A Universal Bijection for Catalan Families

R. Brak

School of Mathematics and Statistics
University of Melbourne

October 25, 2018

Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$1; 2; 5; 14; 42; 132; 429; \square$$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

$$G(z) = \sum_{n \geq 0} C_n z^n; \qquad G = 1 + zG^2$$

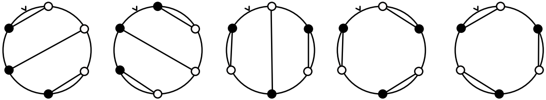
A few Catalan families

Examples of C_3 objects

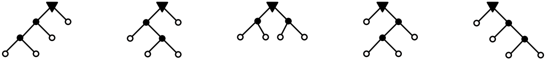
F_1 – Matching brackets and Dyck words

{}{{}}   {{}{}}   {{{}}}   {{}}{}   {}{}{}

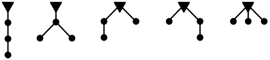
F_2 – Non-crossing chords the circular form of nested matchings



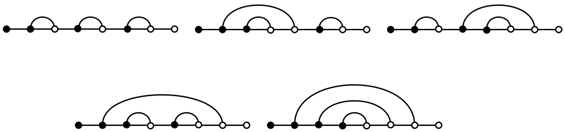
F_3 – Complete Binary trees and Binary trees



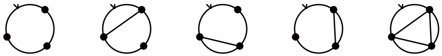
F_4 – Planar Trees



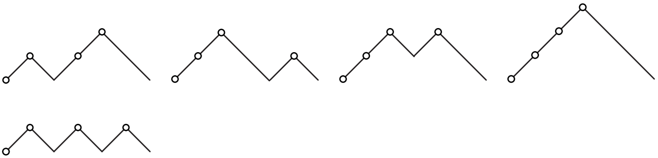
F_5 – Nested matchings or Link Diagrams



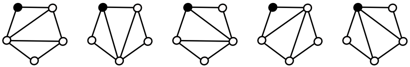
F_6 – Non-crossing partitions



F_7 – Dyck paths



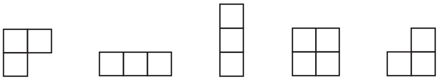
F_8 – Polygon triangulations



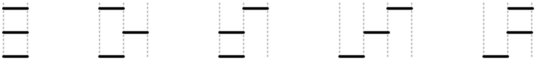
F_9 – 321-avoiding permutations

$$123; \quad 213; \quad 132; \quad 312; \quad 231\,.$$

F_{10} – Staircase polygons



F_{11} – Pyramid of heaps of segments



F_{12} – Two row standard tableau

1	3	5
2	4	6

1	2	5
3	4	6

1	2	3
4	5	6

1	2	4
3	5	6

1	3	4
2	5	6

F_{13} – Non-nested matchings

F_{14} – Frieze Patterns: $n-1$ row periodic repeating rhombus

$$
\begin{array}{ccccccccccccccc}
\square & 1 & & 1 & & \square & & 1 & & \square & & & & & \\
& \square & a_1 & & a_2 & & \square & & a_n & & \square & & & & \\
& & \square & b_1 & & b_2 & & \square & & b_n & & \square & & & \\
& & & & \square & & & & & & \square & & & & \\
& & & & \square & r_1 & & r_2 & & \square & & r_n & & \square & \\
& & & & & \square & 1 & & 1 & & \square & & 1 & & \square
\end{array}
$$

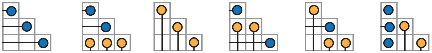
with

$$
\begin{array}{ccc}
 & r & \\
s & & t \\
 & u &
\end{array}
\qquad \text{and} \qquad st - ru = 1
$$

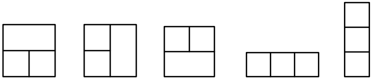
$$12213; \quad 22131; \quad 21312; \quad 13122; \quad 31221 :$$

