# An evolution of a permutation

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April 28, 2014

Joint work with Boris Pittel

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- $C_n$ = number of indecomposable permutations of length $n$ (Sloane, sequence A003319)

$$C_n = n! - \sum_{k=1}^{n-1} C_k \cdot (n-i)!$$

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### Definition

$T(n)$ is a sharp threshold for the property $P$ if for any fixed $\epsilon > 0$

- $m \leq (1 - \epsilon)T(n) \implies \sigma(n, m)$ does not have $P$ whp
- $m \geq (1 + \epsilon)T(n) \implies \sigma(n, m)$ does have $P$ whp

# Permutation Graphs

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  - $V = \{1, 2, \ldots, n\}$
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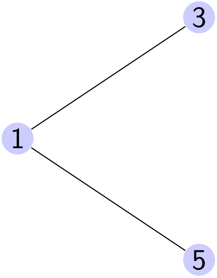

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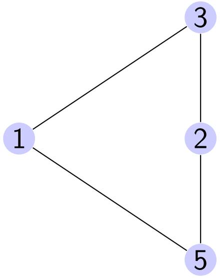

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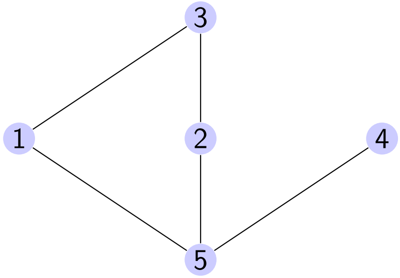

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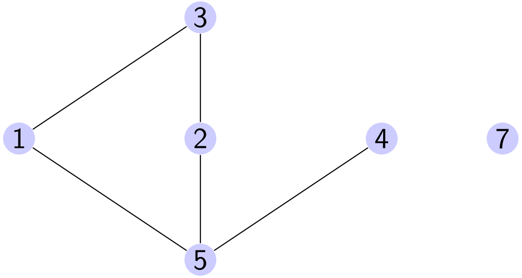

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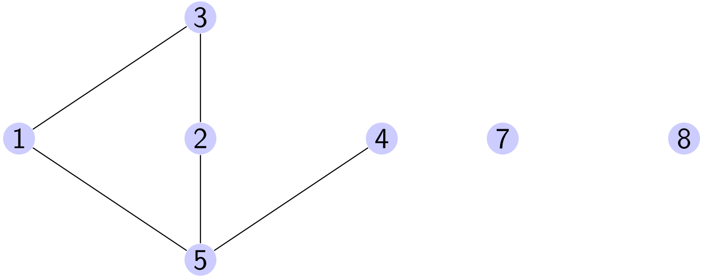

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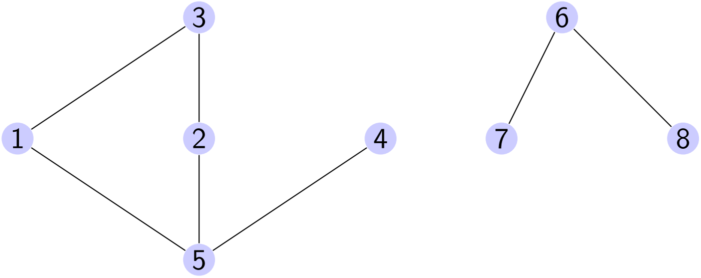

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- Vertex set of a connected component of $G_\pi$ consists of consecutive integers
- (Comtet) If $\sigma$ is chosen u.a.r. from $S_n$, then

$$Pr[\sigma \text{ is indecomposable}] = 1 - 2/n + O(1/n^2)$$

# Connectivity and descent sets

- Connectivity set of $\pi$

$$C(\pi) = \{i \in [n-1] : a_j < a_k \text{ for all } j \le i < k\}$$

$$C(35124786) = \{5\}$$

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## Proposition (Stanley)

Given $I \subseteq [n-1]$,

$$|\{\omega \in S_n : I \subseteq C(\omega)\}| \cdot |\{\omega \in S_n : I \supseteq D(\omega)\}| = n!$$

