Variations on the Erdős-Gallai Theorem

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The original Erdős-Gallai Theorem

The Erdős-Gallai Theorem is a fundamental, classic result that tells you when a sequence of integers occurs as the sequence of degrees of a simple graph. Here, "simple" means no loops or repeated edges. A sequence d of nonnegative integers is said to be graphic if it is the sequence of vertex degrees of a simple graph. A simple graph with degree sequence d is a realisation of d. There are several proofs of the Erdős-Gallai Theorem. A recent one is given in [17]; see also the papers cited therein. We follow the proof of Choudum [4].

Erdős-Gallai Theorem

A sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers in decreasing order is graphic iff its sum is even and, for each integer k with $1 \le k \le n$,

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \qquad (*)$$

Outline of Proof

Necessity is easy:

First, there is an even number of half-edges, so $\sum_{i=1}^{n} d_i$ must be even. Then, consider the set S comprised of the first k vertices. The left hand side of $(*)$ is the number of half-edges incident to S. On the right hand side, $k(k-1)$ is the number of half-edges in the complete graph on S, while $\sum_{i=k+1}^{n} \min\{k, d_i\}$ is the maximum number of edges that could join vertices in S to vertices outside S.

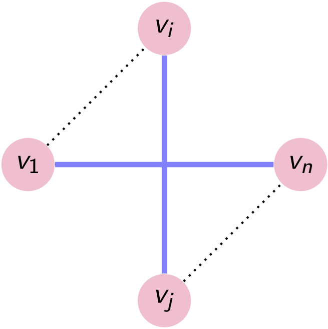
And for the sufficiency..

Sufficiency is by induction on $\sum_{i=1}^n d_i$. It is obvious for $\sum_{i=1}^n d_i = 2$. Suppose that $d = (d_1, \ldots, d_n)$ has even sum and satisfies $(*)$. Consider the sequence d' obtained by reducing both d_1 and d_n by 1. It is not difficult (but tiresome) to show that, when appropriately reordered so as to be decreasing, d' still satisfies $(*)$. So, by the inductive hypothesis, there is a simple graph G' that realises d'; label its vertices $v_1, \ldots, v_n$. We may assume there is an edge in G' connecting v_1 to v_n (otherwise we just add one). Applying the hypothesis to d, using $k = 1$ gives

$$d_1 \leq \sum_{i=2}^{n} \min\{k, d_i\} \leq n - 1,$$

and so $d_1 - 1 < n - 1$. Now in G', the degree of v_1 is $d_1 - 1$. So in G', there is some vertex $v_i \neq v_1$, for which there is no edge from v_1 to v_i. [So $v_i \neq v_n$]. Note that $d'_i > d'_n$. So there is a vertex v_j such that there an edge in G' from v_i to v_j, but there is no edge from v_j to v_n.

The trick



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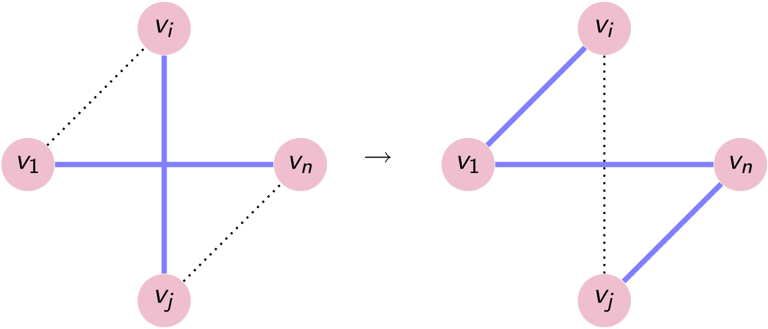

Figure: The Switcheroo

Comments

Remark

Notice that if d is the degree sequence of a simple graph, then d satisfies $(*)$ even if d isn't in decreasing order; indeed, the above proof of the necessity did not use the fact that the sequence is in decreasing order. The converse however is false; the sequence $(1, 3, 3, 3)$ satisfies $(*)$ but it is not the degree sequence of a simple graph.

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Remark

According to Wikipedia: Tibor Gallai (born Tibor Grünwald, July 15, 1912 January 2, 1992) was a Hungarian mathematician. He worked in combinatorics, especially in graph theory, and was a lifelong friend and collaborator of Paul Erdős. He was a student of Dénes König and an advisor of László Lovász. For comments by Erdős on Gallai, see [5, 6, 7].

