Constructing $(0,1)$-matrices with large minimal defining sets

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Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors of non-negative integers such that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Then $\mathcal{A}(R, S)$ is defined to be the set of all $m \times n$ $(0,1)$-matrices with r_i 1's in row i and s_j 1's in column j, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

The Gale-Ryser Theorem for dummies

A matrix M is the unique element of $\mathcal{A}(R, S)$ if and only if a curve $\mathcal{C}$ of non-positive gradient can be drawn with all the 0's above $\mathcal{C}$ and all of the 1's below the curve $\mathcal{C}$.

We say that a partially filled-in $(0, 1)$-matrix D is a *defining set* for M if M is the unique member of $\mathcal{A}(R, S)$ such that $D \subseteq M$.

Given a $(0,1)$-matrix M, the size of the smallest defining set in M is denoted by $\text{scs}(M)$. We define:

$$\text{scs}(\mathcal{A}(R,S)) = \min\{\text{scs}(M) \mid M \in \mathcal{A}(R,S)\}.$$

We also define

$$\text{maxscs}(\mathcal{A}(R,S)) = \max\{\text{scs}(M) \mid M \in \mathcal{A}(R,S)\}.$$

We define $A_{n,x} = \mathcal{A}(R_{n,x}, S_{n,x})$ to be the set of $n \times n$ $(0, 1)$-matrices with constant row and column sum x. Elements of $A_{n,x}$ may also be thought of as *frequency squares*

Theorem. (Cavenagh, 2013). Any defining set D in a matrix from $A_{n,x}$ has size at least $\min\{x^2, (n-x)^2\}$.

Corollary. $\text{scs}(A_{2m,m}) = m^2$.

Our main new result is the following:

Theorem. (Cavenagh and Ramadurai, 2017) If m is a power of two, $\mathrm{maxscs}(A_{2m,m}) = 2m^2 - O(m^{7/4})$.

The analogous question has been considered for Latin squares.

Theorem. (Ghandehari, Hatami and Mahmoodian, 2005): Every Latin square of order n has a defining set of size at most $n^2 - \frac{\sqrt{\pi}}{2} n^{9/6}$ and that for each n there exists a Latin square L with no defining set of size less than $n^2 - (e + o(1))n^{10/6}$.

Theorem. The set D is a defining set of a $(0, 1)$-matrix M if and only if $D \subset M$ and the rows and columns of $M \setminus D$ can be rearranged so that there exista curve $\mathcal{C}$ of non-positive gradient such that there are only 0's in $M \setminus D$ above $\mathcal{C}$ and only 1's in $M \setminus D$ below $\mathcal{C}$.

	0	1	0	0	1	1	0	1
	0	0	1	0	1	0	1	1
	0	0	0	1	0	1	1	1
$x_1 = 0$	1	0	1	1	0	0	1	0
$x_2 = 1$	0	0	1	0	1	0	1	1
$x_3 = 1$	0	0	0	1	0	1	1	1
$x_1 + x_2 = 1$	0	1	1	0	0	1	1	0
$x_1 + x_3 = 0$	1	0	1	0	0	1	0	1
$x_2 + x_3 = 1$	0	0	1	1	1	1	0	0
$x_1 + x_2 + x_3 = 1$	0	1	1	1	0	0	0	1

The matrix $M(0, 1, 1, 1, 0, 1, 1)$, $k = 3$.

Theorem. For each choice of V, $\Delta(V) \leq m^{3/2}$, where $m = 2^k$.

Let $k = 2$. Then B is given below.

	0 0 0	1 0 0	0 1 0	0 0 1	1 1 0	1 0 1	0 1 1	1 1 1
$x_1 = 0$	1	0	1	1	0	0	1	0
$x_2 = 1$	0	0	1	0	1	0	1	1
$x_1 + x_2 = 1$	0	1	1	0	0	1	1	0
$x_3 = 1$	0	0	0	1	0	1	1	1
$x_1 = 1$	0	1	0	0	1	1	0	1
$x_2 = 0$	1	1	0	1	0	1	0	0
$x_1 + x_2 = 0$	1	0	0	1	1	0	0	1
$x_3 = 0$	1	1	1	0	1	0	0	0

Lemma. Let R and C be any subsets of the rows and columns, respectively, of B. Then the difference between the number of 0's and the number of 1's in the submatrix defined by $R \times C$ is at most $2m^{3/2} + m$.