The Catalan Problem

- Over 200 families of Catalan objects:
  Richard Stanley: "Catalan Numbers" (2015)
- Regular trickle of new families ...
- Alternative Tableau (2015) – related to Weyl algebra

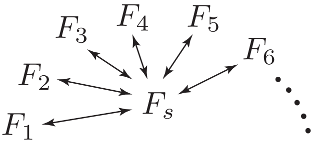


- Floor plans (2018)

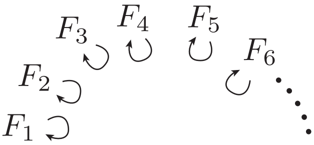


- How to prove Catalan: Focus on bijections
- **Problem I**: Too many bijections.
- Assume 200 families: $F_1; F_2; F_3; \square$
- $\square$ $\binom{200}{2} = 19\,900$ possible bijections

- Better: Biject to a common family

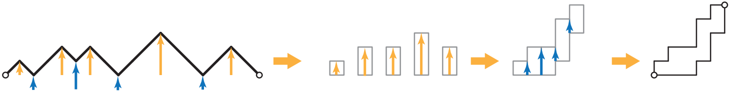


- Which family F_s?
- Even better:



- **Problem II**: Proofs can be lengthy

  Dyck words ↔ Staircase polygons (Delest & Viennot 1984)



- **Problem III**: Uniqueness:

  If $|A| = |B| = n$ then $n!$ possible bijections.

- Why choose any one?

The Magma

- Solution to all three problems:
- Replace "bijection" by "isomorphism"
- What algebra?
- Magma

Definition (Magma – Bourbaki 1970)

A **magma** defined on M is a pair $(M;\square)$ where $\square$ is a map

$$\square\,\square\, M \times M \to M$$

called the **product map** and M a non-empty set, called the **base set**.

- No conditions on map.

Additional definitions

- **Unique factorisation magma**: if product map $\square$ is injective.
- **Magma morphism**: Two magmas, $(M; \square)$ and $(N; \bullet)$ and a map

$$\square\; M \to N$$

satisfying

$$(m \,\square\, m') = (m) \bullet (m') :$$

- **Irreducible elements**: elements not in the image (range) of the product map.

Example magma

$\star$	1	2	3	4	5	$\ldots$
1	5	7	10	3	16	22
2	6	9	4	15	21	$\iddots$
3	8	4	14	20	27	$\iddots$
4	11	13	19	26		$\iddots$
5	12	18	25			$\iddots$
$\vdots$	17	24	$\iddots$	$\iddots$	$\iddots$	$\iddots$

- ii) Not a unique factorisation magma: $4 = 2 \star 3 = 3 \star 2$.
- iii) Two "irreducible" elements: 1, 2 absent.

Standard Free magma

Definition

Let X be a non-empty finite set. Define the sequence $W_n(X)$ of sets of nested 2-tuples recursively by:

$$W_1(X) = X$$

$$W_n(X) = \square_{p=1}^{n-1} W_p(X) \times W_{n-p}(X) ; \qquad n > 1;$$

$$\mathrm{W}_X = \square_{n \geq 1} W_n(X) :$$

Let $\mathrm{W}_X = \square_{n \geq 1} W_n(X)$.
Define the product map $\;\mathrm{W}_X \times \mathrm{W}_X \to \mathrm{W}_X$ by

$$m_1 \quad m_2 \; \square \; (m_1; m_2)$$

The pair $(\mathrm{W}_X; \;\;)$ is called the **standard free magma** generated by X.

- Elements of W_X for $X = \{\ \}$:

$$;\quad (\ ;\ );\quad (\ ;(\ ;\ ));\quad ((\ ;\ );\ );$$
$$(\ ;(\ ;(\ ;\ )));\quad ((\ ;(\ ;\ ));\ );\quad (\ ;((\ ;\ );\ ));$$
$$(((\ ;\ );\ );\ );\quad ((\ ;\ );(\ ;\ ))\ :::$$

- Three ways to write products:

$$;\qquad \text{and}\quad (\quad (\quad))$$

all give

$$(\ ;(\ ;\ )):$$

Norm

We need one additional ingredient to make connection with Catalan numbers.

