The Conjecture - Polynomial Phenomenon
The Erdős-Hajnal Conjecture - Polynomial Phenomenon

For every undirected graph $H$ there exist $\epsilon(H), c(H) > 0$ such that every graph $G$ not containing $H$ as an induced subgraph contains a clique or a stable set of size at least $c(H)|G|^{\epsilon(H)}$. 
The Erdős-Hajnal Conjecture

For every undirected graph $H$ there exist $\epsilon(H), c(H) > 0$ such that every graph $G$ not containing $H$ as an induced subgraph contains a clique or a stable set of size at least $c(H)|G|^{\epsilon(H)}$.

The Erdős-Hajnal Conjecture - directed version

For every tournament $H$ there exist $\epsilon(H), c(H) > 0$ such that every tournament $T$ not containing $H$ as an induced subtournament contains a transitive subtournament of size at least $c(H)|T|^{\epsilon(H)}$. 
The Erdős-Hajnal Conjecture

For every undirected graph $H$ there exist $\epsilon(H), c(H) > 0$ such that every graph $G$ not containing $H$ as an induced subgraph contains a clique or a stable set of size at least $c(H)|G|^{\epsilon(H)}$.

The Erdős-Hajnal Conjecture - directed version

For every tournament $H$ there exist $\epsilon(H), c(H) > 0$ such that every tournament $T$ not containing $H$ as an induced subtournament contains a transitive subtournament of size at least $c(H)|T|^{\epsilon(H)}$. 
What was known...
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Directed case revolution
Directed case revolution
Directed case revolution

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Tournaments satisfying the Conjecture in the linear sense

Definition

Tournament $H$ is a celebrity if there exists $c(H) > 0$ such that every $H$-free $n$-vertex tournament contains a transitive subtournament of order at least $c(H)n$.

Theorem (Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, Thomasse ’11)

*Tournament $H$ is a celebrity iff either:*

- *it is not strongly connected and is of the form $H_1 \rightarrow H_2$, where $H_1, H_2$ are celebrities or,*
- *is strongly connected and is of the form $\Delta(1, T_k, D)$, where $D$ is a celebrity and $T_k$ is a transitive tournament on $k$ vertices.*
Tournaments satisfying the Conjecture in the linear sense

**Definition**

A dichromatic number $\chi(T)$ of the tournament $T$ is the smallest number of colors that can be used to color its vertices in such a way that there does not exist a monochromatic directed cycle.

**Theorem (Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, Thomasse ’11)**

*Tournament $H$ is a celebrity iff there exists $d(H)$ such that every $H$-free tournament $T$ satisfies:*

$$\chi(T) \leq d(H).$$
Directed case revolution
Directed case revolution
Directed case revolution

$\Delta (2,2,2)$

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Tournaments satisfying the Conjecture in the pseudolinear sense

**Definition**

Tournament $H$ is a pseudocelebrity if it is not a celebrity, but there exist $c(H), d(H) > 0$ such that every $n$-vertex $H$-free tournament $T$ satisfies: $\chi(T) \leq c(H) \log^{d(H)}(n)$.

**Theorem (Choromanski, Chudnovsky, Seymour '12)**

**Tournament $H$ is a pseudocelebrity if it is of the form:**

- $H_1 \implies H_2$, where both $H_i$s are pseudocelebrities or one is a celebrity and the other one is a pseudocelebrity or
- $\Delta(1, T_k, H)$ or $\Delta(2, T_k, T_k)$, where $H$ is a pseudocelebrity.
Directed case revolution
Directed case revolution

Δ(2,2,2)
Directed case revolution

\[ \Delta(2,2,2) \]
Directed case revolution

The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings.
Galaxies, constellations, nebulae...
Galaxies, constellations, nebulae...
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Galaxies, constellations, nebulae...
Infinitely many prime Erdős-Hajnal tournaments

**Theorem (Berger, Choromanski, Chudnovsky ’12)**

*Every galaxy satisfies the Erdős-Hajnal Conjecture. In particular, every directed path satisfies the Conjecture.*

**Theorem (Choromanski ’12)**

*Tournament $C_5$ satisfies the Conjecture.*

**Corollary**

*Every tournament on at most five vertices satisfies the Conjecture.*
Galaxies, constellations, nebulae...
Galaxies, constellations, nebulae...
Galaxies, constellations, nebulae...
The Erdős-Hajnal Conjecture, structured nonlinear graph-based hashing and anonymization via perfect matchings

Going beyond galaxies

Theorem (Choromanski ’12)

*Every constellation satisfies the Erdős-Hajnal Conjecture.*
Galaxies, constellations, nebulae...
The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings

Galaxies, constellations, nebulae...
Combining tournaments...

**RIGHT – SIDED**
Combining tournaments...
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Combining tournaments...
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Combining tournaments...
Combining tournaments...
Combining tournaments...
Combining tournaments...
Combining tournaments...