# Permutations with given number of cycles

- $\pi(n, m)$ = permutation chosen u.a.r from all permutations of $\{1, \ldots, n\}$ with $m$ cycles
- $p(n, m) = Pr[\pi(n, m)$ is connected]

Theorem (R. Cori, C. Matthieu, and J.M. Robson - 2012)

(i) $p(n, m)$ *is decreasing in* $m$

(ii) $p(n, m) \to f(c)$ *as* $n \to \infty$ *and* $m/n \to c$

# Erdős-Rényi Graphs

- $G(n, m)$ : Uniform over all graphs on $[n]$ with exactly $m$ edges
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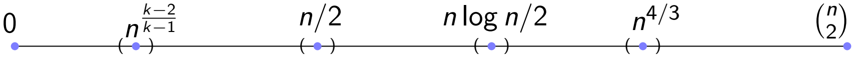
  - Connectedness probability of $G(n, m)$ increases with $m$
  - Sharp threshold: $n \log n/2$
- Graph Process $\widetilde{G}_n$
  - Start with $n$ isolated vertices
  - Add an edge chosen u.a.r. at each step
  - $G(n, m)$ is the snapshot at the $m$-th step of the process
  - $G(n, m) \subset G(n, m+1)$

# Erdős-Rényi Graph $G(n, m)$




- $n^{(k-2)/(k-1)}$: components of size $k$
- $n/2$: giant component
- $n \log n/2$: connectedness
- $n^{4/3}$: 4-clique

Question: Is there a similar process for $\sigma(n, m)$ (or $G_{\sigma(n,m)}$) such that

1. Uniform distribution is achieved after each step
2. Existing inversions (edges of $G_{\sigma(n,m)}$) are preserved

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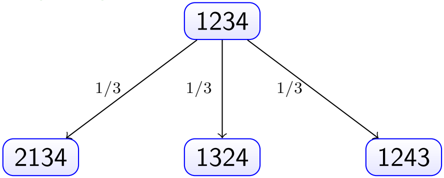
Example (n=4)



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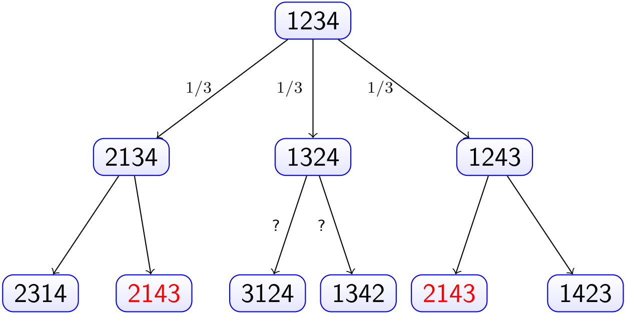

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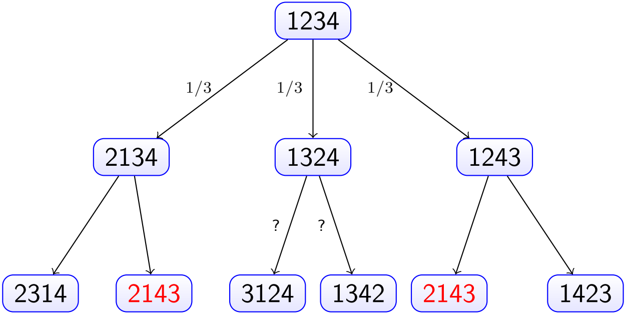


- Preserves the existing inversions (edges in the permutation)
- No uniformity

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Answer: YES

# Inversion Sequences

- Inversion sequence of $\pi = a_1 a_2 \dots a_n$ is $(x_1, \dots, x_n)$

$$x_j = \#\{i : i < j \text{ and } a_i > a_j\}$$

- $0 \leq x_j \leq j - 1$
- permutations of $[n] \leftrightarrow (x_1, \dots, x_n)$ where $0 \leq x_i \leq i - 1$