Non-simple graphs

Theorem

If $d = (d_1, \ldots, d_n)$ *is in decreasing order, then*

(a) d *is the sequence of vertex degrees of a graph iff its sum is even,*

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(b) *d is the sequence of vertex degrees of a graph without loops iff its sum is even and $d_1 \leq \sum_{i=2}^{n} d_i$,*

(c) *d is the sequence of vertex degrees of a graph without multiple edges iff its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} d_i \leq k(k+1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \qquad (\dagger)$$

Remark

Part (a) is obvious. Part (b) is well known [11].

Proof of (a) and (b)

(a) really is obvious : at each vertex v_i, attach $\lfloor d_i/2 \rfloor$ loops. There are an even number of vertices for which the degrees d_i are odd: group these into pairs and join the vertices of each pair by an edge.

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(b) We argue by induction on $\sum_{i=1}^{n} d_i$. Suppose the degree sum is even. If $d_1 = \sum_{i=2}^{n} d_i$, just put in d_i edges between v_1 and v_i, for each i. If $d_1 < \sum_{i=2}^{n} d_i$, notice that d_1 is at least 2 less than the sum of the other degrees, since d_1 and $\sum_{i=2}^{n} d_i$ are either both odd or both even. Drop off 1 from the degrees of the 2 vertices of lowest degree, v_{n-1} and v_n. By the inductive hypothesis, there is a realisation without loops of $(d_1, \dots, d_{n-2}, d_{n-1} - 1, d_n - 1)$. Then add an edge between v_{n-1} and v_n.

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Conversely, if there is a realisation without loops of $(d_1, \dots, d_n)$, then, as before, the degree sum is even. Let $(d'_1, \dots, d'_n)$ be the degree sequence of the graph obtained by deleting all the edges not adjacent to v_1. So $d'_1 = d_1$ and $d'_i \leq d_i$ for all $i \geq 2$. Clearly $d'_1 = \sum_{i=2}^n d'_i$, so $d_1 \leq \sum_{i=2}^n d_i$.

Proof of (c)

The proof of sufficiency is by induction on $\sum_{i=1}^{n} d_i$. It is obvious for $\sum_{i=1}^{n} d_i = 2$. Suppose a decreasing sequence $d = (d_1, \ldots, d_n)$ has even sum and satisfies (†). As in Choudum's proof of the Erdős-Gallai Theorem, consider the sequence d' obtained by reducing both d_1 and d_n by 1. Let d'' denote the sequence obtained by reordering d' so as to be decreasing.

One can show (again tiresome) that when reordered in decreasing order, d' satisfies (†) and hence by the inductive hypothesis, there is a graph G' without multiple edges that realises d'. Let the vertices of G' be labelled $v_1, \ldots, v_n$. We may assume there is an edge in G' connecting v_1 to v_n (otherwise we can just add one). If there is no loop at either v_1 or v_n, remove the edge between v_1 and v_n, and add loops at both v_1 and v_n.

So we may assume there is a loop at either v_1 or v_n.

First assume there is a loop in G' at v_1.

Applying the hypothesis to d, using $k = 1$ gives

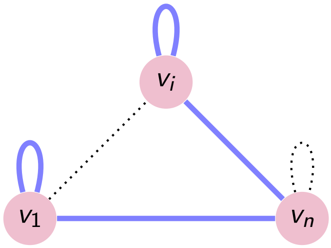
$$d_1 \leq 2 + \sum_{i=2}^{n} \min\{k, d_i\} \leq n + 1,$$

and so $d_1 - 3 < n - 1$. Now in G', the degree of v_1 is $d_1 - 1$ and so apart from the loop at v_1, there are a further $d_1 - 3$ edges incident to v_1. So in G', there is some vertex $v_i \neq v_1$, for which there is no edge from v_1 to v_i. [So $v_i \neq v_n$]. Note that $d'_i > d'_n$. If there is a loop in G' at v_n, or if there is no loop at v_i nor at v_n, then there is a vertex v_j such that there an edge in G' from v_i to v_j, but there is no edge from v_j to v_n. Then just do the Switcheroo: remove the edge $v_i v_j$, and put in edges $v_1 v_i$ and $v_j v_n$.

If there is no loop in G' at v_n, but there is a loop at v_i, we consider the two cases according to whether or not there is an edge between v_i and v_n.

If there is an edge between v_i and v_n, ...

If there is an edge between v_i and v_n, then remove this edge, add the edge $v_1 v_i$ and add a loop at v_n.