RIGHT – SIDED

LEFT – SIDED

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Combining tournaments...
Combining tournaments...

RIGHT – SIDED

LEFT – SIDED

BOTH – SIDED
Combining tournaments...

**RIGHT – SIDED**

**LEFT – SIDED**

**BOTH – SIDED**

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Combining tournaments...

RIGHT - SIDED

LEFT - SIDED

BOTH - SIDED
Combining tournaments...

$H_1 \oplus H_2 \oplus H_3$
Combining tournaments...
Combining tournaments...
Combining tournaments...

$H_4$
Combining tournaments...
Combining tournaments...
Combining tournaments...
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Combining tournaments...

**Theorem (Choromanski ’13)**

*There exists a generic procedure for constructing larger prime tournaments satisfying the conjecture from smaller ones.*
Galaxies, constellations, nebulae...
Galaxies, constellations, nebulae...

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The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings
Definition

Tournament is a **nebula** if it has an ordering of vertices under which the graph of backward edges is a collection of vertex disjoint stars.
Nebulae...

Definition

Tournament is a **nebula** if there exists its ordering of vertices under which the graph of backward edges is a collection of vertex disjoint stars.

Definition

Tournament is a **left/right nebula** if it has an ordering of vertices under which the graph of backward edges is a collection of vertex disjoint left/right stars.
Let $N_l$ be a left nebula and $N_r$ be a right nebula. Then there exists $\epsilon(N_l, N_r) > 0$ such that every $\{N_l, N_r\}$-free $n$-vertex tournament contains a transitive subtournament of order at least $n^{\epsilon(N_l, N_r)}$. 
Nebulae of small stars
Nebulae of small stars
Nebulae of small stars
Nebulae of small stars

Theorem (Choromanski ’14)

Let $N^s_l$ be a small left nebula and $N^s_r$ be a small right nebula. Then there exists $\epsilon(N^s_l, N^s_r) > 0$ such that every $\{N^s_l, N^s_r\}$-free $n$-vertex tournament contains a transitive subtournament of order at least $n^{\epsilon(N^s_l, N^s_r)}$. 
Hardcore nebulae...

**Definition**

Let $S$ be a left/right star. We call the set of vertices of $S$ other than its first and last vertex a **core**.

**Definition**

A tournament is called a **hardcore nebula** if it is a collection of vertex-disjoint left and right stars such that the only vertices between core vertices of any given star in the collection are core vertices.
Hardcore nebulae...

Theorem (Choromanski ’14)

Let $HN_l$ be a left hardcore nebula and $HN_r$ be a right hardcore nebula. Then there exists $\epsilon(HN_l, HN_r) > 0$ such that every $(HN_l, HN_r)$-free $n$-vertex tournament contains a transitive subtournament of order at least $n^{\epsilon(HN_l, HN_r)}$. 
Galaxies, constellations, nebulae...
The Erdős-Hajnal Conjecture
Structured nonlinear graph-based hashing
Anonymization via perfect matchings

Galaxies, constellations, nebulae...
On the Erdős-Hajnal Conjecture for six-vertex tournaments

Figure: Tournament $L_1$ on the left and tournament $L_2$ on the right. Both are obtained from $C_5$ by adding one extra vertex.
Theorem (Berger, Choromanski, Chudnovsky ’15)

Every tournament on six vertices other than $K_6$ satisfies the Erdős-Hajnal Conjecture.
Galaxies, constellations, nebulae...
Galaxies, constellations, nebulae...
Galaxies, constellations, nebulae...
Ultraconstellations - main results

Theorem (Choromanski '15)

Let \( H \) be an ultraconstellation and let \( \theta_H \) be its ultraconstellation ordering of vertices. Then there exists \( \epsilon(H) > 0 \) such that every \( \{(H, \theta_H), (H, \theta_{cH})\} \)-free ordered tournament \( (T, \theta_T) \) contains a transitive subtournament of order at least \( |T|^{\epsilon(H)} \).

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Ultraconstellations - main results

Corollary 1

*Gives the proof of the standard directed version of the Conjecture for the class of tournaments that contains as special cases all known infinite families of prime tournaments satisfying the Conjecture and defined by a single ordering.*

Corollary 2

*Implies all known results regarding excluding pairs of prime tournaments.*
Undirected setting - excluding $H$ and $H^c$
Undirected setting - excluding $H$ and $H^c$

**Theorem (Bousquet, Lagoutte, Thomasse ’13)**

Let $k, l > 0$. Define the class $\mathcal{H}_{k,l}$ of tournaments as those tournaments that are $\{P_k, P_l^c\}$-free, where $P_k$ is a path of $k$ vertices and $P_l^c$ is an antipath of $l$ vertices. Then $\mathcal{H}_{k,l}$ has polynomial-size transitive subtournaments.
The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and anonymization via perfect matchings
Excluding double-hooks
Excluding double-hooks
Excluding double-hooks
Excluding double-hooks

