2} \rho v^2 - \rho u - \frac{1}{2\mu_0} B^2 \right) dV$$

where $\rho$ is the density, $v$ is the velocity, $u$ is the internal energy, and $B$ is the magnetic field.

Then, we have:

$$L = \sum_a m_a \left( \frac{1}{2} v_a^2 - u_a - \frac{B_a^2}{2\mu_0 \rho_a} \right)$$

**Euler-Lagrange equations give discrete form of:**

$$\frac{d\rho}{dt} = -\rho (\nabla \cdot v)$$

$$\frac{dv}{dt} = -\frac{1}{\rho} \nabla \cdot \left[ \left( P + \frac{1}{2} \frac{B^2}{\mu_0} \right) I - \frac{BB}{\mu_0} \right] - \frac{B(\nabla \cdot B)}{\mu_0 \rho}$$

$$\frac{du}{dt} = -\frac{P}{\rho} (\nabla \cdot v)$$

$$\frac{d}{dt} \left( \frac{B}{\rho} \right) = \left( \frac{B}{\rho} \cdot \nabla \right) v$$

**Method is dissipationless**

Include $\nabla \cdot B$ source term for stability

These equations are equivalent to the 8-wave formulation of Powell et al. 1994

Price & Monaghan (2004a,b, 2005)

See review by Price (2012)

J. Comp. Phys. 231, 759
\[
\delta L = m_a \mathbf{v}_a \cdot \delta \mathbf{v}_a - \sum_b m_b \left[ \frac{\partial u_b}{\partial \rho_b} \right]_s \delta \rho_b + \frac{1}{2\mu_0} \left( \frac{B_b}{\rho_b} \right)^2 \delta \rho_b - \frac{1}{\mu_0} \mathbf{B}_b \cdot \delta \left( \frac{\mathbf{B}_b}{\rho_b} \right)
\]

\[
\delta \rho_b = \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \cdot \nabla_b W_{bc} \quad \text{and} \quad \frac{du}{d\rho} \bigg|_s = \frac{P}{\rho^2}.
\]

\[
\delta \left( \frac{\mathbf{B}_b}{\rho_b} \right) = \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \frac{\mathbf{B}_b}{\rho_b^2} \cdot \nabla_b W_{bc}
\]

\[
\frac{dv^i_a}{dt} = \sum_b m_b \left[ \left( \frac{S^{ij}}{\rho^2} \right)_a + \left( \frac{S^{ij}}{\rho^2} \right)_b \right] \nabla^j W_{ab}, \quad \text{Obtain \textit{CONSERVATIVE} momentum equation}
\]

\text{Consistent with Janhunen formulation of \textit{div B} terms}

\textit{Price & Monaghan (2005)}
PHILLIPS & MONAGHAN (1985): SPH WITH MHD IN CONSERVATIVE FORM IS UNSTABLE WHEN BETA < 1
WITH SOURCE TERM IN MOMENTUM EQUATION

Circularly polarised Alfvén wave
ZERO DISSIPATION – II. ADVECTION OF A CURRENT LOOP

1000 crossings (Rosswog & Price 2007)

2 crossings (Gardiner & Stone 2005)

Fig. 3. Gray-scale images of the magnetic pressure $B^2 + B^2$ at $t = 2$ for an advected field loop ($v_0 = \sqrt{3}$) using the $\delta_r$ (top left), $\delta_i$ (top right) and $\delta_i$ (bottom) CT algorithm.

Fig. 8. Magnetic field lines at $t = 0$ (left) and $t = 2$ (right) using the CTU + CT integration algorithm.
ZERO DISSIPATION III
SHOCK CAPTURING IN SPH

\[
\frac{dU_a}{dt} = \sum_b m_b \frac{\rho_{ab}}{\alpha \nu_{\text{sig}} (U_a - U_b)} \hat{r}_{ab} \cdot \nabla W_{ab}
\]

- Formulate dissipative terms similar to approximate Riemann solvers

\[
\left( \frac{dv_i}{dt} \right)_{\text{diss}} = \sum_j m_j \frac{\alpha \nu_{\text{sig}} (v_i - v_j)}{\rho_{ij}} \hat{r}_{ij} \cdot \nabla_{i} W_{ij},
\]

\[
\left( \frac{de_i}{dt} \right)_{\text{diss}} = \sum_j m_j \frac{(e_i^* - e_j^*)}{\rho_{ij}} \hat{r}_{ij} \cdot \nabla_{i} W_{ij},
\]

\[
e = \frac{1}{2} v^2 + u
\]