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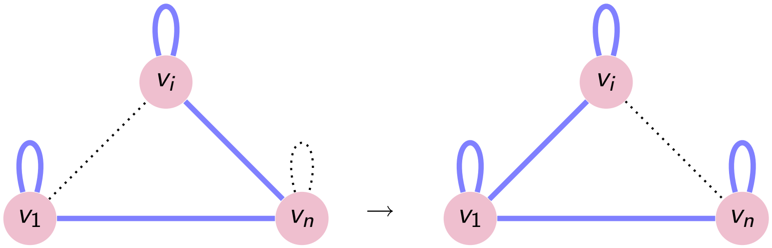

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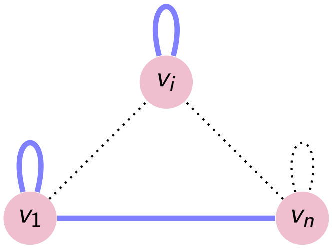

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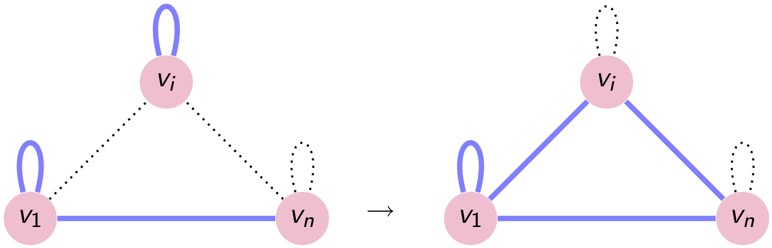
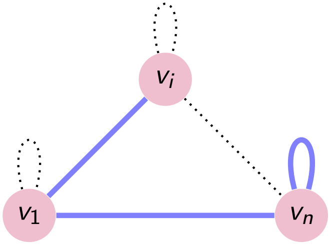


Finally, suppose there is a loop at v_n, but not at v_1.

Apart from the loop, there are a further $d_n - 3$ edges incident to v_n. Since $d_1 \geq d_n$, we have $d_1 - 1 > d_n - 3$, and so there is a vertex v_i such that there an edge in G' from v_1 to v_i, but there is no edge from v_i to v_n. Remove $v_1 v_i$, put in $v_i v_n$ and add a loop at v_1.



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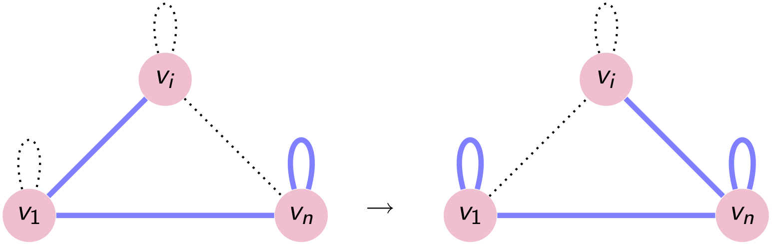

Connected graphs, Trees and Forests

Theorem

Let $d = (d_1, \ldots, d_n)$ *be a sequence of positive integers in decreasing order. Then* d *is the sequence of vertex degrees of a connected simple graph iff the Erdős-Gallai condition* $(*)$ *is satisfied and furthermore* $\sum_{i=1}^{n} d_i \geq 2(n-1)$*.*

Proof of sufficiency

We induct on $\sum_{i=1}^{n} d_i$. The result is clearly true for $\sum_{i=1}^{k} d_i = 2$.

Suppose that d satisfies $(*)$ and $\sum_{i=1}^{n} d_i \geq 2(n-1)$.

By the Erdős-Gallai Theorem, d is realised by a simple graph, and by Choudum's proof, so too can the sequence d' obtained from d by decreasing both d_1 and d_n by 1. If $d_n > 1$, then $\sum_{i=1}^{n} d_i \geq 2n$ and so $\sum_{i=1}^{n} d'_i \geq 2(n-1)$. If $d_n = 1$, then the sequence d' has at most $n' = n-1$ positive members and so $\sum_{i=1}^{n'} d'_i = \sum_{i=1}^{n} d_i - 2 \geq 2(n'-1)$. Hence, in either case, by the inductive hypothesis, d' is realized by a connected simple graph G'.

Now conlude as in Choudum's proof of the Erdős-Gallai Theorem; the key point is that the Switcheroo won't disconnect a connected graph.

For the converse, ...

Suppose that d is the sequence of vertex degrees of a connected simple graph G. So d satisfies $(*)$. If $d_n > 1$, then $\sum_{i=1}^{n} d_i \geq 2n$, which gives the required result. If $d_n = 1$, let G' be the connected simple graph obtained by removing vertex v_n and the single edge attached to it. So G' has $n' = n - 1$ vertices, and degree sequence d' where $\sum_{i=1}^{n'} d'_i = \sum_{i=1}^{n} d_i - 2$. By induction, $\sum_{i=1}^{n'} d'_i \geq 2(n' - 1)$, and hence $\sum_{i=1}^{n} d_i \geq 2(n - 1)$. $\square$

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Recall that a forest is a simple graph having no cycle, and a tree is a connected forest.