Theorem (Choromanski, Falik, Liebenau, Patel, Pilipczuk '15)

Let $H_t$ be an double $t$-hook. Then for every $m$ there exists $\epsilon(m)$ such that every $\{H_t, H_t^c : t = m, m+1, \ldots\}$-free undirected $n$-vertex graph $G$ contains a clique or a stable set of size at least $n^{\epsilon(m)}$. 
Excluding double-hooks

**Theorem (Choromanski, Falik, Liebenau, Patel, Pilipczuk '15)**

Let $H_t$ be an double $t$-hook. Then for every $m$ there exists $\epsilon(m)$ such that every $\{H_t, H^c_t : t = m, m + 1, \ldots\}$-free undirected $n$-vertex graph $G$ contains a clique or a stable set of size at least $n^{\epsilon(m)}$.

**Corollary**

For every hook $H$ there exists $\epsilon(H) > 0$ such that every $\{H, H^c\}$-free undirected $n$-vertex graph $G$ contains a clique or a stable set of size at least $n^{\epsilon(H)}$ (that extends the result of Bousquet, Lagoutte and Thomasse).
The Erdős-Hajnal Conjecture, structured nonlinear graph-based hashing and anonymization via perfect matchings

Excluding double-hooks

Theorem (Choromanski, Falik, Liebenau, Patel, Pilipczuk ’15)

Let $H_t$ be a double $t$-hook. Then for every $m$ there exists $\epsilon(m)$ such that every $\{H_t, H_t^c : t = m, m + 1, \ldots\}$-free undirected $n$-vertex graph $G$ contains a clique or a stable set of size at least $n^{\epsilon(m)}$.

Corollary

For every tree $H$ on at most six vertices there exists $\epsilon(H) > 0$ such that every $\{H, H^c\}$-free undirected $n$-vertex graph $G$ contains a clique or a stable set of size at least $n^{\epsilon(H)}$. 
Asymptotics of the EH coefficients
Asymptotics of the EH coefficients

\[ \varepsilon = 1 \]
Asymptotics of the EH coefficients

\[ \varepsilon = 1 \]
Asymptotics of the EH coefficients

\[ O\left(\frac{1}{h}\right) \]

\[ \varepsilon = 1 \]
Theorem (Choromanski ’10)

There exists $\eta > 0$ such that if we denote by $H^{n,\eta}$ the set of all $n$-vertex tournaments $H$ with $\epsilon(H) \leq \frac{4}{|H|}(1 + \frac{\eta \sqrt{\log(|H|)}}{\sqrt{|H|}})$, and by $H^n$ the set of all $n$-vertex tournaments then

$$\lim_{n \to \infty} \frac{|H^{n,\eta}|}{|H^n|} = 1.$$
Asymptotics of the EH coefficients

\[ O\left(\frac{1}{h}\right) \]

\[ \varepsilon = 1 \]
Asymptotics of the EH coefficients

\[ \frac{1}{2^{50h^2} + 1} \leq \mathcal{O}\left(\frac{1}{h}\right) \quad \varepsilon = 1 \]
The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings

Freed from the Regularity Lemma

Theorem (Choromanski, Jebara '13)

*Every known prime tournament $H$ satisfying the Conjecture satisfies also: $\epsilon(H) \geq \frac{1}{2^{50}|H|^2+1}$.***
Asymptotics of the EH coefficients

\[
1 \leq 2 \frac{2^{50h^2} + 1}{\frac{1}{h}} \leq O\left(\frac{1}{h}\right)
\]

\[\varepsilon = 1\]
Asymptotics of the EH coefficients

\[
\frac{1}{2^{2^{50n^2}+1}} \quad O\left(\frac{1}{n}\right) \quad \varepsilon = 1
\]
Asymptotics of the EH coefficients

\[ \frac{1}{h^5 \log(h)} \]

\[ O\left(\frac{1}{h}\right) \]

\[ \varepsilon = 1 \]
Theorem (Choromansi‘14)

There exists $C > 0$ such that every known prime tournament $H$ satisfying the Conjecture satisfies also:

$$\epsilon(H) \geq \frac{C}{|H|^5 \log(|H|)}.$$

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The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings
Asymptotics of the EH coefficients

\[
\frac{1}{2^{50h^2 + 1}} \ll \Omega\left(\frac{1}{h^5 \log(h)}\right)
\]

\[\mathcal{O}\left(\frac{1}{h}\right)\]

\[\varepsilon = 1\]
Asymptotics of the EH coefficients

\[ \frac{1}{2^{2^{50} h^2 + 1}} \leq \Omega \left( \frac{1}{h^5 \log(h)} \right) \leq O \left( \frac{1}{h} \right) \]

\[ \varepsilon = 1 \]
Asymptotics of the EH coefficients

\[
\frac{1}{2^{2^{50h^2}+1}} \leq \frac{1}{\epsilon^5 \log(h)} \leq \Omega\left(\frac{1}{h \log(h)}\right) \leq O\left(\frac{1}{h}\right)
\]

\[\epsilon = 1\]
Tight bounds on EH coefficients for stars
Tight bounds on EH coefficients for stars