- Enforce positive definite contribution to entropy

\[
\left( \frac{dU}{dt} \right)_{\text{diss}} = -\sum_j m_j \frac{\rho_j}{\rho_i} \left\{ \frac{1}{2} \alpha \nu_{\text{sig}} (v_i \cdot \hat{r}_j)^2 + \alpha v_{\text{sig}}^2 (u_i - u_j) \right\} \hat{r}_{ij} \cdot \nabla_{i} W_{ij}.
\]

Viscous heating  Thermal conduction

- Gives artificial dissipation terms equivalent to artificial viscosity, conductivity and (in MHD) resistivity

- Viscosity terms = Navier Stokes equations with \( \nu = \frac{\alpha}{10} \nu_{\text{sig}} h \)

BUT dissipation terms are first order
ARTIFICIAL VISCOSITY
ARTIFICIAL CONDUCTIVITY

Chow & Monaghan (1997)
Price (2008), JCP

No conductivity

With conductivity

\[ \mu \]

\[ P \]
APPROACHES TO $\text{DIV } B = 0$

1. Ignore
2. Prevent
3. Clean
POWELL'S 8 WAVE METHOD = IGNORE BUT PRESERVE

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B)
\]

\[
\frac{dB}{dt} = (B \cdot \nabla)v - B(\nabla \cdot v)
\]

consistent \(\nabla \cdot B\) terms

Preserve

Divergence advection test from Dedner et al. (2002)

Smear

Application to SPH: Price & Monaghan (2005)
USE OF POWELL-ONLY DIV B CONTROL IN SPMHD

Good results on test problems... but not so good for star formation the Orszag-Tang vortex problem in SPMHD (Price & Monaghan 2005, Rosswog & Price 2007)

**PREVENT: DIV B = 0 BY CONSTRUCTION IN SPH**

\[ \mathbf{B} = \nabla \alpha \times \nabla \beta \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]

\[ \frac{d\alpha}{dt} = 0 \]
\[ \frac{d\beta}{dt} = 0 \]

*Euler potentials (e.g. Stern, 1976)*

(advection of magnetic field lines by Lagrangian particles)
Price & Bate (2007): Effect of Magnetic Fields on Single and Binary Star Formation

...problem forming discs and binaries in the presence of magnetic fields?

see also Allen et al. (2003), Galli et al. (2006), Mellon & Li (2008), Hennebelle & Fromang (2008), Commerçon et al. (2010), Krasnopolsky et al. (2010), Seifried et al. (2012), Santos-Lima et al. (2012), Joos et al. (2013) and many others
LIMITATIONS OF THE EULER POTENTIALS APPROACH

- advection of magnetic fields: no change in topology ($A.B = 0$)
- does not follow wind-up of magnetic fields
- difficult to model resistive effects — reconnection processes not treated correctly

\[ \mathbf{B} = \nabla \alpha \times \nabla \beta \]

\[ \frac{d\alpha}{dt} = 0 \]
\[ \frac{d\beta}{dt} = 0 \]
HYPERBOLIC/PARABOLIC DIVERGENCE CLEANING

Dedner et al. (2002)
Price & Monaghan (2005)
Mignone & Tzeferacos (2010)

\[
\frac{\partial B}{\partial t} = - \nabla \psi
\]
\[
\frac{\partial \psi}{\partial t} = - c_h^2 (\nabla \cdot B) - \frac{\sigma c_h}{c_p} \psi
\]

Use dimensionless parameter

Critical damping on resolution length

Hyperbolic term only

Use dimensionless parameter

Critical damping on resolution length

Wavelength of critical damping

Hyperbolic term only
WHEN CLEANING ATTACKS

5.3. Static cleaning test: free boundaries

A further variant of the divergence advection test we consider replaces the periodic boundaries by a free boundary, since many applications of SPMHD involve free boundaries (e.g. the merger of two neutron stars [36], or studies of galaxy interactions [15,16]).