Theorem

Let $d = (d_1, \ldots, d_n)$ be a sequence of positive integers in decreasing order. Then $d = (d_1, \ldots, d_n)$ can be realised by a forest (resp. tree) iff $\sum_{i=1}^{k} d_i$ is even and $\sum_{i=1}^{k} d_i \leq 2(n-1)$ (resp. $\sum_{i=1}^{k} d_i = 2(n-1)$).

Bipartite graphs

First, notice that for multigraphs, the problem of realizing sequences as bipartite graphs is trivial. Indeed, it is obvious that a sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers is the sequence of vertex degrees of a bipartite graph (possibly with multiple edges) iff d can be written as the union of two disjoint parts $e = (e_1, \ldots, e_l)$ and $f = (f_1, \ldots, f_r)$ having the same sum. For simple bipartite graphs, the problem is more difficult. The Gale–Ryser Theorem [10, 13] is a natural generalization of the Erdős-Gallai Theorem.

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Gale–Ryser Theorem

A pair $e = (e_1, \ldots, e_l)$ and $f = (f_1, \ldots, f_r)$ of sequences of positive integers in decreasing order can be realized as the degree sequences of the parts of a bipartite graph iff they have the same sum and for all $1 \leq k \leq l$,

$$\sum_{i=1}^{k} e_i \leq \sum_{i=1}^{r} \min\{k, f_i\}. \qquad (\triangle)$$


Manfred Schocker (1970-2006)

There is a direct proof of the Gale–Ryser Theorem, similar to, and a little easier than Choudum's proof of the Erdős-Gallai Theorem.

Manfred Schocker [15] proved that the Erdős-Gallai and Gale–Ryser theorems are equivalent, in the sense that each one can be deduced from the other. The original papers by Gale and Ryser were both formulated in terms of matrices, rather than graphs, and their proofs were also matrix based. This is also the case with Schocker's proofs.

There is another way

There is another way of deducing Gale–Ryser from Erdős-Gallai. Given $e = (e_1, \ldots, e_l)$ and $f = (f_1, \ldots, f_r)$ satisfying the Gale–Ryser condition, define d as follows: suppose $l \geq r$, and set

$$d = (e_1 + l - 1, e_2 + l - 1, \ldots, e_l + l - 1, f_1, \ldots, f_r).$$

A little (tiresome) work shows that d verifies the Erdős-Gallai condition, and so d can be realized by a simple graph G', with respective vertices $v_1, \ldots, v_l, w_1, \ldots, w_r$, say. The degree sequence d has been chosen so that the restriction of G' to $v_1, \ldots, v_l$ is a complete graph; deleting these edges we obtain the required realization of e, f.

For other proofs and related results, see [2, Chapter 7] and [18, Section 4.3].

Other results

There are a number of results related to the Erdős-Gallai Theorem; see [16] and [9, Theorem 5.1]. In fact, some of them predate the Erdős-Gallai Theorem. Several of these are of the following form: if d' is obtained from d by reducing certain degrees, then d' is graphic if d is graphic. For example, there is the Kleitman–Wang theorem [12]:

Theorem

If $d = (d_1, \ldots, d_n)$ is graphic, then so is the sequence d' obtained by deleting one of the d_i and subtracting 1 from each of the d_i largest terms remaining.

And questions

There are many papers sequences that can be realised by planar graphs. A lot of things are known, but we are long way from having a complete solution. In particular, there are specific concrete sequences for which it is not known whether they are graphical. In [14], they write: "The great variety of seemingly unrelated degree sequences which are not planar graphical (... for example, that $6^{p-4}3^4$ is planar graphical if and only if $p \geq 8$ and p is even) strongly suggests that the complete solution to the above problem is out of reach."

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There are many other natural questions. Basically, given any property of graphs, you can ask which degree sequences can be realised by graphs having that property. There are lots of papers of this kind: on Hamiltonian graphs, bipartite graphs, triangle free graphs, etc. For example, see [8].

And a final, remarkable result

The Gale–Ryser Theorem is not the final word on the degree sequences of bipartite graphs. In a recent paper, Alon, Ben-Shimon and Krivelevich proved the following remarkable result [1, Corollary 2.2]:

Theorem

Let $a \geq 1$ be a real. If $d = (d_1, \ldots, d_n)$ is a list of nonnegative integers in decreasing order and

$$d_1 \leq \min\left\{a \cdot d_n, \frac{4an}{(a+1)^2}\right\},$$

then there exists a simple bipartite graph with degree sequence d on each side. In particular, this holds for $d_1 \leq \min\{2d_n, \frac{8n}{9}\}$.

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