Theorem (Choromanski ’12)

For every star $H$ there exist $C_1, C_2 > 0$ such that

$$
\frac{C_1}{h \log(h)} \leq \epsilon(H) \leq \frac{C_2 \log(h)}{h},
$$

where $h = |H|$. 
Asymptotics of the EH coefficients

\[
\frac{1}{2^{2^{50h^2}+1}} \quad \Omega\left(\frac{1}{h^5 \log(h)}\right) \quad \frac{1}{h \log (h)} \quad O\left(\frac{1}{h}\right) \quad \varepsilon = 1
\]
Asymptotics of the EH coefficients

\[ \Omega\left(\frac{1}{h^5 \log(h)}\right) \]

\[ \Omega\left(\frac{1}{h \log(h)}\right) \]

\[ O\left(\frac{1}{h}\right) \]

\[ \varepsilon = 1 \]

\[ 2^{2^{50h^2}+1} \]
Asymptotics of the EH coefficients

\[ \frac{1}{h^5 \log(h)} \]

\[ \frac{1}{h \log(h)} \]

\[ \frac{1}{h \log^2(h)} \]

\[ 2^{2^{50h^2 + 1}} \]

\[ \mathcal{O}(\frac{1}{h}) \]

\[ \mathcal{O}(\frac{1}{h \log(h)}) \]

\[ \mathcal{O}(\frac{1}{h \log^2(h)}) \]

ε = 1

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The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings
Tight bounds on EH coefficients for directed paths

Theorem (Choromanski ’15)

For every directed path $P_h$ there exist $C_1, C_2 > 0$ such that

$$\frac{C_1}{h \log^2(h)} \leq \epsilon(H) \leq \frac{C_2 \log(h)}{h}.$$
Partition numbers and EH-coefficients

Theorem (Choromanski ’12)

There exists $C_1 > 0$ such that for a tournament $H$ the following holds:

$$\epsilon(H) \leq C_1 \frac{\log(\log(p(H)))}{\log(p(H))}.$$

There exists $C_2 > 0$ such that if $H$ is prime then the following holds:

$$\epsilon(H) \leq C_2 \frac{\log(|H|)}{|H|}.$$
Powers of graphs
Small homogeneous sets imply small EH-coefficients

**Theorem (Choromanski ’12)**

If $H$ is a tournament without homogeneous sets of size larger than $\frac{\sqrt{|H|}}{2}$ then

$$\limsup_{p(H) \to \infty} \frac{\epsilon(H)}{\log(p(H))} \cdot \frac{p(H)^{1/2 - \delta}}{p(H)} < \infty,$$

for every $\delta > 0$. 

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Neural networks - an overview
Neural networks - an overview

A SIMPLE NEURAL NETWORK

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Neural networks - an overview
Neural networks - an overview
Neural networks - an overview

\[ \chi \]

\[
\begin{bmatrix}
    a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & a_{07} & a_{08} & a_{09} \\
    a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
    a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
    a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
    a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
    a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
    a_{60} & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
    a_{70} & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
    a_{80} & a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
    a_{90} & a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{bmatrix}
\]
Neural networks - an overview

$$X = \begin{bmatrix}
    a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & a_{07} & a_{08} & a_{09} \\
    a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
    a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
    a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
    a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
    a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
    a_{60} & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
    a_{70} & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
    a_{80} & a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
    a_{90} & a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{bmatrix}$$
Neural networks - an overview

\[ \begin{bmatrix}
  a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & a_{07} & a_{08} & a_{09} \\
  a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
  a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
  a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
  a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
  a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
  a_{60} & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
  a_{70} & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
  a_{80} & a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
  a_{90} & a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{bmatrix} \rightarrow \begin{bmatrix}
  f(z_0) \\
  f(z_1) \\
  f(z_2) \\
  f(z_3) \\
  f(z_4) \\
  f(z_5) \\
  f(z_6) \\
  f(z_7) \\
  f(z_8) \\
  f(z_9)
\end{bmatrix} \]
Neural networks - an overview

\[ \mathbf{X} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & a_{07} & a_{08} & a_{09} \\
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 a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
 a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
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 a_{90} & a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} \end{bmatrix} \]

\[ \mathbf{Z} = \begin{bmatrix} f(z_0) \\
 f(z_1) \\
 f(z_2) \\
 f(z_3) \\
 f(z_4) \\
 f(z_5) \\
 f(z_6) \\
 f(z_7) \\
 f(z_8) \\
 f(z_9) \end{bmatrix} \]