## Example

- $(x_1, x_2, x_3, x_4, x_5) = (0, 1, 0, 3, 3)$
- $\pi = 4, 3, 5, 1, 2$

# Evolution of a Permutation: Model 2

Increase one of the components in the inversion sequence by 1

- Not all the inversions are protected
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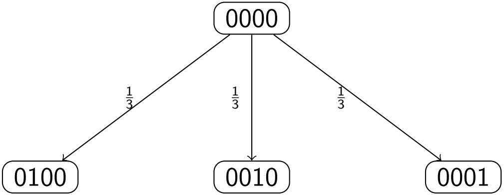

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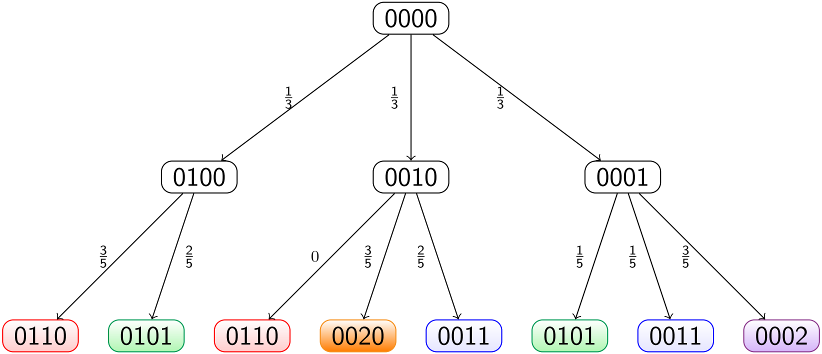
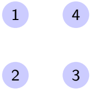


| Inv. Sequence | Permutation | Graph |
|---|---|---|
| 0000 | 1234 |  |

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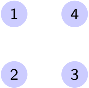
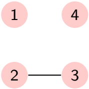
| 0000 | 1234 |  |
| 0010 | 1324 |  |

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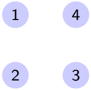
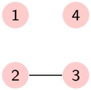
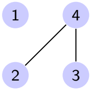
| 0000 | 1234 |   |
| 0010 | 1324 |   |
| 0011 | 1423 |   |

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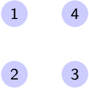
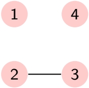
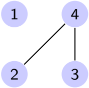
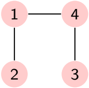
| 0000 | 1234 |  |
| 0010 | 1324 |  |
| 0011 | 1423 |  |
| 0021 | 2413 |  |

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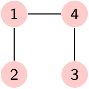
| 0021 | 2413 |  |

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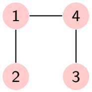
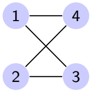
| 0021 | 2413 |  |
| 0022 | 3412 |  |

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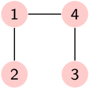
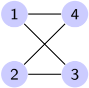
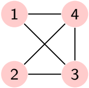
| 0021 | 2413 |  |
| 0022 | 3412 |  |
| 0122 | 4312 |  |

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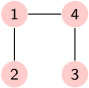
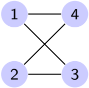
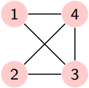
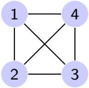
| 0021 | 2413 |   |
| 0022 | 3412 |   |
| 0122 | 4312 |   |
| 0123 | 4321 |   |

$f(n,k) =$ number of permutations of $[n]$ with $k$ inversions

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$$\begin{aligned} f(n,k) &= [z^k] \prod_{j=0}^{n-1} (1 + z + \cdots + z^j) \\ &= [z^k](1-z)^{-n} \prod_{j=1}^{n} (1-z^j) \end{aligned}$$

# The Process

- Start with $(0, 0, \ldots, 0)$
- Each time increase exactly one of the components by 1
- $\mathbf{X}(k) = (X_1(k), \ldots, X_n(k))$ after step $k$ is uniformly distributed

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Example

$$(0,0,0,0) \longrightarrow (0,0,1,0) \longrightarrow (0,0,1,1) \longrightarrow (0,0,2,1) \longrightarrow (0,0,2,2) \longrightarrow (0,1,2,2) \longrightarrow (0,1,2,3)$$