5.3.1. Setup

The setup is identical to the divergence advection problem (Section 5.1) with \( r_0 = 1 = \sqrt{8} \), except that the domain is a circular area of fluid with \( q = 1 \) for \( r < 1 \) and \( q = 0 \) (no particles) for \( r > 1 \), set up using a total of 1976 particles placed on a cubic lattice. The divergence perturbation is introduced at the centre of the circle, and the velocity field is set to zero. Rather than impose an external confining potential, we solve only Eqs. (16) and (17) without the full MHD equations, as in Section 5.2.

5.3.2. Results

Fig. 6 shows the results of purely hyperbolic cleaning (\( r = 0 \)) for this case. As in Fig. 4, the top row shows the unconstrained and non-conservative difference/difference formulation, while the bottom row shows results using the conservative difference/symmetric combination. Similar results are also found in this case, with divergence errors piling up at the free boundary in the non-conservative formulation leading to numerical instability, but our constrained formulation remaining stable.

5.4. 2D Blast wave in a magnetised medium

We now turn to tests that are more representative of the dynamics encountered in typical astrophysical simulations, beginning with a blast wave expanding in a magnetised medium. In this case the initial magnetic field is divergence-free, meaning that the only divergence errors are those created by numerical errors during the course of a simulation – rather than the artificial errors we have induced in the previous tests. Based on the results from the previous tests, in this and subsequent tests we apply cleaning only using constrained, energy-conserving formulations – that is, with conjugate operators for \( r/C_1 \) and \( r/w \). We use this problem to the examine the effectiveness of the divergence cleaning in the presence of strong shocks, as well as to investigate whether cleaning should be performed using the difference or symmetric \( r/C_1 \) operator. As with the divergence advection test, a key goal is to find optimal values for the damping parameter \( r \).

5.4.1. Setup

The implementation of the blast wave follows that of Londrillo and Del Zanna [18]. The domain is a unit square with periodic boundaries, set up with 512 \( \times \) 590 particles on a hexagonal lattice with \( q = 1 \). The fluid is at rest with magnetic field \( B_x = 10 \). The pressure of the fluid is set to \( P = 1 \), with \( c = 1 \) except a region in the centre of radius 0.125 has its pressure increased by a factor of 100 by increasing its thermal energy. An adiabatic equation of state is used.

Divergence advection test (Dedner et al. 2002) with 10:1 jump in density
“CONSTRAINED” HYPERBOLIC/PARABOLIC DIVERGENCE CLEANING

Tricco & Price (2012); Tricco, Price & Bate (2016)

- Define energy associated with cleaning field

\[ E = \int \left[ \frac{1}{2} \frac{B^2}{\mu_0} + \frac{1}{2} \frac{\psi^2}{\mu_0 c_h^2} \right] dV \]

- Enforce energy conservation in hyperbolic terms

\[
\frac{dE}{dt} = \int \left[ \frac{B}{\mu_0} \cdot \left( \frac{dB}{dt} \right)_\psi + \frac{\psi}{\mu_0 c_h^2} \frac{d\psi}{dt} - \frac{\psi^2}{2\mu_0 \rho c_h^2} \frac{d\rho}{dt} - \frac{\psi^2}{\mu_0 c_h^3} \frac{dc_h}{dt} \right] dV = 0
\]

\[
\frac{dB}{dt} = -\nabla \psi
\]

Requires particular choice of operators here

\[
\frac{d\psi}{dt} = -c_h^2 (\nabla \cdot B) - \frac{\sigma c_h}{h} \psi - \frac{1}{2} \psi (\nabla \cdot v)
\]

- Can enforce exact energy conservation in SPH discretisation
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WHAT IF THE CLEANING SPEED VARIES?

Tricco, Price & Bate (2016)
J. Comp. Phys. 322, 326

Hyperbolic terms conserve energy even with variable wave speed!

\[
\frac{dB}{dt} = -\nabla \psi \\
\frac{d}{dt} \left( \frac{\psi}{c_h} \right) = -c_h (\nabla \cdot B) - \frac{\psi}{2c_h} (\nabla \cdot \mathbf{v}) - \frac{\sigma}{h} \frac{\psi}{c_h}
\]

With thanks to Gabor Tóth for inspiration!
APPLICATION TO FINITE VOLUME SCHEMES

Original method (Dedner et al. 2003):

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ E \\ B \\ \psi \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} (E + P + \frac{1}{2} B^2) I - BB \\ v(E + P + \frac{1}{2} B^2) - B(v \cdot B) \\ vB - Bv \\ c_h^2 B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\psi}{\tau} \end{bmatrix}
\]

