Combinatorial representations

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December 2011

Joint work with Max Gadouleau and Søren Riis
see arXiv 1109.1216

Matroids

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A matroid can be described in many different ways: by the independent sets, the bases, the minimal dependent sets, the rank function ...

Matroid representations

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Let E be the ground set and $\mathcal{B}$ the family of bases of a matroid M of rank r. A vector representation of M is an assignment of a vector $v_i \in F^r$ to each $i \in E$, such that, for $i_1, \ldots, i_r \in E$,

$$(v_{i_1}, \ldots, v_{i_r}) \text{ is a basis for } F^r \Leftrightarrow \{i_1, \ldots, i_r\} \in \mathcal{B}.$$

. . . in dual form

Now regard the representing vectors $v_1, \ldots, v_r$ as lying in the dual space of F^r. To emphasise this I will write f_i instead of v_i; thus f_i is a function from F^r to F.

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Notation: if $f_{i_1}, \ldots, f_{i_r} : F^r \to F$, then we regard the r-tuple $(f_{i_1}, \ldots, f_{i_r})$ as being a function from F^r to F^r.

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Notation: if $f_{i_1}, \ldots, f_{i_r} : F^r \to F$, then we regard the r-tuple $(f_{i_1}, \ldots, f_{i_r})$ as being a function from F^r to F^r.
Now a vector representation of the matroid M is an assignment of a linear map $f_i : F^r \to F$ to each $i \in E$, so that

$$(f_{i_1}, \ldots, f_{i_r}) : F^r \to F^r \text{ is a bijection } \Leftrightarrow \{i_1, \ldots, i_r\} \in \mathcal{B}.$$

. . . generalised

Let $\mathcal{B}$ be any family of r-subsets of a ground set E, and let A be an alphabet of size q. A combinatorial representation of $(E, \mathcal{B})$ over A is an assignment of a function $f_i : A^r \to A$ to each point $i \in E$ so that

$$(f_{i_1}, \ldots, f_{i_r}) : A^r \to A^r \text{ is a bijection } \Leftrightarrow \{i_1, \ldots, i_r\} \in \mathcal{B}.$$

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Thus any vector representation of a matroid, dualised, is a combinatorial representation.
If $X = \{i_1, \ldots, i_r\}$, we denote $(f_{i_1}, \ldots, f_{i_r})$ by f_X.

An example

Let $n = 4$ and $\mathcal{B} = \{\{1,2\},\{3,4\}\}$. A combinatorial representation over a 3-element set $\{a,b,c\}$ is given by taking f_1 and f_2 to be the two coordinate functions (that is, $f_1(x,y) = x$ and $f_2(x,y) = y$), and f_3 and f_4 by the tables

b	a	a
b	c	b
c	c	a

and

b	b	c
a	c	c
a	b	a

.

Note that $(E, \mathcal{B})$ is not a matroid.

A normalisation

Suppose that $b = \{i_1, \ldots, i_r\} \in \mathcal{B}$. Define functions g_i, for $i \in E$, by

$$g_i(x_1, \ldots, x_r) = f_i(y_1, \ldots, y_r),$$

where $(y_1, \ldots, y_r)$ is the inverse image of $(x_1, \ldots, x_r)$ under the bijection f_b. These functions also define a combinatorial representation, with the property that g_{i_j} is the jth coordinate function. So, where necessary, we may suppose that the first r elements of E form a basis and the first r functions are the coordinate functions. This transformation can be viewed as a change of variables.

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Proof.

We verify the exchange axiom. Let $B_1, B_2 \in \mathcal{B}$; we may assume that the elements of B_1 are the coordinate functions. Now consider the $r-1$ functions f_i for $i \in B_2$, $i \neq k$, for some fixed $k \in B_2$. These define a surjective function from F^r to F^{r-1}. Take any non-zero vector in the kernel, and suppose that its lth coordinate is non-zero. Then it is readily checked that the functions with indices in $B_2 \setminus \{k\} \cup \{l\}$ give a bijection from F^r to F^r; so this set is a basis. □

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Let $(E, \mathcal{B}_1)$ and $(E, \mathcal{B}_2)$ be families of r-sets, which have representations over alphabets of cardinalities q_1 and q_2 respectively. Then $(E, \mathcal{B}_1 \cap \mathcal{B}_2)$ has a representation over an alphabet of size $q_1 q_2$.

Now, to prove the theorem, we observe that

$$\mathcal{B} = \bigcap_{C \notin \mathcal{B}} \left(\binom{E}{r} \setminus \{C\} \right)$$

so it is enough to represent the family consisting of all but one of the r-sets; and it is straightforward to show that this family is indeed a representable matroid.

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Note that our proof shows that in fact every set family has a representation by "matrix functions". More on this later.

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so it is enough to represent the family consisting of all but one of the r-sets; and it is straightforward to show that this family is indeed a representable matroid.
Note that our proof shows that in fact every set family has a representation by "matrix functions". More on this later.