Goal: Finding $\mathbf{p}(\mathbf{X}(k))$, a (conditional) probability distribution for the $(k+1)$st addition OR

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Transition matrix $\rho_{n,k}$

- $f(n,k) \times f(n,k+1)$ matrix
- rows are indexed by inversion sequences with sum $k$
- columns are indexed by inversion sequences with sum $k+1$

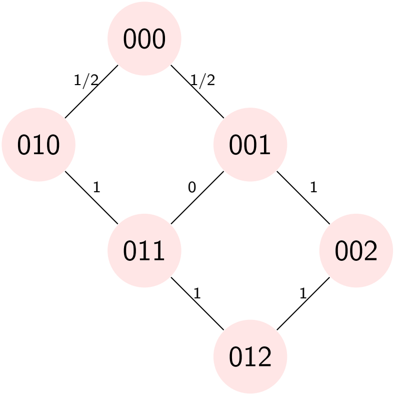
## Example (n=3)

$f(3,0) = 1$, $f(3,1) = 2$, $s(3,2) = 2$, and $s(3,3) = 1$.

$$\rho_{3,0} = \begin{array}{c} \\ 000 \end{array} \begin{array}{c} \begin{array}{cc} 010 & 001 \end{array} \\ \left[\begin{array}{cc} 1/2 & 1/2 \end{array}\right] \end{array}$$

$$\rho_{3,1} = \begin{array}{c} \\ 010 \\ 001 \end{array} \begin{array}{c} \begin{array}{cc} 011 & 002 \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \end{array}$$

$$\rho_{3,2} = \begin{array}{c} \\ 011 \\ 002 \end{array} \begin{array}{c} 012 \\ \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \end{array}$$



### Theorem

*Transition matrices exist for all* $n$ *and for all possible values of* $m$*.*

## Theorem

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## Sketch Proof

- Induction on $n$
- Order the sequences with reverse lexicographic order

| | $y_n = 0$ | $y_n = 1$ | $y_n = 2$ | $\dots$ | $y_n = n-2$ | $y_n = n-1$ |
|---|---|---|---|---|---|---|
| $x_n = 0$ | $\rho'_{n-1,m}$ | $\beta_1 I$ | | | | |
| $x_n = 1$ | | $\rho'_{n-1,m-1}$ | $\beta_2 I$ | | | |
| $x_n = 2$ | | | $\rho'_{n-1,m-2}$ | $\ddots$ | | |
| $\vdots$ | | | | $\ddots$ | $\ddots$ | |
| $x_n = n-2$ | | | | | $\rho'_{n-1,m-n+2}$ | $\beta_{n-1} I$ |
| $x_n = n-1$ | | | | | | $\rho'_{n-1,m-n+1}$ |

- $\rho'(n-1, m-j) = (1 - \beta_{j+1})\rho_{n-1,m-j}$
- Find constants $\beta_1, \dots, \beta_{n-1}$ such that all the column sums are equal to $f(n,m)/f(n,m+1)$

| | 0120 | 0111 | 0021 | 0102 | 0012 | 0003 |
|---|---|---|---|---|---|---|
| 0110 | $1-\beta_1$ | $\beta_1$ | 0 | 0 | 0 | 0 |
| 0020 | $1-\beta_1$ | 0 | $\beta_1$ | 0 | 0 | 0 |
| 0101 | 0 | $1-\beta_2$ | 0 | $\beta_2$ | 0 | 0 |
| 0011 | 0 | 0 | $1-\beta_2$ | 0 | $\beta_2$ | 0 |
| 0002 | 0 | 0 | 0 | $\frac{1-\beta_3}{2}$ | $\frac{1-\beta_3}{2}$ | $\beta_3$ |

- column sums must be $5/6$

| | 0120 | 0111 | 0021 | 0102 | 0012 | 0003 |
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| 0101 | 0 | $1-\beta_2$ | 0 | $\beta_2$ | 0 | 0 |
| 0011 | 0 | 0 | $1-\beta_2$ | 0 | $\beta_2$ | 0 |
| 0002 | 0 | 0 | 0 | $\frac{1-\beta_3}{2}$ | $\frac{1-\beta_3}{2}$ | $\beta_3$ |