- **Definition (Norm)**

  Let $(M; *)$ be a magma. A **norm** is a super-additive map

  $$\|\cdot\| : M \to \mathbb{N}$$

  .

- Super-additive: For all $m_1; m_2 \in M$ $\|m_1 * m_2\| \geq \|m_1\| + \|m_2\|$.
- If $(M; *)$ has a norm it will be called a **normed magma**.
- Standard Free magma norm: if $m \in W_n$ then $\|m\| = n$.
- eg. $\|(\bullet; (\bullet; \bullet))\| = 3$.

With a norm we now get:

Proposition (Segner 1761)

Let $\mathrm{W}(X)$ *be the standard free magma generated by the finite set* X*. If*

$$W_\ell = \{m \in \mathrm{W} : |m| = \ell\}; \qquad \ell \geq 1;$$

then

$$|W_\ell| = |X|^\ell \, C_{\ell-1} = |X|^\ell \, \frac{1}{\ell}\binom{2\ell-2}{\ell-1}; \tag{2}$$

and for a single generator, $X = \{"\}$*, we get the Catalan numbers:*

$$|W_\ell| = C_{\ell-1} = \frac{1}{\ell}\binom{2\ell-2}{\ell-1}: \tag{3}$$

Main theorem

- **Theorem (RB)**

  Let $(M;\square)$ *be a unique factorisation normed magma. Then* $(M;\square)$ *is isomorphic to the standard free magma* $W(X)$ *generated by the irreducible elements of* M*.*

  - Proof
    - Use norm to prove reducible elements have finite recursive factorisation.
    - Use injectivity to get bijective map to set of reducible elements.
    - Morphism straightforward.

Definition (Catalan Magma)

A unique factorisation normed magma with only one irreducible element is called a **Catalan magma**.

Consequences...

- If we can define a product

$$\square_i \; \square \; \mathsf{F}_i \times \mathsf{F}_i \to \mathsf{F}_i$$

  on a set F_i and:
  - show $\square_i$ is injective,
  - has one irreducible element
  - and define a norm, then
- F_i is a Catalan magma and F_i isomorphic to $W('')$:

$$\Gamma_i \; \square \; \mathsf{F}_i \to W('')$$

- and thus
  - Γ_i is in bijection,
  - norm partitions F_i into Catalan number sized subsets,
  - the bijection is recursive,
  - and embedded bijections, Narayana statistic correspondence, ...

- The proof is constructive and thus gives

$$\Gamma_i \ \square\ F_i \to W('')$$

explicitly.

- Furthermore, the bijection is "universal" – same (meta) algorithm for all pairs of families.

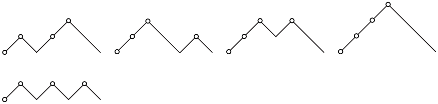
-

$$\begin{array}{ccc} F_i & \longrightarrow & W_{''_i} \\ \Big\downarrow{\scriptstyle \Gamma_{i,j}} & & \Big\downarrow{\scriptstyle i,j} \\ F_i & \longleftarrow & W_{''_j} \end{array} \qquad (4)$$

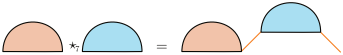
- Morphism implies recursive: $\Gamma(m_1 \,\square\, m_2) = \Gamma(m_1) \bullet \Gamma(m_2)$.