Constrained hyperbolic cleaning with variable wave speed:

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ E' \\ B \\ \phi \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} (E' + P + \frac{1}{2} B^2) I - BB \\ v(E' + P + \frac{1}{2} B^2) - B(v \cdot B) + \psi B \\ vB - Bv \\ \phi v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\psi^2}{c_h^2 \tau} \\ \psi \sqrt{\mathbf{v} \cdot \mathbf{B}} \\ -\frac{c_h \psi}{\sqrt{\rho}} \left( \nabla \cdot \mathbf{B} - \frac{\phi}{\tau} \right) \end{bmatrix}
\]

\[
E' = E + \frac{1}{2} \frac{\psi^2}{c_h^2} \\
\phi = \frac{\psi}{c_h \sqrt{\rho}}
\]

Price, this week
See also: Derigs, Gassner, Walch & Winters (2017)
+ Florian Hindenlang talk
APPLICATION TO FINITE VOLUME CODES

Original GLM

Constrained hyperbolic cleaning

Pure hyperbolic case
**CAN WE ENFORCE \( \text{DIV} \mathbf{B} = 0 \) EXACTLY?**

![Graph showing the iteration count versus \( h |\text{div}B|/|B| \) for different values of \( \sigma \).](image)

*Tricco, Price & Bate (2016)*

*J. Comp. Phys. 322, 326*

Achievable in principle, not currently practical
SHOCK DISSIPATION SWITCHES

- Cullen & Dehnen (2010) switch for shock viscosity

\[ A = \max \left[ -\frac{d}{dt} (\nabla \cdot \mathbf{v}), 0 \right] \quad \alpha_{loc} = \min \left( \frac{10h^2 A}{c_s^2 + h^2 A}, 1 \right) \]

- Tricco & Price (2013) switch for resistivity

\[ \alpha^B = \min \left( \frac{h|\nabla B|}{|B|}, 1 \right) \]

- Revised further in Phantom - 2nd order artificial resistivity, vanishes when \( v = \text{const} \)

Figure 3. Shock tube test 5A from RJ95 performed in 2D with left state \((\rho, P, v_x, v_y, B_x) = (1, 1, 0, 0, 0, 1)\) and right state \((\rho, P, v_x, v_y, B_x) = (0.125, 0.1, 0, 0, -1)\) with \(B_x = 0.75\) at \(t = 0.1\). Black circles represent the particles and the red line represents the solution obtained with the \textsc{athena} code using \(10^4\) grid cells.
Performed with all dissipation, shock capturing and divergence cleaning turned on.

Advection of current loop (Gardiner & Stone 2005, 2008)

Convergence on circularly polarised Alfvén wave with ALL dissipation turned on.
JETS FROM THE FIRST CORE

Price, Tricco & Bate (2012); see also Machida et al. (2008)
PROTOTERLAR JETS: SECOND COLLAPSE

Bate, Tricco & Price (2014)

Time: 24964 yrs

Performed with radiation magnetohydrodynamics (gray FLD: Whitehouse & Bate 2004a,b; Whitehouse, Bate & Monaghan 2006)
MAGNETICALLY LAUNCHED OUTFLOWS

First core (100 x 100 au)

Second (protostellar) core (10 x 10 au)
STRONG MAGNETIC FIELDS IMPLANTED IN STARS AT BIRTH

Figure A1. The time evolution of the maximum density during RMHD calculations of the collapse of a rotating molecular cloud core with an initial mass-to-flux ratio of $\mu = 5$ times critical performed using resolutions of 1 (red dotted line), 3 (black solid line) and 10 (blue dashed line) million particles. The free-fall time of the initial cloud core, $t_{ff} = 7.71 \times 10^{11}$ s (24,430 yr). In the right-hand panel, the time has been set to zero when the stellar core begins to form (i.e. when the maximum density reaches $10^{-4} g cm^{-3}$).

Figure A2. The evolution of the maximum gas temperature (left) and maximum magnetic field strength (right) versus maximum density for RMHD calculations of the collapse of a rotating molecular cloud core with an initial mass-to-flux ratio of $\mu = 5$ times critical performed using resolutions of 1 (red dotted line), 3 (black solid line) and 10 (blue dashed line) million particles. With lower resolution the fields strength in the collapse is reduced due to increased numerical resistivity.