\[ f(x) = \frac{1}{1 + e^{-x}} \]
Neural networks - an overview

$$\mathbf{X} = \begin{bmatrix}
  a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & a_{07} & a_{08} & a_{09} \\
  a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
  a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
  a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
  a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
  a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
  a_{60} & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
  a_{70} & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
  a_{80} & a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
  a_{90} & a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{bmatrix}

\begin{bmatrix}
  f(z_0) \\
  f(z_1) \\
  f(z_2) \\
  f(z_3) \\
  f(z_4) \\
  f(z_5) \\
  f(z_6) \\
  f(z_7) \\
  f(z_8) \\
  f(z_9)
\end{bmatrix}

f(1) = \frac{1}{1 + e^{-x}}
Introducing structured approach
Introducing structured approach
Introducing structured approach

\[ f \left( \frac{1}{1 + e^{-x}} \right) \]

Krzysztof Choromanski
Google Research, New York City

The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings
Introducing structured approach

The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings

Krzysztof Choromanski
Google Research, New York City
Introducing structured approach
Structured nonlinear hashing with PHDs

\[ x \in \mathbb{R}^n \]
Structured nonlinear hashing with PHDs

\[ x \in \mathbb{R}^n \]
Structured nonlinear hashing with PHDs

\[ D \in \text{diag}_{\text{rand}}(n) \]

\[ x \in \mathbb{R}^n \]
Structured nonlinear hashing with PHDs

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\[ x \in \mathbb{R}^n \]
Structured nonlinear hashing with PHDs

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\[ x \in \mathbb{R}^n \]
Structured nonlinear hashing with PHDs

\[ D \in \text{diag}_{\text{rand}}(n) \]

\[ x \in \mathbb{R}^n \]

\[ H \in \text{Had}_{\text{norm}}(n) \]
Structured nonlinear hashing with PHDs

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Krzysztof Choromanski, Google Research, New York City
Structured nonlinear hashing with PHDs

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\[ x \in \mathbb{R}^n \]

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Krzysztof Choromanski
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The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings
Structured nonlinear hashing with PHDs

\[ D \in \text{diag}_{\text{rand}}(n) \]

\[ x \in \mathbb{R}^n \]

\[ H \in \text{Had}_{\text{norm}}(n) \]

\[ y \in \{-1,+1\}^m \]
Ψ-regular structured gaussian matrices

Matrix $P$ is Ψ-regular random matrix if it has the following form

$$
\begin{pmatrix}
\sum_{l \in S_{1,1}} g_l & \ldots & \sum_{l \in S_{1,j}} g_l & \ldots & \sum_{l \in S_{1,n}} g_l \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{l \in S_{i,1}} g_l & \ldots & \sum_{l \in S_{i,j}} g_l & \ldots & \sum_{l \in S_{i,n}} g_l \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{l \in S_{k,1}} g_l & \ldots & \sum_{l \in S_{k,j}} g_l & \ldots & \sum_{l \in S_{k,n}} g_l
\end{pmatrix}
$$

where $S_{i,j} \subseteq \{1, \ldots, t\}$, $|S_{i,1}| = \ldots = |S_{i,n}|$, $S_{i,j} \cap S_{i,u} = \emptyset$ for $j \neq u$, and furthermore:

- for a fixed column $C$ of $P$ and fixed $l \in \{1, \ldots, t\}$ random variable $g_l$ appears in at most $\Psi + 1$ entries from $C$. 
Preserving angles with structured gaussian matrices
Preserving angles with structured gaussian matrices
Preserving angles with structured gaussian matrices
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Preserving angles with structured gaussian matrices
Coloring graphs of structured matrices

Let us fix two rows of $P$ of indices $1 \leq k_1 < k_2 \leq k$ respectively. We define a graph $G_P(k_1, k_2)$ as follows:

- $V(G_P(k_1, k_2)) = \{\{j_1, j_2\} : \exists l \in \{1, ..., t\} s.t. g_l \in S_{k_1,j_1} \cap S_{k_2,j_2}, j_1 \neq j_2\}$,
- there exists an edge between vertices $\{j_1, j_2\}$ and $\{j_3, j_4\}$ iff $\{j_1, j_2\} \cap \{j_3, j_4\} \neq \emptyset$.

