Cycles of length 3 and 4 in tournaments

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Context

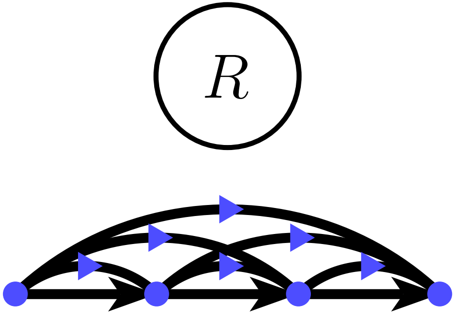
Mantel 1907: Any graph with more than $\lfloor n^2/4 \rfloor$ copies of K_2 contains a copy of K_3.

Erdős-Rademacher problem: If a graph exceeds $\lfloor n^2/4 \rfloor$ copies of K_2, *how many* copies of K_3 are forced?
A: Asymptotically solved by Razborov 2008, using flag algebras.

Topic of this talk: Analogous problem for **tournaments**.

Tournaments

Complete graph with every edge given a direction.
e.g. random tournament, transitive tournament



Main question

Q: What is the minimum number of K_3's in a graph with a given number of K_2's?

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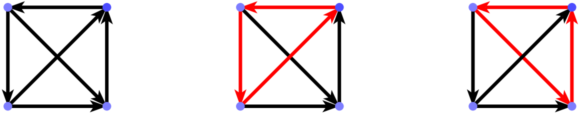
density:
$c_\ell(T) :=$ probability that a random mapping from $V(C_\ell)$ to $V(T)$ is a homomorphism i.e. arcs of C_ℓ map to arcs of T.

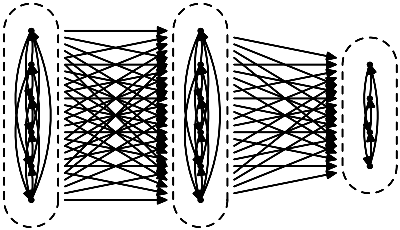


$$c_3(T) = (3+3)/4^3 = 3/32$$

Q: Given $c_3(T)$, *asymptotically* minimise $c_4(T)$.

An extremal construction?

Fix $z \in [0,1]$. Create as many blocks of vertices of size zn as possible, and put the remaining $\leq zn$ vertices in a single block. Edges within blocks behave randomly, edges between blocks go to the right.



"random blow-up of a transitive tournament"

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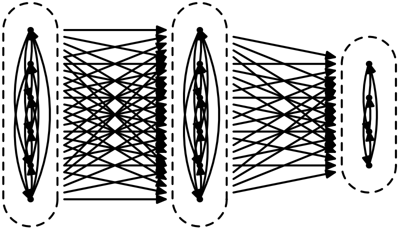

"random blow-up of a transitive tournament"

$$c_3(T) = \frac{1}{8}\left(\lfloor z^{-1} \rfloor z^3 + \left(1 - \lfloor z^{-1} \rfloor z\right)^3\right) + o(1)$$

$$c_4(T) = \frac{1}{16}\left(\lfloor z^{-1} \rfloor z^4 + \left(1 - \lfloor z^{-1} \rfloor z\right)^4\right) + o(1)$$

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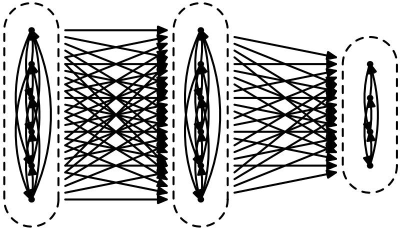

"random blow-up of a transitive tournament"

Conjecture (Linial & Morgenstern 2016)

For every tournament T,

$$c_4(T) \geq g(c_3(T)) + o(1).$$

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The above conjecture is true for $c_3(T) \geq 1/72$*. Furthermore, we characterise the extremal tournaments when* $c_3(T) \geq 1/32$*.*