- column sums must be 5/6

| | 0120 | 0111 | 0021 | 0102 | 0012 | 0003 |
|---|---|---|---|---|---|---|
| 0110 | 5/12 | 7/12 | 0 | 0 | 0 | 0 |
| 0020 | 5/12 | 0 | 7/12 | 0 | 0 | 0 |
| 0101 | 0 | 3/12 | 0 | 9/12 | 0 | 0 |
| 0011 | 0 | 0 | 3/12 | 0 | 9/12 | 0 |
| 0002 | 0 | 0 | 0 | 1/12 | 1/12 | 10/12 |

### Definition

An index $t$ ($t \geq 1$) is a decomposition point if $(X_{t+1}, \ldots, X_n)$ is an inversion sequence, i.e., if

$$X_{t+1} \leq 0, \quad X_{t+2} \leq 1, \quad \ldots \quad X_n \leq n - t - 1$$

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- number of components = number of decomposition points $+1$

### Corollary

$\Pr[\sigma(n, m)$ *is indecomposable*$]$ *is non-decreasing in* $m$

$C(\sigma) :=$ number of components in $G_{\sigma(n,m)}$

## Theorem

*If*

(i) $m = \frac{6n}{\pi^2}\left[\log(n) + 0.5 \log\log(n) + \log(12/\pi) - 12/\pi^2 + x_n\right]$

(ii) $x_n = o(\log\log\log n)$

*then*

$$d_{TV}[C(\sigma) - 1, Poisson(e^{-x_n})] \leq (\log n)^{-1+\epsilon} \text{ for any } \epsilon > 0.$$

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## Remarks

1. If $x_n \to c$, then $C(\sigma) - 1 \xrightarrow{d}$ Poisson$(e^{-c})$
2. $T(n) = \frac{6n}{\pi^2}[\log n + 0.5\log\log n]$ is a sharp threshold for connectedness of $G_{\sigma(n,m)}$

# Idea of the Proof for $x_n \to c$

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1. Need $D_n$, the number of decomposition points
   - $\nu = 2m \log n/n$
   - Mark $t$ if $(X_{t+1}, \ldots, X_{t+\nu})$ is an inversion sequence
   - $M_n$ = number of marked points
2. Whp $M_n = D_n$ as $n \to \infty$
3. $Pr[t \text{ is marked}] \sim e^{-c}/n$
4. $E_k = E\left[\binom{M_n}{k}\right] \to \frac{(e^{-c})^k}{k!}$
5. $M_n \to \text{Poisson}(e^{-c})$ in distribution

- $L_{\min}$ = size of the smallest component
- $L_{\max}$ = size of the largest block (component)

**Theorem**

*If*

- $m = \frac{6n}{\pi^2}\left[\log(n) + 0.5 \log\log(n) + \log(12/\pi) - 12/\pi^2 - x_n\right]$
- $x_n = o(\log\log\log n)$ *and* $x_n \to \infty$

*then*

1. $\lim_{n\to\infty} Pr[L_{min} \geq ne^{-2x_n} y] = e^{-y},$ *for any constant* $y \geq 0$
2. $\lim_{n\to\infty} P[L_{max} \leq ne^{-x_n}(x_n + z)] = e^{-e^{-z}},$ *for constant* $z \geq 0$

Note: Expected number of decomposition points $\sim e^{x_n}$

## Remark

Divide the interval $[0,1]$ into $k$ intervals with $k-1$ randomly chosen points.