Figure A3. Snapshots of the density on slices parallel to the rotation axis showing the development of the outflows that are launched from the first hydrostatic cores (left-hand panels) and stellar cores (right-hand panels) in calculations with initial mass-to-flux ratios of $\mu = 5$ times critical performed with different resolutions as labelled above each panel.
Figure 3. $z$-column integrated $\langle B \rangle$ and $|B|$, defined $<B> = \frac{\int B \, dz}{\int dz}$, for Flash (top) and Phantom (bottom) at resolutions of $256^3$ for $t/t_c = 2, 4, 6, 8$. The density field has similar structure in both codes at early times, but diverge at late times due to the non-deterministic behaviour of the turbulence. The magnetic field is strongest in the densest regions, while the mean magnetic field strength throughout the domain increases with time. Similar growth rates. In contrast, the Phantom results have growth rates that increase with resolution by nearly a factor of two for each doubling of resolution.

Analytic studies of the exponential growth rate of the small-scale dynamo have shown that for $Pm \gg 1$, the growth rate scales with $Rm^{1/2}$, while for $Pm \ll 1$, it scales with $Re^{1/2}$ (Schober et al. 2012a; Bovino et al. 2013). Theoretical predictions of the growth rate for $Pm \sim 1$, which is the Prandtl number regime for numerical codes in the absence of explicit dissipation terms, are more uncertain. The growth rate in the transition region between $0.1 < Pm < 10$ was probed by Federrath et al. (2014) using Flash simulations with explicit viscous and resistive dissipation. They found that the magnetic energy growth rate for $Pm \sim 1$ exhibited a steep dependence on $Pm$ and only agreed qualitatively with the analytical expectations of Schober et al. (2012a) and Bovino et al. (2013). Conversely, the growth rate for $Pm \ll 1$ quantitatively agreed with analytical expectations, with, by comparison, relatively little variation with respect to $Pm$.
MAGNETIC FIELDS IN TIDAL DISRUPTION EVENTS

Bonnerot, Price, Rossi, Lodato (2017), MNRAS

Artificial dynamo with Powell-terms only
Strong coupling approximation: $\rho \approx \rho_n; \ \rho_i \ll \rho$

$$\frac{dB}{dt} = -B(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla) \mathbf{v}$$

$$-\nabla \times \left[ \eta_O \mathbf{J} + \eta_H \mathbf{J} \times \hat{\mathbf{B}} - \eta_A (\mathbf{J} \times \hat{\mathbf{B}}) \times \hat{\mathbf{B}} \right]$$

\[ \text{Ohmic} \quad \text{Hall} \quad \text{Ambipolar} \]

Spatial discretisation exactly conserves energy

Guaranteed positive definite contribution to entropy

RKC super-timestepping for ambipolar/Ohmic terms

(Alexiades et al. 1996; O’Sullivan & Downes 2006)
Figure 2. Face-on gas column density using ideal MHD. The initial rotation is counterclockwise and the initial magnetic field is directed out of the page (i.e. $B_0 \cdot \Omega_0 > 0$). Each model is initialized with $\sim 3 \times 10^5$ particles within the sphere. From left to right, the columns represent snapshots at a given time (in units of the free-fall time, $t_{\text{ff}} = 2.4 \times 10^4$ yr). The rows represent models with different initial magnetic field strengths given in terms of $\mu_0$ (i.e. the initial mass-to-flux ratio normalised to the critical mass-to-flux ratio). The top row has no initial magnetic field and the bottom row has the strongest magnetic field (i.e. increasing magnetic field strength corresponds to a decreasing value of $\mu_0$). The white circles represent the sink particle with the radius of the circle representing the accretion radius of the sink particle. Each frame represents a disc growing and shrinking with time. At any given time, the models with stronger magnetic fields have smaller and less massive discs than the models with the weaker initial magnetic field. The hydrodynamic model yields the largest and most massive disc in our entire suite of simulations. Discs while magnetohydrodynamical collapses hinder or suppress the formation of discs, with smaller discs forming in simulations with stronger initial magnetic fields – assuming a disc forms at all.

In agreement with (e.g.) Allen et al. (2003), PB07, Mellon & Li (2008), and Hennebelle & Fromang (2008), this demonstrates the magnetic braking catastrophe.