Example: Dyck path Magma

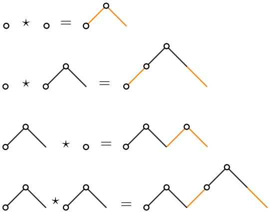
- Dyck Paths



- Product



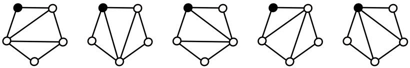
- Generator: ” = ○ (a vertex).
- Examples



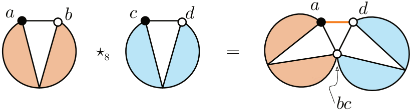
- Norm = Number of up steps + 1

Example: Triangulation Magma

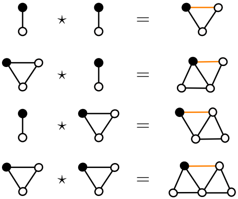
- Polygon Triangulation's



- Product:



- Generator = 
- Examples:



- Norm = (Number of triangles) + 1

Example: Frieze pattern Magma (Conway and Coxeter 1973)

F_{14} – Frieze Patterns: $n-1$ row periodic repeating rhombus

$$
\begin{array}{cccccccccccccccc}
\square & 1 & & 1 & & \square & & 1 & & \square & & & & & & \\
& \square & a_1 & & a_2 & & \square & & a_n & & \square & & & & & \\
& & \square & b_1 & & b_2 & & \square & & b_n & & \square & & & & \\
& & & & \square & & & & & & \square & & & & & \\
& & & & \square & r_1 & & r_2 & & \square & & r_n & & \square & & \\
& & & & & \square & 1 & & 1 & & \square & & 1 & & \square &
\end{array}
$$

with $\begin{array}{ccc} & r & \\ s & & t \\ & u & \end{array}$ and $st - ru = 1$

$$12213; \quad 22131; \quad 21312; \quad 13122; \quad 31221:$$

- Product: $a_1; a_2; :::; a_n \square b_1; b_2; :::; b_m = c_1; c_2; :::; c_{n+m-1}$ where

$$c_i = \begin{cases} a_1 + 1 & i = 1 \\ a_i & 1 < i < n \\ a_n + b_1 + 1 & i = n \\ b_i & n < i < n + m - 1 \\ b_m + 1 & i = n + m - 1 \end{cases} \tag{5}$$

- Generator: $'' = 00$.
- Examples:

$$00 \square 00 = 111$$

$$00 \square 111 = 1212$$

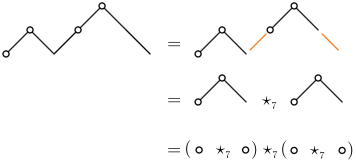
$$111 \square 00 = 2121$$

$$111 \square 111 = 21312$$

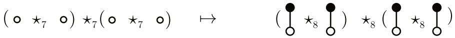
- Norm = (Length of sequence) $- 1$

Bijections

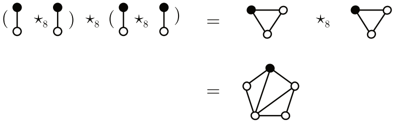
- First, factorise path to its generators



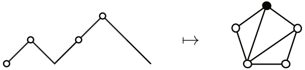
- then change generators and product rules: $\square_7 \rightarrow \square_8$:



- then re-multiply:



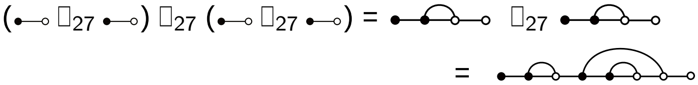
- which gives the bijection



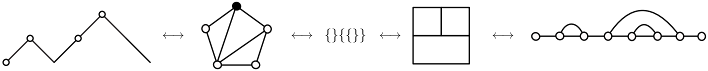
- Similarly, if we perform the same multiplications for matching brackets:

$$(\square\ \square_1\ \square)\ \square_1\ (\square\ \square_1\ \square) = \{\}\ \square_1\ \{\} = \{\}\{\{\}\}$$

- or for nested matchings,



- Thus we have the bijections:



Conclusion

- Magmatisation of Catalan families gives “universal” recursive bijection.
- Also, embedded bijections, Narayanaya statistic etc.
- Adding a unary map gives Fibonacci, with binary map gives Motzkin, Schröder paths etc.
- Current projects:
  - Extending to coupled algebraic equations eg. pairs of ternary trees
  - Reformulating the “symbolic” method.
- Reference: arXiv:1808.09078 [math.CO]

– Thank You –