$L_{min}, L_{max}$ = smallest and largest intervals, respectively

- $Pr[L_{min} \geq y/k^2] \to e^{-y}$ as $k \to \infty$
- $Pr[L_{max} \leq \frac{\log k + z}{k}] \to e^{-e^{-z}}$ as $k \to \infty$

Question: Conditioned on $\{\text{the number of blocks in } \sigma(n,m) = k\}$, do we have

$$(L_1/n, \ldots, L_k/n) \to (\eta_1, \ldots, \eta_k) \text{ as } n \to \infty$$

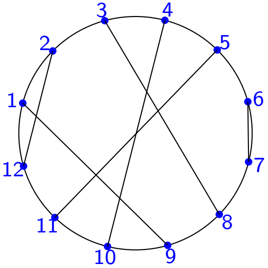
where

- $L_j$ = size of the $j^{\text{th}}$ block in $\sigma(n,m)$
- $\eta_j$ = size of the $j^{\text{th}}$ interval in $[0,1]$?

# Chord Diagrams and Intersection Graphs

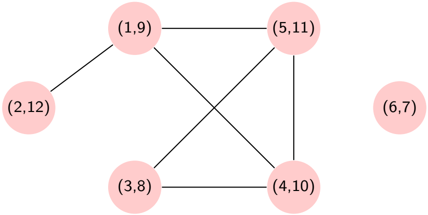
Chord Diagram

matching of $2n$ points




Intersection Graph

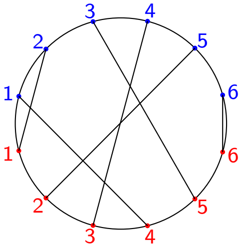
$V =$ chords, $E =$ crossings




- Number of chord diagrams:
  $(2n-1)!! = (2n-1)(2n-3)\cdots(3)\cdot(1)$

# Permutations as Chord Diagrams

- Relabel the points on the lower semicircle
- Draw the chords from the upper semicircle to the lower semicircle



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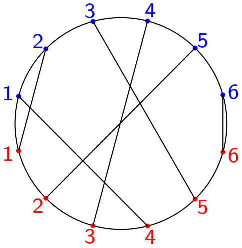

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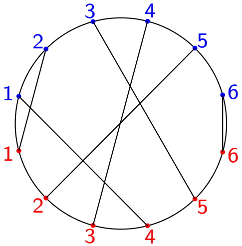

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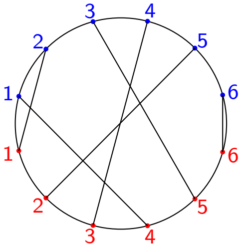

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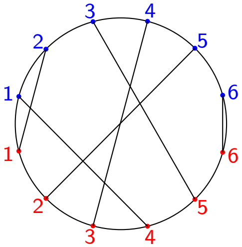

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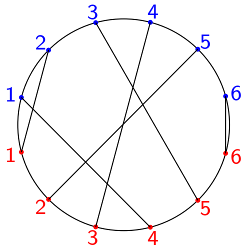

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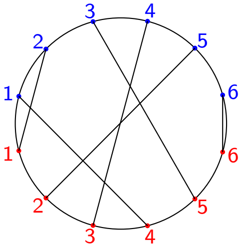

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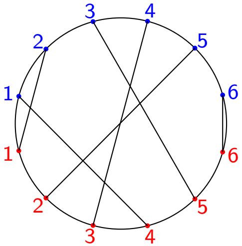

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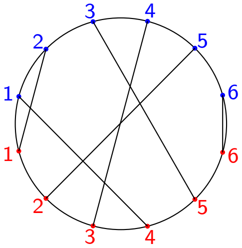



Permutation= 254136

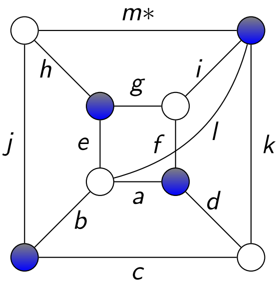
# pointed hypermaps $\leftrightarrow$ indecomposable permutations

## Definition

A labeled pointed hypermap on $[n]$ is a triple $(\sigma, \theta, r) \in S_n \times S_n \times [n]$ such that $< \sigma, \theta >$ acts transitively on $[n]$.

## Example

$$\sigma = (abel)(cdk)(fgi)(hjm)$$
$$\theta = (adf)(bjc)(egh)(ilkm)$$
$$r = m$$



THANK YOU!