4.1.1 Resolution

Fig. 4 shows a comparison of the discs formed at resolutions of $\sim 3 \times 10^5$ particles in the collapsing sphere (top row) and $\sim 10^6$ particles (bottom row) using $\mu_0 = 7.5$. This magnetic field strength was used so that disc characteristics could be compared. The $\sim 10^6$ particle model took $\sim 3.5$ times longer to run, which is reasonable given the increase in resolution.

The two resolutions follow the same general trend, with large discs forming. For $1.10^{11} \lesssim t/t_{\text{ff}} \lesssim 1.21$, the star + disc mass, disc mass and disc radius typically differ by less than 20 per cent. Thus, these results are relatively robust to the resolution increase presented here.
NON-IDEAL MHD: ALIGNED INITIAL FIELD

Wurster, Price & Bate (2016)

Figure 6. Face-on gas column density, as in Fig. 2 but for non-ideal MHD including the effect of Ohmic resistivity, the Hall effect and ambipolar diffusion.

The top plot has the magnetic field initialised with $B_0 \cdot \Omega_0 > 0$, and the bottom plot with $B_0 \cdot \Omega_0 < 0$. Compared to ideal MHD, discs size smaller for $B_0 \cdot \Omega_0 > 0$, but larger for $B_0 \cdot \Omega_0 < 0$. This indicates that the Hall effect is the most important non-ideal MHD term for disc formation.

Our $\sim 3 \times 10^5$ particle models meet the resolution criteria set out by Bate & Burkert (1997) (c.f. Section 3), and our brief resolution study indicates that our results agree at both resolutions. Thus, to save computational costs of the $B_0 \cdot \Omega_0 < 0$ models with weaker magnetic fields, the bottom panel in Fig. 6 shows the lower resolution models. For consistency, we thus present all the models in Sections 4.1 and 4.3 at the lower resolution. The remainder of this study is performed using the $\sim 10^6$ particle models, with the exception of our discussion of the cosmic ionisation rate. Note that the non-ideal MHD model with $\mu_0 = 5$ and $B_0 \cdot \Omega_0 < 0$ has only evolved to $t \approx 1.18 t_{ff}$. Wurster, Price & Bate (2016)
Figure 6. Face-on gas column density, as in Fig. 2 but for non-ideal MHD including the effect of Ohmic resistivity, the Hall effect and ambipolar diffusion. The top plot has the magnetic field initialised with $B_0 \cdot \Omega_0 > 0$, and the bottom plot with $B_0 \cdot \Omega_0 < 0$. Compared to ideal MHD, disc size is small for $B_0 \cdot \Omega_0 > 0$, but large for $B_0 \cdot \Omega_0 < 0$. This indicates that the Hall effect is the most important non-ideal MHD term for disc formation.

Our simulations at resolutions within a factor of two than its counterpart, and the evolution indicates that it will not dissipate. Our simulations at resolutions within a factor of two than its counterpart, and the evolution indicates that it will not dissipate.

Our $\sim 3 \times 10^5$ particle models meet the resolution criteria set out by Bate & Burkert (1997) (c.f. Section 3), and our brief resolution study indicates that our results agree at both resolutions.

Thus, to save computational costs of the $B_0 \cdot \Omega_0 < 0$ models with weaker magnetic fields, the bottom panel in Fig. 6 shows the lower resolution models. For consistency, we thus present all the models in Sections 4.1 and 4.3 at the lower resolution. The remainder of this study is performed using the $\sim 10^6$ particle models, with the exception of our discussion of the cosmic ionisation rate. Note that the non-ideal MHD model with $\mu_0 = 5$ and $B_0 \cdot \Omega_0 < 0$ has only evolved to $t \approx 1.18 t_{\text{ff}}$. See also Tsukamoto et al. (2015).
Can non-ideal MHD save discs?

Hydro

μ₀ = 10

μ₀ = 7.5

μ₀ = 5

Figure 5. Edge-on gas column density using ideal MHD and zoomed out to (3000 AU)² and using a density range shifted down by a factor of ten to visualise the full extent of the outflows launched shortly after the collapse (t ≈ 1.02t_ff) in the magnetic models. The models with strongermagnetic fields have faster and more collimated outflows.

It is reasonable to only compare star+disc masses. At the remaining two magnetic field strengths, the non-ideal MHD models have larger disc masses and radii, and the specific angular momentum is similar or slightly larger. The ideal MHD models have stronger magnetic fields in the disc; this is expected given the inclusion of the two dissipative terms in the non-ideal MHD models. On average, gas pressure dominates the magnetic pressure in the disc.

