On the size Ramsey number

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March 19th 21st 2018

joint work with: (1) Matthew Jenssen, Yoshiharu Kohayakawa, Natasha Morrison, Guilherme O. Mota, Damian Reding, Barnaby Roberts & (2) Meysam Miralaei, Damian Reding, Mathias Schacht, Anusch Taraz
Mit welcher Linie fliegt ihr denn?

Tigerair
Ramsey graphs
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For every graph $F$, there is an integer $n \in \mathbb{N}$ such that $K_n \rightarrow F$
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Observation: For every graph $F$ we have

$$\hat{r}(F) \leq \binom{r(F)}{2}.$$
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There is a constant $C > 0$ such that for every $n \in \mathbb{N}$ it holds that $\hat{r}(P_n) < Cn$. For short:

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**Natural question:** For the size Ramsey numbers of which graph sequences do we have linearity in the number of vertices?
Linearity of the size Ramsey number

The path $P_n$, the cycle $C_n$, and trees $T$ on $n$ vertices of bounded maximum degree $\Delta(T) \leq k$.

**Question (Beck; 1990)**
Is $\hat{r}(F)$ linear for all graphs with bounded maximum degree?

**Theorem (Rödl, Szemerédi; 2000)**
There is a sequence of graphs $H$ with $v(H) = n$, $\Delta(H) = 3$ and such that $\hat{r}(H) = \Omega(n \log \frac{1}{60} n)$.

**Theorem (Kohayakawa, Rödl, Schacht, Szemerédi; 2011)**
For every $\Delta \in \mathbb{N}$ there is a constant $c = c(\Delta)$ such that the following holds: For every graph $F$ with $v(F) = n$ and maximum degree $\Delta$ it holds that $\hat{r}(F) \leq cn^{2 - \frac{1}{\Delta}} \log \frac{1}{\Delta} n$. 
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$$\hat{r}(C_n) \leq \begin{cases} 
113482 \cdot 10^6 n & \text{if } n \text{ is odd} \\
2514 \cdot 10^6 n & \text{if } n \text{ is even} 
\end{cases}$$

for large enough $n$. 
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Theorem (Rödl, Szemerédi; 2000)

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\[\Rightarrow\] both papers: universality result
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Powers of paths

Definition:
Let \( F = (V, E) \) be a graph. The \( k \)-th power \( F^k \) of \( F \) is defined on the same vertex set as \( F \) with edges given as follows:

\[ uv \in E(F^k) : \iff \text{dist}_{F}(u, v) \leq k \]

Problem (Conlon; 2016)
Find out whether the size Ramsey number of \( P_k n \) is linear in \( n \).

Theorem (C., Jenssen, Kohayakawa, Morrison, Mota, Reding, Roberts; 2017+)
\( \hat{r}(P_k n) = O(n) \) and \( \hat{r}(C_k n) = O(n) \)
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![Diagram of a path $P_7$]
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start with a graph $H$ on $\Theta(n)$ vertices, with $\Delta(H) = \Theta(1)$ and the property

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(between large sets there is always an edge)
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\[ H \]

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\[ H^k \quad (k = 2) \]

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(it will turn out later that $H^k \to (P_n, P_n^k)$)
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(it will turn out later that $H^k \rightarrow (P_n, P_n^k)$)

Take "complete-t-blow-up" $(H^k)_t$ of $H^k$:

- replace each vertex by a clique of size $r(K_t)$, called cluster
Proof for $\hat{r}(P_n^k) = O(n)$

start with a graph $H$ on $\Theta(n)$ vertices, with $\Delta(H) = \Theta(1)$ and the property
"for every disjoint sets $S, T \subset V(H)$ with $|S|, |T| \geq \gamma n$ it holds that $e_H(S, T) > 0$"

take $k$-the power $H^k$ of $H$
(it will turn out later that $H^k \to (P_n, P_n^k)$)

take "complete-t-blow-up" $(H^k)_t$ of $H^k$:
- replace each vertex by a clique of size $r(K_t)$, called cluster
- replace each edge by a complete bipartite graph
Proof for $\hat{r}(P_n^k) = O(n)$

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take "complete-t-blow-up" $(H^k)_t$ of $H^k$:

- replace each vertex by a clique of size $r(K_t)$, called cluster
- replace each edge by a complete bipartite graph

