The question we have is:

Given the continuous time prototype filter

\[ H_c(s) = \frac{s + 2}{s^2 + 9s + 20} \]

use the technique of impulse invariance to create a discrete time equivalent. Assume a sampling rate of 60 kHz. Provide the transfer function and difference equation of the resulting discrete system. **Hint:** Recall that the Laplace transform of the first order term \( \frac{K}{s + \alpha} \) is

\[ Ke^{-\alpha t}u(t) \]

To answer this we do the following:

We perform partial fraction expansion to get the system into a summation of simple continuous exponentionals. Then, given a sampling rate, we find the bases of discrete exponentionals that will give the same values as the continuous ones at the sampling points. Then, these discrete exponentionals are recombined to get a transfer function and then we can get a difference equation.

The first step is to perform partial fraction expansion of the above equation. We must factorise the denominator, which is easily seen to be \((s + 5)(s + 4)\). We thus have the following:

\[
\frac{A}{s + 4} + \frac{B}{s + 5} = \frac{s + 2}{s^2 + 9s + 20}
\]

We must solve for \(A\) and \(B\), so we have:

\[ A(s + 5) + B(s + 4) = s + 2 \]  

(2)

Now, if we set \(s = -5\) we will zero the first term, and be left with:

\[ B(-5 + 4) = -5 + 2 \]  

(3)

or

\[ B = 3 \]  

(4)
Now we set $s = -4$ to zero the second term, and we get

$$A(-4 + 5) = -4 + 2 \quad (5)$$

which becomes

$$A = -2 \quad (6)$$

So we end up with:

$$H(s) = \frac{-2}{s+4} + \frac{3}{s+5} \quad (7)$$

If we take the inverse LaPlace transform we get:

$$h(t) = -2e^{-4t} + 3e^{-5t} \quad (8)$$

Now we recall that the basic discrete exponential is of the form $Ka^n$. Remembering that the $n$ is actually a $nTs$ where $Ts$ is the time period of one sample, and is equal to $1/Fs$ where $Fs$ is our sampling rate. Therefore to find a discrete equivalent of the first term $-2e^{-4t}$ we have

$$a^1 = e^{-4Ts} \quad (9)$$

And $a^1 = a$, so we have $a = e^{-4Ts}$. Similary for the second term, we have $b = e^{-5Ts}$. Now we have the following discrete impulse response:

$$h[n] = -2(e^{-4Ts})^n + 3(e^{-5Ts})^n \quad (10)$$

Referring to our useful formulae we see that the z-transform of $Ka^n$ is $\frac{Kz}{z-a}$, so take the z-transform of our equation and get:

$$H(z) = \frac{-2z}{z-(e^{-4Ts})} + \frac{3z}{z-(e^{-5Ts})} \quad (11)$$

Now simply recombine these two fractions to get a transfer function. Once back into transfer function form, we must make the substitions for $Ts$. In this question our sampling rate was 60 khz, so $Ts = 1/60000$, i.e. $e^{-4Ts} = e^{-60/60000}$ and $e^{-5Ts} = e^{-75/60000}$. Once we have the transfer function:

$$H(z) = \frac{z^2 + z(2(e^{-4Ts}) - 3(e^{-5Ts}))}{z^2 - z(e^{-5Ts} + e^{-4Ts}) + e^{-9Ts}} \quad (12)$$

And now we convert this to a difference equation in the normal fashion.