4.3.1 Resolution

As with our ideal MHD simulations, we analyse the effect of increasing the resolution from ∼ 3 × 10⁵ particles in the initial gas cloud to ∼ 10⁶ particles. Given the h² dependence that the smoothing length has on the non-ideal MHD timestep, the increase in run-time is considerable for the models that form discs and include the Hall effect (since super-timestepping cannot be used). It takes the non-ideal MHD model with μ₀ = 5, B₀ · Ω₀ < 0 and ∼ 10⁶ particles ∼ 19 times longer to reach t = 1.15t_ff than its ∼ 3 × 10⁵ particle counterpart; this is the time when the disc dissipated in the ∼ 3 × 10⁵ model. For comparison, it takes the B₀ · Ω₀ < 0 model with ∼ 10⁶ particles ∼ 6.8 times longer to reach t = 1.21t_ff than its ∼ 3 × 10⁵ counterpart.

Wurster, Price & Bate (2016)
OUTFLOWS: NON-IDEAL MHD / ALIGNED INITIAL FIELD

Can non-ideal MHD save discs?

Figure 9. Edge-on gas column density, as in Fig. 5 but for non-ideal MHD including the effect of Ohmic resistivity, the Hall effect and ambipolar diffusion. The top plot has the magnetic field initialised with $B_0 \cdot \Omega_0 > 0$, and the bottom plot with $B_0 \cdot \Omega_0 < 0$. Outflows for simulations that form small discs.

4.4 Non-ideal MHD — outflows

Fig. 9 shows the edge-on column density for the non-ideal MHD calculations, showing the models with $B_0 \cdot \Omega_0 > 0$ and $B_0 \cdot \Omega_0 < 0$ in the top and bottom plots, respectively. The most interesting aspect is that outflows appear to anticorrelate with the presence of discs. That is, outflows carry away angular momentum, which hinders the formation of discs. This is counterintuitive since one would normally expect outflows to be launched from a disc. Here, as in Price et al. (2012), the outflows are powered by a rotating, sub-Keplerian flow, and carry away sufficient angular momentum to prevent the formation of a Keplerian disc. Non-ideal MHD, in general, appears to suppress the formation of outflows. This is quantified further in Section 4.6, where we discuss the influence of individual non-ideal MHD terms.

Wurster, Price & Bate (2016)
Outflows are anti-correlated with disc formation!

Outflows: Non-Ideal MHD / Anti-Aligned

- $\mu_0 = 10$
- $\mu_0 = 7.5$
- $\mu_0 = 5$

Figure 9 shows the edge-on column density for the non-ideal MHD calculations, showing the models with $B_0 \cdot \Omega_0 > 0$ and $B_0 \cdot \Omega_0 < 0$ in the top and bottom plots, respectively. The most interesting aspect is that outflows appear to anticorrelate with the presence of discs. That is, outflows carry away angular momentum, which hinders the formation of discs. This is counterintuitive since one would normally expect outflows to be launched from a disc. Here, as in Price et al. (2012), the outflows are powered by a rotating, sub-Keplerian flow, and carry away sufficient angular momentum to prevent the formation of a Keplerian disc. Non-ideal MHD, in general, appears to suppress the formation of outflows. This is quantified further in Section 4.6, where we discuss the influence of individual non-ideal MHD terms.
### WHICH NON-IDEAL EFFECTS ARE IMPORTANT?

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<td><strong>Ambipolar</strong></td>
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- Hall effect is dominant during disc formation
- Produces counter-rotating envelope when $B$ and rotation are misaligned
- Maybe why half of all stars have planets?
ARE FOSSIL FIELDS POSSIBLE IN NON-IDEAL MHD?

Wurster, Price & Bate (2018)
CONCLUSIONS

➤ Enforcing $\text{div } B = 0$ is main issue in accurate SPMHD simulations

➤ Current best approach to enforcing $\text{div } B = 0$ in SPMHD is to use “constrained” hyperbolic/parabolic cleaning

➤ Phantom SPMHD code now public

➤ Non-ideal MHD, in particular the Hall effect, plays a crucial role in the formation of protostellar discs

➤ Can seemingly rule out fossil field hypothesis for origin of magnetic fields in stars