Note: $e((H^k)_t) = O(n)$

$(t = t(k) \text{ is chosen as a large constant})$
Proof for $\hat{r}(P_n^k) = O(n)$

start with a graph $H$ on $\Theta(n)$ vertices, with $\Delta(H) = \Theta(1)$ and the property
"for every disjoint sets $S, T \subset V(H)$ with $|S|, |T| \geq \gamma n$ it holds that $e_H(S,T) > 0$"

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(it will turn out later that $H^k \to (P_n, P_n^k)$)

take "complete-t-blow-up" $(H^k)_t$ of $H^k$:
- replace each vertex by a clique
of size $r(K_t)$, called cluster
- replace each edge by a complete bipartite graph

**Theorem:**
$(H^k)_t$ is Ramsey for $P_n^k$. 
Proof for $\hat{r}(P^k_n) = O(n)$

Theorem: $(H^k)^t$ is Ramsey for $P^k_n$.

take "complete-t-blow-up" $(H^k)^t$ of $H^k$:
• replace each vertex by a clique
  of size $r(K_t)$, called cluster
• replace each edge by a complete
  bipartite graph

$\bullet \quad (H^k)^t$
Proof for $\hat{r}(P_n^k) = O(n)$

Proof: Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given.

take "complete-t-blow-up" $((H^k)_t)$ of $H^k$:
- replace each vertex by a clique
  of size $r(K_t)$, called cluster
- replace each edge by a complete
  bipartite graph

Theorem:

$(H^k)_t$ is Ramsey for $P_n^k$.
Proof for $\hat{r}(P_{n}^{k}) = O(n)$

**Theorem:** $(H^{k})_{t}$ is Ramsey for $P_{n}^{k}$.

Proof: Let a 2-colouring $\chi$ of $E((H^{k})_{t})$ be a given. In each cluster find a m.c. copy of $K_{t}$.

- take "complete-t-blow-up" $(H^{k})_{t}$ of $H^{k}$:
  - replace each vertex by a clique of size $r(K_{t})$, called cluster
  - replace each edge by a complete bipartite graph

**Theorem:**

$(H^{k})_{t}$ is Ramsey for $P_{n}^{k}$.
Proof for $\hat{r}(P^n_k) = O(n)$

**Theorem:** $(H_k)^t$ is Ramsey for $P^n_k$.

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)^t)$ be a given. In each cluster find a m.c. copy of $K_t$.

- take "complete-t-blow-up" $(H^k)^t$ of $H^k$:
  - replace each vertex by a clique of size $r(K_t)$, called cluster
  - replace each edge by a complete bipartite graph

**Theorem:**

$(H^k)^t$ is Ramsey for $P^n_k$. 

Proof

![Diagram of a graph and its Ramsey properties](image-url)
Proof for $\hat{r}(P_{n}^{k}) = O(n)$

**Theorem:** $(H^{k})_{t}$ is Ramsey for $P_{n}^{k}$.

**Proof:** Let a 2-colouring $\chi$ of $E((H^{k})_{t})$ be a given. In each cluster find a m.c. copy of $K_{t}$. 

(between large sets there is always an edge)

\[ H \]

\[ H^{k} \quad (k = 2) \]

\[ (H^{k})_{t} \]

colouring $\chi$
Proof for $\hat{r}(P_n^k) = O(n)$

**Theorem:** $(H^k)_t$ is Ramsey for $P_n^k$.

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$. W.l.o.g. half of all clusters have a blue copy of $K_t$. 

Diagram: 
- $H$  
- $H^k$ ($k = 2$)  
- $(H^k)_t$  
- Colouring $\chi$
Proof for $\hat{r}(P^k_n) = O(n)$

**Theorem:** $(H^k_t)$ is Ramsey for $P^k_n$.

**Proof:** Let a 2-colouring $\chi$ of $E((H^k_t))$ be a given. In each cluster find a m.c. copy of $K_t$.

W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$, reduce: $H$ to $F := H[W]$

$H^k$ to $F^k$

$(H^k_t)$ to $F'$ induced by the blues $K_t'$s

(colouring $\chi$)

(between large sets there is always an edge)
Theorem: \( (H^k)_t \) is Ramsey for \( P^k_n \).