Question

Given a set family, what are the cardinalities of alphabets over which it has a combinatorial representation?

Graphs

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As a warm-up, let us consider the complete graph. It is readily checked from the definitions that a representation of K_n over an alphabet of size q is the same thing as a set of $n - 2$ mutually orthogonal Latin squares of order q; these are known to exist for all sufficiently large q.

Pairwise balanced designs

A pairwise balanced design, or PBD, consists of a set X and a collection $\mathcal{L}$ of subsets of X (each of size greater than 1) such that every two points of X are contained in a unique "line" in $\mathcal{L}$. If the line sizes all belong to the set K of positive integers, we call it a PBD(K).

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A set K of positive integers is PBD-closed if, whenever there exists a PBD(K) on a set of size v, then $v \in K$.

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A set K of positive integers is PBD-closed if, whenever there exists a PBD(K) on a set of size v, then $v \in K$.
Given K, we define

$$\begin{aligned} \alpha(K) &= \gcd\{k-1 : k \in K\}, \\ \beta(K) &= \gcd\{k(k-1) : k \in K\}. \end{aligned}$$

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This is the essential tool in the proof of our theorem.

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We claim that the set K of alphabet sizes for which the given graph Γ has an idempotent representation is PBD-closed.
Let $(X, \mathcal{L})$ be a PBD, and suppose that Γ has a representation (f^L) with alphabet L, for every line $L \in \mathcal{L}$. Define a representation (f) of Γ over X by the rule that $f_i(x,x) = x$, while if $x \neq y$ then

$$f_i(x,y) = f_i^L(x,y),$$

where L is the unique line containing x and y. It is readily checked that this is a combinatorial representation.

Now it is straightforward to see that the set K of alphabet sizes over which Γ has a combinatorial representation satisfies $\alpha(K) = 1$ and $\beta(K) = 2$. (Using the proof of the first theorem, we see that K contains a sufficiently high power of any prime.)

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Question

Does an analogous result hold for families of r-sets with $r > 2$?

Matrix representations

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In particular, if two families have linear representations over F, then their intersection has a "representation by two-rowed matrices", each point associated with a function from $(F^r)^2$ to F^2.

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There are families which do not have representations by two-rowed matrices. An example is given by

$$E = \{1, \ldots, 7\}, \mathcal{B} = \{\{1,2\}, \{3,4\}, \{5,6\}, \{5,7\}, \{6,7\}\}.$$

The proof of non-representability uses the Ingleton inequality.

Rank functions

A rank function for a set family $(E, \mathcal{B})$ is a function $\mathrm{rk} : 2^E \to [0, r]$ satisfying

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The first three conditions are equivalent to the definition of a polymatroid.

Rank functions from representations

Theorem

Let $f = (f_i)$ be a representation of $(E, \mathcal{B})$ over an alphabet X of size q. Then the function r_f, defined by $r_f(S) = H(f_S)$, is a rank function for $(E, \mathcal{B})$.

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Here H is the q-ary entropy function given by

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The converse is false; there are rank functions which do not arise from any combinatorial representation.

Bounds for rank functions

If we set $r_m(X) = \max_{B \in \mathcal{B}} |B \cap X|$ and $r_M(X) = \min\{r, |X|\}$, (so that r_M is the rank function for the uniform matroid of rank r), then it is easy to see that $r_m(X) \leq \mathrm{rk}(X) \leq r_M(X)$.

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On the other hand, we have:

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Any family $(E, \mathcal{B})$ has a rank function which takes integer or half-integer values (or indeed, values in the rationals with denominator dividing p, for any $p > 1$).

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An example of such a function is given by

$$\mathrm{rk}(X) = \begin{cases} |X| & \text{if } |X| \leq r-1 \text{ or } X \in \mathcal{B}, \\ r - 1/p & \text{if } |X| = r,\ X \notin \mathcal{B}, \\ r & \text{if } |X| \geq r+1. \end{cases}$$

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We see that the function r_M is the supremum of all rank functions for $(E, \mathcal{B})$, and can be approached arbitrarily closely.

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Let $(E, \mathcal{B})$ *be a set family of rank* r*. Then there is a set* X *with* $|X| = r$ *and* $r_m(X) = (r + I)/2$*, where*

$$I = \min_{B \in \mathcal{B}} \max_{C \in \mathcal{B}, C \neq B} |B \cap C|.$$

Moreover, for any rank function rk*, we have*

$$\mathrm{rk}(X) - r_m(X) \geq (r - I)/4.$$

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So a basis disjoint from all other bases leads to large differences between any rank function and the lower bound r_m.

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Not every closure operator (satisfying the first three conditions) comes from a rank function.

Closure in a representation

If the rank function arises from a combinatorial representation $f = (f_e : e \in E)$, then we have

$$\mathrm{cl}(X) = \{e \in E : f_X \text{ refines } f_e\}.$$

(We say that f_1 refines f_2 if $f_1(x) = f_1(y)$ implies $f_2(x) = f_2(y)$.)