**Definition**

Let $P$ be a $\Psi$-regular matrix. We define the $P$-chromatic number $\chi(P)$ as:

$$\chi(P) = \max_{1 \leq k_1 < k_2 \leq k} \chi(G(k_1, k_2)).$$
Coloring graphs of structured matrices

\[
\begin{pmatrix}
g_1 & g_2 & \ldots & g_n \\
g_2 & g_3 & \ldots & g_1 \\
\vdots & & & \vdots \\
g_i & g_{i+1} & \ldots & \\
g_j & g_{j+1} & \ldots & \\
\vdots & & & \vdots \\
\end{pmatrix}
\]

\[j = i + 2\]

Figure: Structured graph for the circulant matrix - a set of disjoint cycles.
Theorem (Choromanski '15)

Take the extended $\Psi$-regular hashing model $\mathcal{M}$. Let $N$ be the size of the dataset. Denote by $k$ the size of the hash and by $n$ the dimensionality of the data. Let $f(n)$ be an arbitrary positive function. Then for every $a, \epsilon > 0$ the following is true:

$$\mathbb{P}\left(\left|\tilde{\theta}_{p,r}^n - \frac{\theta_{p,r}}{\pi}\right| \leq \epsilon\right) \geq \left[1 - 4\left(\binom{N}{2}\right) e^{-\frac{f^2(n)}{2}} - 4\chi(\mathcal{P})\binom{k}{2} e^{-\frac{2a^2 t}{f^4(t)}}\right] \Lambda,$$

where $\Lambda = 1 - \frac{1}{\pi} \sum_{j=\epsilon k}^{k} \frac{1}{\sqrt{j}}\left(\frac{ke}{j}\right)^j \mu^j (1 - \mu)^{k-j} + 2e^{-\frac{\epsilon^2 k}{2}}$ and

$$\mu = \frac{8k(a\chi(\mathcal{P}) + \psi \frac{f^2(n)}{n})}{\theta_{p,r}}.$$
Corollary

Take the extended Ψ-regular hashing model $\mathcal{M}$. Assume that the projection matrix $\mathcal{P}$ is Toeplitz gaussian. Let $N$ be the size of the dataset. Denote by $k$ the size of the hash and by $n$ the dimensionality of the data. Then for every $\epsilon > 0$ the following is true:

$$\mathbb{P}\left( \left| \tilde{\theta}_{p,r}^{n} - \frac{\theta_{p,r}}{\pi} \right| \leq k^{-\frac{1}{3}} \right) \geq 1 - O \left( \frac{N^2}{n^{4.5}} + k^2 e^{-\Omega\left( \frac{n^{1/3}}{\log^2(n)} \right)} \right) \Lambda,$$

where $\Lambda = \left[ 1 - \left( \frac{k^7}{n} \right)^{\frac{1}{3}} \right]$. 
Theorem (Choromanski ’15)

Take the short $\Psi$-regular hashing model $\mathcal{M}$, where $\mathcal{P}$ is a Toeplitz gaussian matrix. Denote by $k$ the size of the hash. Then the following is true

$$\text{Var}(\tilde{\theta}_{p,r}^{n}) \leq \frac{1}{k} \frac{\theta_{p,r}(\pi - \theta_{p,r})}{\pi^2} + \left(\frac{\log(k)}{k^2}\right)^{\frac{1}{3}},$$

and thus for any $c > 0$:

$$\mathbb{P} \left( \left| \tilde{\theta}_{p,r}^{n} - \frac{\theta_{p,r}}{\pi} \right| \geq c \left( \frac{\sqrt{\log(k)}}{k} \right)^{\frac{1}{3}} \right) = O \left( \frac{1}{c^2} \right).$$
Adaptive anonymity with $b$-matchings

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<th>0</th>
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<tr>
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<tr>
<td>Paul</td>
<td>0</td>
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<tr>
<td>Peter</td>
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<td>Carl</td>
<td>1</td>
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<tr>
<td>Olaf</td>
<td>0</td>
<td>1</td>
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</tbody>
</table>

Figure: The $b$-matching $k$-anonymity. The comparability graph is not a disjoint union of complete bipartite graphs. The parameters of the model are: $n = 6$, $f = 4$, $k = 2$. Presented solution achieves $\#(*) = 8$. The standard $k$-anonymity would achieve $\#(*) = 10$. 
Combinatorics of adaptive anonymity via \(b\)-matching

**Definition**

Let \( G(A, B) \) be a bipartite graph with color classes: \( A, B \), where \(|A| = |B| = n\). For a vertex \( v \in V(G(A, B)) \) we denote by \( N(v) \) the set of its neighbours in \( G(A, B) \). For a subset \( S \subseteq V(G(A, B)) \) we denote: \( N(S) = \bigcup_{v \in S} N(v) \).

**Definition**

A *perfect matching* in the graph \( G \) is the set of its pairwise vertex-disjoint edges that cover all its vertices.

**Hall’s Theorem**

Bipartite graph \( G(A, B) \) has a perfect matching if and only if \(|N(S)| \geq |S|\) for every \( S \subseteq A \).
Definitions again...

Definition

Assume that $G(A, B)$ has a perfect matching. Let $M$ be some fixed canonical matching in $G(A, B)$. Then for $S \subseteq A$ we denote $m(S) = \bigcup_{s \in S} m(s)$, where $(s, m(s)) \in M$.