**Proof:** Let a 2-colouring \( \chi \) of \( E((H^k)_t) \) be a given. In each cluster find a m.c. copy of \( K_t \).

W.l.o.g. half of all clusters have a blue copy of \( K_t \).

Set \( W := \{ v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t \} \),

reduce: \( H \) to \( F := H[W] \)

\( H^k \) to \( F^k \)

\((H^k)_t \) to \( F' \) induced by the blues \( K'_t \)s
Proof for $\hat{r}(P^k_n) = O(n)$

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$. W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$, reduce: $H$ to $F := H[W]$, $H^k$ to $F^k$, $(H^k)_t$ to $F'$ induced by the blues $K'_t$'s

**Theorem:**

$(H^k)_t$ is Ramsey for $P^k_n$. 

F $\subset$ H

(between large sets there is always an edge)

$F^k \subset H^k$

$F' \subset (H^k)_t$

colouring $\chi$
Proof for $\hat{r}(P_k^n) = O(n)$

Theorem: $(H^k)_t$ is Ramsey for $P_k^n$.

Proof: Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$. W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$, reduce: $H$ to $F := H[W]$, $H^k$ to $F^k$, $(H^k)_t$ to $F'$ induced by the blues $K_t's$.

Define an auxiliary colouring $\chi'$ of $E(F^k)$:
Proof for $\hat{r}(P_n^k) = O(n)$

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$.

W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$,

reduce: $H$ to $F := H[W]$

$H^k$ to $F^k$

$(H^k)_t$ to $F'$ induced by the blues $K'_t$s

Define an auxiliary colouring $\chi'$ of $E(F^k)$:

colour $e = uv$ blue if between the corresponding blue $K_t$-copies in $F'$

there is a blue copy of $K_{2k,2k}$, otherwise colour $e = uv$ red.

**Theorem:**

$(H^k)_t$ is Ramsey for $P_n^k$. 
Proof for $\hat{r}(P_n^k) = O(n)$

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$.

W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$,

reduce: $H$ to $F := H[W]$  
$H^k$ to $F_k^k$  
$(H^k)_t$ to $F'$ induced by the blues $K_t's$  

Define an auxiliary colouring $\chi'$ of $E(F^k)$: 

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**Theorem:** 

$(H^k)_t$ is Ramsey for $P_n^k$. 

Proof for $\hat{r}(P^k_n) = O(n)$

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$.

W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$,

reduce: $H$ to $F := H[W]$

$H^k$ to $F^k$

$(H^k)_t$ to $F'$ induced by the blues $K_t'$

Define an auxiliary colouring $\chi'$ of $E(F^k)$:

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**Theorem:**

$(H^k)_t$ is Ramsey for $P^k_n$. 
Proof for $\hat{r}(P_n^k) = O(n)$

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$.

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Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$,

reduce: $H$ to $F := H[W]$

$H^k$ to $F^k$

$(H^k)_t$ to $F'$ induced by the blues $K'_t$s

Define an auxiliary colouring $\chi'$ of $E(F^k)$:

- colour $e = uv$ blue if between the corresponding blue $K_t$-copies in $F'$
- otherwise colour $e = uv$ red.

**Theorem:**

$(H^k)_t$ is Ramsey for $P_n^k$. 

**Diagram:**

- $F \subset H$
- $F^k \subset H^k$
- colouring $\chi'$
- $F' \subset (H^k)_t$
- colouring $\chi$
Proof for $\hat{r}(P_k^n) = O(n)$

**Proof:** Let a 2-colouring $\chi$ of $E((H^k)_t)$ be a given. In each cluster find a m.c. copy of $K_t$.

W.l.o.g. half of all clusters have a blue copy of $K_t$.

Set $W := \{v \in V(H) : \text{cluster } C(v) \text{ has blue } K_t\}$,

reduce:

- $H$ to $F := H[W]$
- $H^k$ to $F^k$
- $(H^k)_t$ to $F'$ induced by the blues $K'_t$s

Define an auxiliary colouring $\chi'$ of $E(F^k)$:

- colour $e = uv$ blue if between the corresponding blue $K_t$-copies in $F'$
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$(H^k)_t$ is Ramsey for $P_k^n$. 
Proof for $\hat{r}(P_k^n) = O(n)$

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$H^k$ to $F^k$

$(H^k)_t$ to $F'$ induced by the blues $K'_t$s

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- colour $e = uv$ blue if between the corresponding blue $K_t$-copies in $F'$
- otherwise colour $e = uv$ red.