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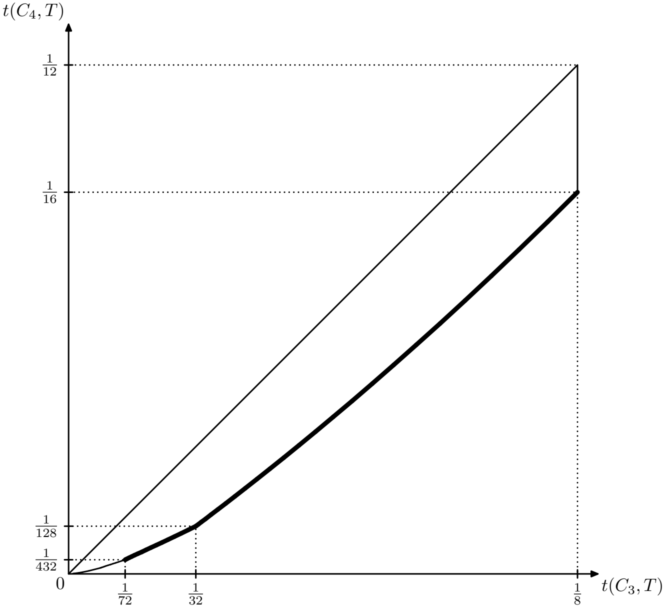
Notes:

- Behaviour appears similar to the Razborov result
- Proof uses spectral methods instead of flag algebras
- The space of extremal tournaments is surprisingly large!

The c_3-c_4 profile



Aside: The upper bound

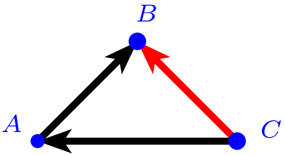
Upper bound is $c_4(T) \leq \frac{2}{3}c_3(T) + o(1)$.

Bottom left construction ($c_3 = 0, c_4 = 0$): transitive tournament

Upper right construction ($c_3 = 1/8, c_4 = 1/12$): the "circular" tournament, edges directed from v_i to $v_{i+1}, \ldots v_{i+n/2}$ for each i (indices modulo n)

The spectral approach

tournament matrix: non-negative square matrix satisfying
$A + A^T =$ matrix of ones.
tournament $\mapsto$ tournament matrix by taking the usual (directed) adjacency matrix and replacing the diagonal entries with $1/2$.



$$\begin{bmatrix} 1/2 & 1 & 0 \\ 0 & 1/2 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

The spectral approach

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Fact: If A is the tournament matrix corresponding to a T of order n and $\ell \geq 3$, then the number of homomorphisms from C_ℓ to T is $\mathrm{Tr}(A^\ell) + O(n^{\ell-1})$.

density:

$$\sigma_\ell(A) := \frac{1}{n^\ell} \operatorname{Tr} A^\ell \leftrightarrow c_\ell(T)$$

Fact:

$$Tr(A^\ell) = \sum_{i=1}^{n} \lambda_i^\ell,$$

where the λ_i are the eigenvalues of A.

Rephrasing the problem

Minimise $c_4(T)$ for fixed $c_3(T)$
$\iff$ Minimise $\mathrm{Tr}(A^4)$ for fixed $\mathrm{Tr}(A^3)$
$\iff$ Minimise the sum of 4th powers of the eigenvalues of A, given a fixed the sum of 3rd powers

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Lemma (Linial & Morgenstern)

Let $x_1, \ldots, x_n$ be non-negative **real** numbers summing to $1/2$. Then

$$x_1^4 + \cdots + x_n^4 \geq g(x_1^3 + \ldots x_n^3).$$

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Problem: What if the eigenvalues are complex?