Definition

Let $G(A, B)$ be a bipartite graph with $|A| = |B| = n$ and let $M$ be its canonical matching. We say that a set $S \subseteq V(A)$ is closed if $N(S) = m(S)$.
Simple theorems

Lemma

If $G(A, B)$ is an arbitrary $d$-regular graph and the adversary does not know in advance any edges of the matching he is looking for then every person is $d$-anonymous.
Simple theorems

Lemma

If $G(A, B)$ is an arbitrary $d$-regular graph and the adversary does not know in advance any edges of the matching he is looking for then every person is $d$-anonymous.

Proof:

It suffices to prove that for every edge $e$ of $G(A, B)$ there exists a perfect matching in $G(A, B)$ that uses $e$. This is a direct implication of Hall’s Theorem. You keep finding matchings one by one, removing edges of the matchings found so far from the graph.
The Erdős-Hajnal Conjecture
Structured nonlinear graph-based hashing
Anonymization via perfect matchings

Simple theorems - sustained attack with $d$-anonymity

Lemma

If $G(A, B)$ is clique-bipartite $d$-regular graph and the adversary knows in advance $c$ edges of the matching then every person is $(d - c)$-anonymous.
Simple theorems - sustained attack with $d$-anonymity

**Proof:**
Follows immediately from the following lemma:

**Lemma**

Assume that $G(A, B)$ is clique-bipartite $d$-regular graph (i.e. it is a union of disjoint complete bipartite graphs). Denote by $M$ some perfect matching in $G(A, B)$. Let $C$ be some subset of the edges of $M$ and let $c = |C|$. Fix some vertex $v \in A$ not matched in $C$. Then there are at least $(d - c)$ edges adjacent to $v$ such that for each edge $e$ like that there exists some perfect matching $M^e$ in $G(A, B)$ that uses both $e$ and $C$. 
Main Result - Adversary with extra Knowledge

**Theorem (Choromanski ’11-15)**

Let $G(A,B)$ be a $k$-regular bipartite graph with color classes: $A$ and $B$. Assume that $|A| = |B| = n$. Denote by $M$ some perfect matching $M$ in $G(A,B)$. Let $C$ be some subset of the edges of $M$ and let $c = |C|$. Take some $\xi \geq c$. Denote $\hat{n} = n - c$. Fix any function $\phi: \mathbb{N} \to \mathbb{R}$ satisfying $\forall k (\xi \sqrt{2k + \frac{1}{4}} < \phi(k) < k)$. Then for all but at most 

$$\delta = \frac{2ck^2\hat{n}\xi(1 + \frac{\phi(k)}{\phi^2(k)} + \frac{\sqrt{\phi^2(k) - 2\xi^2k}}{2\xi k})}{\phi^3(k)(1 + \frac{1}{\phi^2(k)})(1 - \frac{\phi^2(k) - 2\xi^2k}{\phi^2(k)})(\frac{1}{\xi} - \frac{c}{\phi(k)} + \frac{k(1 - \frac{\xi}{\phi(k)})}{\phi(k)}) + \frac{ck}{\phi(k)}},$$

vertices $v \in A$ not matched in $C$ the following holds:
Main Result - Adversary with extra knowledge

**Theorem (Choromanski ’11-15)**

The size of the set of edges $e$ adjacent to $v$ and with the additional property that there exists some perfect matching $M^v$ in $G(A, B)$ that uses $e$ and edges from $C$ is at least $(k - c - \phi(k))$. 
Main Result - proof

**Definition**

Take a bipartite graph $G_{del} = G(A_{del}, B_{del})$ with color classes $A_{del}, B_{del}$, obtained from $G(A, B)$ by deleting all the vertices of $C$. For a vertex $v \in A_{del}$ and an edge $e$ adjacent to it in $G_{del}$ we say that $e$ is *bad in respect to* $v$ if there is no perfect matching in $G(A, B)$ that uses $e$ and all the edges from $C$.

**Definition**

We say that a vertex $v \in A_{del}$ is *bad* if there are at least $\phi(d)$ edges bad with respect to $v$. 
Main Result - proof

Lemma

For every edge $e$ which is bad with respect to a vertex $v$ there exists a closed set $S^e_v$ such that $v \notin S^e_v$ and $v$ is adjacent to some vertex in $m(S^e_v)$.