**Theorem:**

$(H^k)_t$ is Ramsey for $P_k^n$. 

Proof for $\hat{r}(P_n^k) = O(n)$

$F \subset H$

(between large sets there is always an edge)

$F^k \subset H^k$

colouring $\chi'$

$F' \subset (H^k)_t$

colouring $\chi$

no blue $K_{2k,2k}$
Proof for $\hat{r}(P_n^k) = O(n)$

Lemma 1 (Key Lemma):
Any 2-colouring $\chi'$ of $E(F^k)$ has a blue copy of $P_n$ or a red copy of $P_n^k$.

$F \subset H$

(between large sets there is always an edge)

$F^k \subset H^k$

colouring $\chi'$

$F' \subset (H^k)_t$

no blue $K_{2k,2k}$

colouring $\chi$
Proof for $\hat{r}(P_n^k) = O(n)$

**Lemma 1 (Key Lemma):**
Any 2-colouring $\chi'$ of $E(F^k)$ has a blue copy of $P_n$ or a red copy of $P_n^k$.

**Lemma 2:**
If $\chi'$ on $E(F^k)$ has a blue copy of $P_n$, then $\chi$ has a blue copy of $P_n^k$ in $H_t^k$.

**Lemma 3:**
If $\chi'$ on $E(F^k)$ has a red copy of $P_n^k$, then $\chi$ has a red copy of $P_n^k$ in $H_t^k$.
Proof for $\hat{r}(P_n^k) = O(n)$

Any 2-colouring $\chi'$ of $E(F_k)$ has a blue copy of $P_n$ or a red copy of $P_k^n$.

Lemma 1 (Key Lemma):
Any 2-colouring $\chi'$ of $E(F_k)$ has a blue copy of $P_n$ or a red copy of $P_k^n$.

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If $\chi'$ on $E(F_k)$ has a blue copy of $P_n$, then $\chi$ has a blue copy of $P_k^n$ in $H_t^k$.

Lemma 3:
If $\chi'$ on $E(F_k)$ has a red copy of $P_k^n$, then $\chi$ has a red copy of $P_k^n$ in $H_t^k$. 
Proof for $\hat{r}(P^n_k) = O(n)$

\[ F \subset H \]

(between large sets there is always an edge)

\[ F^k \subset H^k \]

Lemma 1 (Key Lemma):
Any 2-colouring $\chi'$ of $E(F^k)$ has a blue copy of $P_n$ or a red copy of $P^k_n$.

\[ F' \subset (H^k)_t \]

Lemma 2:
If $\chi'$ on $E(F^k)$ has a blue copy of $P_n$, then $\chi$ has a blue copy of $P^k_n$ in $H^k_t$.

Lemma 3:
If $\chi'$ on $E(F^k)$ has a red copy of $P^k_n$, then $\chi$ has a red copy of $P^k_n$ in $H^k_t$.
Proof for $\hat{r}(\mathcal{P}_n^k) = O(n)$

Lemma 1 (Key Lemma):
Any 2-colouring $\chi'$ of $E(F^k)$ has a blue copy of $P_n$ or a red copy of $P_n^k$.
Proof for $\hat{r}(P_n^k) = O(n)$

**Theorem (Pokrovskiy, 2017):**
Let $E(K_m)$ be coloured with red/blue. Then $V(K_m)$ can be covered by $k$ blue paths and a red balanced complete $(k + 1)$-partite graph (vertex-disjoint).

**Lemma 1 (Key Lemma):**
Any 2-colouring $\chi'$ of $E(F^k)$ has a blue copy of $P_n$ or a red copy of $P_n^k$. 

(between large sets there is always an edge)
Proof for $\hat{r}(P_n^k) = O(n)$

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**Lemma 1 (Key Lemma):**
Any 2-colouring $\chi'$ of $E(F^k)$ has a blue copy of $P_n$ or a red copy of $P_n^k$.