Taking a step back

general case: A has eigenvalues

- ρn, the spectral radius
- $r_1 n, \ldots, r_k n$, the remaining real eigenvalues
- $(a_1 \pm \iota b_1)n, \ldots, (a_\ell \pm \iota b_\ell)n$, conjugate pairs of complex eigenvalues

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Optimization problem Spectrum

Parameters:	reals $c_3 \in [0, 1/8]$ and $\rho \in [0, 1/2]$
	non-negative integers k and ℓ
Variables:	real numbers $r_1, \ldots, r_k$, $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$
Constraints:	$0 \le r_1, \ldots, r_k \le \rho$
	$0 \le a_1, \ldots, a_\ell$
	$\rho + \sum_{i=1}^{k} r_i + 2\sum_{i=1}^{\ell} a_i = 1/2$
	$\rho^3 + \sum_{i=1}^{k} r_i^3 + 2\sum_{i=1}^{\ell} \left(a_i^3 - 3a_i b_i^2\right) = c_3$
Objective:	minimize $\rho^4 + \sum_{i=1}^{k} r_i^4 + 2\sum_{i=1}^{\ell} \left(a_i^4 - 6a_i^2 b_i^2 + b_i^4\right)$

Structure of optimal solutions

Key lemma

Let $r_1, \ldots r_k, a_1, \ldots a_\ell, b_1, \ldots, b_\ell$ be an optimal solution to the optimisation problem. Then one of the following holds:

1. There exist positive reals r' and r'' such that $r_1, \ldots, r_k \in \{0, r', r'', \rho\}$ and $(a_1, b_1), \ldots, (a_\ell, b_\ell) \in \{(0,0), (r', 0), (r'', 0)\}$.
2. There exist reals a' and $b' \neq 0$ such that $r_1, \ldots, r_k \in \{0, \rho\}$ and $(a_1, b_1), \ldots, (a_\ell, b_\ell) \in \{(0,0), (a', b'), (a', -b')\}$.

Structure of optimal solutions

Key lemma

Let $r_1, \dots r_k, a_1, \dots a_\ell, b_1, \dots, b_\ell$ be an optimal solution to the optimisation problem. Then one of the following holds:

1. All eigenvalues are real
2. There exist reals a' and $b' \neq 0$ such that $r_1, \dots, r_k \in \{0, \rho\}$ and $(a_1, b_1), \dots, (a_\ell, b_\ell) \in \{(0,0), (a', b'), (a', -b')\}$.

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1. All eigenvalues are real
2. Up to multiplicity, there's only real eigenvalue and one pair of complex eigenvalues

A key ingredient

Our optimisation problem involves:

- An objective function f (sum of 4th powers of e-values)
- Two constraint functions $g_1 = 1/2$ and $g_2 = c_3$ (sum of e-values and 3rd powers of e-values)
- Some boundary conditions on the r_i and a_i

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- Some boundary conditions on the r_i and a_i

The method of *Lagrange multipliers* tells us that the extrema of f in the feasible set occur at

- the boundary of the feasible set, or
- where $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ for some constants λ_1, λ_2.

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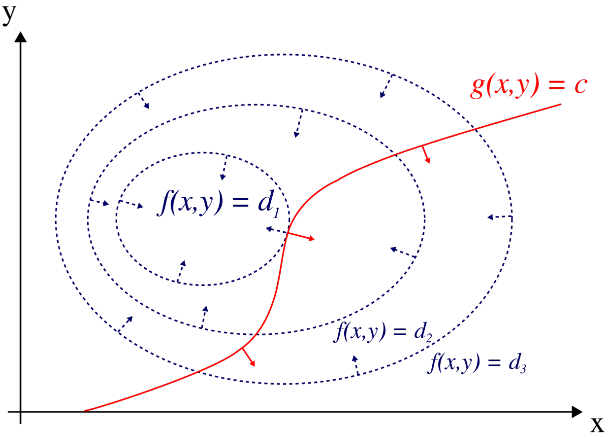

Theorem revisited

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If $c_3(T) \geq 1/72$, *then* $c_4(T) \geq g(c_3(T)) + o(1)$.

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Furthermore, we characterise the extremal tournaments for $c_3(T) \geq 1/32$*.*

To get all the extremal tournaments on n vertices:

- Associate each vertex v_i with a real number $p_i \in [0, 1/2]$
- Direct the edge $v_i v_j$ from i to j with probability $1/2 + p_i - p_j$.
- The resulting tournament has $c_4(T) = g(c_3(T)) + o(1)$ w.h.p.

Open problems

- Obvious: Prove the conjecture for remaining values of c_3, and find all the extremal examples
- Related: Study *profiles* of graphs and tournaments