Definition

Fix some bad vertex $v$ and some set $E$ of its bad edges of size $\phi(d)$. Let $S^E_v = \bigcup_{e \in E} S^e_v$. Note that $S^E_v$ is closed as a sum of closed sets. We also have: $v \notin S^E_v$. Besides every edge from $E$ touches some vertex from $m(S^E_v)$. We say that the set $S$ is $\phi(d)$-bad with respect to a vertex $v \in A_{del} - S$ if it is closed and there are $\phi(d)$ bad edges with respect to $v$ that touch $S$. So we conclude that $S^E_v$ is $\phi(d)$-bad with respect to $v$. 
Main Result - proof

**Definition**

Denote by $S_v^m$ a minimal $\phi(d)$-bad set with respect to $v$. 
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof

![Diagram with images and connections between them]
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof

Lemma

Let $v_1, v_2$ be two bad vertices. If $v_2 \in S^m_{v_1}$ then $S^m_{v_2} \subseteq S^m_{v_1}$. 
Main Result - proof
Main Result - proof
Main Result - proof

[Diagram of a graph with nodes and edges connecting them, representing a proof of the Erdős-Hajnal Conjecture.]
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof

Forbidden edge
Main Result - proof
Main Result - proof
Main Result - proof

Lemma

Denote $P = \{ S^m_v \, : \, v \in X \}$. As a poset with an ordering induced by the inclusion relation, it does not have antichains of size larger than \( \frac{cd}{\phi(d)} \).
Main Result - proof

**Lemma**

Denote $P = \{ S_v^m : v \in X \}$. As a poset with an ordering induced by the inclusion relation, it does not have antichains of size larger than $\frac{cd}{\phi(d)}$.

**Corollary**

Using Dilworth’s lemma about chains and antichains in the poset and the previous lemma we can conclude that a set $P = \{ S_v^m : v \in A \}$ has a chain of length at least $\frac{\hat{n}\phi(d)}{cd}$.
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof
Main Result - proof

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The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing, and b-matching anonymization via perfect matchings

Main Result - proof

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Google Research, New York City
Main Result - proof
Main Result - proof

Gap Lemma

If $|X_i| > c$ then $|X_i| \geq \frac{\phi(k)}{c} - c$.

Short subsequences of small values

For every $i$ and $l > \frac{\phi(d) - \sqrt{\phi^2(d) - 2\xi^2 d}}{\xi}$ in the sequence $(X_{i+1}, \ldots, X_{i+l})$ there exists at least one element of size more than $c$. 

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Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ m(S^m_{v_i}) \]
Main Result - proof - Gap Lemma

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The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and b-matching anonymization via perfect matchings
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ \geq (k - \phi(k)) \]
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ X_4 \quad \quad \quad X_3 \quad \quad \quad \geq (k - \frac{\phi(k)}{c}) \]
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ \geq k \mid S_{v_i}^m \mid -ck \]

\[ \geq (k - \phi(k)) \]

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Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ \geq k \mid S_{v_i}^m \mid -ck \]

\[ \phi(k) \]

\[ > c(k - \frac{\phi(k)}{c}) \]

\[ \geq (k - \frac{\phi(k)}{c}) \]

\[ \geq k \mid S_{v_i}^m \mid -ck \]
Main Result - proof - Gap Lemma

\[ \geq k \left| S^m_{v_i} \right| - ck \]

\[ > c(k - \phi(k)) \]

\[ \geq k \left| S^m_{v_i} \right| - ck \]
Main Result - proof - Gap Lemma

\[
\begin{align*}
\geq k & \mid S_{v_i}^m \mid - ck \\
\phi(k) & > c(k - \frac{\phi(k)}{c}) \\
\geq k & \mid S_{v_i}^m \mid - ck
\end{align*}
\]
Main Result - proof - Gap Lemma

\[ S^m_{v_i} \geq k \left( \phi(k) - \frac{\phi(k)}{c} \right) \]

\[ \geq k \left| S^m_{v_i} \right| - ck \]
Main Result - proof - Gap Lemma

\[ \phi(k) \geq (k - \frac{\phi(k)}{c}) \]

\[ \geq k \left| S_{v_i}^m \right| - ck \]

\[ > k \left| S_{v_i}^m \right| \]
The Erdős-Hajnal Conjecture, structured non-linear graph-based hashing and anonymization via perfect matchings

Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ X_4 \quad X_3 \quad X_2 \leq k - \frac{\phi(k)}{c} \]
Main Result - proof - Gap Lemma
Main Result - proof - Gap Lemma

\[ \frac{\phi(k)}{c} \leq X_2 \leq k - \frac{\phi(k)}{c} \]
Main Result - proof - Gap Lemma
Main Result - proof - short subsequences
Main Result - proof - short subsequences
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Main Result - proof - short subsequences
Main Result - proof - short subsequences
Main Result - proof - short subsequences
Main Result - proof - short subsequences
Main Result - proof - short subsequences

$X_4 \leq c \leq c \leq X_1$
Main Result - proof - short subsequences
Main Result - proof - short subsequences
Main Result - proof - short subsequences
Main Result - proof - short subsequences

\[ \geq \phi(k) - c \]

\[ X_4 \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 \]

\[ \leq c \quad \leq c \]
Main Result - proof - short subsequences
Main Result - proof - short subsequences

\[ \geq \phi(k) - c - 2c \]

\[ \leq c \quad \leq c \]
Main Result - proof - short subsequences
Main Result - proof - short subsequences

\[ \geq l\phi(k) - c - 2c - 3c - \ldots - (l-1)c \]