**Proof:** Extend $\chi'$ by colouring non-edges on $V(F^k)$ red.
Proof for $\hat{r}(P^k_n) = O(n)$

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(between large sets there is always an edge)
Proof for $\hat{r}(P_n^k) = O(n)$

Theorem (Pokrovskiy, 2017): Let $E(K_m)$ be coloured with red/blue. Then $V(K_m)$ can be covered by $k$ blue paths and a red balanced complete $(k+1)$-partite graph (vertex-disjoint).

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Proof: Extend $\chi'$ by colouring non-edges on $V(F^k)$ red. Suppose that there is no blue copy of $P_n$. By Pokrovskiy's Theorem find a huge red balanced complete $(k+1)$-partite graph.
Proof for $\hat{r}(P_n^k) = O(n)$

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Let $E(K_m)$ be coloured with red/blue. Then $V(K_m)$ can be covered by $k$ blue paths and a red balanced complete $(k+1)$-partite graph (vertex-disjoint).

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\[ F \subset H \]

(between large sets there is always an edge)

\[ F^k \subset H^k \]

colouring $\chi'$

\[ F' \subset (H^k)_t \]

colouring $\chi$

\[ \text{no blue } K_{2k,2k} \]
Proof for $\hat{r}(P_n^k) = O(n)$

**Theorem (Pokrovskiy, 2017):**
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**Proof:** Extend $\chi'$ by colouring non-edges on $V(F^k)$ red.
Suppose that there is no blue copy of $P_n$.
By Pokrovskiy's Theorem find a huge red balanced complete $(k + 1)$-partite graph. Using only the edges from $F \subset H$ find a copy of $P_n$. 

Let $E(K_m)$ be coloured with red/blue. Then $V(K_m)$ can be covered by $k$ blue paths and a red balanced complete $(k + 1)$-partite graph (vertex-disjoint).
Proof for $\hat{r}(P_n^k) = O(n)$

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By Pokrovskiy's Theorem find a huge red balanced complete $(k + 1)$-partite graph. Using only the edges from $F \subset H$ find a copy of $P_n$. 

Proof for $\hat{r}(P_n^k) = O(n)$

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**Proof:** Extend $\chi'$ by colouring non-edges on $V(F^k)$ red.
Suppose that there is no blue copy of $P_n$.
By Pokrovs'kiy's Theorem find a huge red balanced complete $(k+1)$-partite graph. Using only the edges from $F \subset H$ find a copy of $P_n$.
Then use the edges from $F^k$ to finish a red copy of $P_n^k$. 

Proof for $\hat{r}(P_n^k) = O(n)$

**Theorem (Pokrovskiy, 2017):**
Let $E(K_m)$ be coloured with red/blue. Then $V(K_m)$ can be covered by $k$ blue paths and a red balanced complete $(k + 1)$-partite graph (vertex-disjoint).

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**Diagram:**
- $F \subset H$
- $(F^k) \subset H^k$
- $F' \subset (H^k)_t$

(colouring $\chi'$)

(colouring $\chi$)

(no blue $K_{2k,2k}$)

(between large sets there is always an edge)
Open problems and work in progress

Problem 1 (What about grids?):
Let $G_{k,n}$ be the grid graph of size $k \times n$, i.e. the cartesian product of the paths $P_k$ and $P_n$.
If $k$ is a fixed constant, then the previous result implies $\hat{r}(G_{k,n}) = O(n)$.

Now, let $G_{d,n}$ be the $d$-dimensional grid, obtained by taking the cartesian product of $d$ copies of $P_n$.

Question (CJKMMRR; 2017+): For any $d \geq 2$, is it true that $\hat{r}(G_{d,n}) = O(n^d)$?

Theorem (C., Miralaei, Reding, Schacht, Taraz; 2018+): $\hat{r}(G_{2,n}) = O(n^3 + o(1))$. 
Open problems and work in progress

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Open problems and work in progress

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Open problems and work in progress

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Now, let $G_n^d$ be the $d$-dimensional grid, obtained by taking the cartesian product of $d$ copies of $P_n$.

![Grid graphs](image)
Open problems and work in progress

Problem 1 (What about grids?): Let $G_{k,n}$ be the grid graph of size $k \times n$, i.e. the cartesian product of the paths $P_k$ and $P_n$. If $k$ is a fixed constant, then the previous result implies

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Now, let $G^d_n$ be the $d$-dimensional grid, obtained by taking the cartesian product of $d$ copies of $P_n$.

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Open problems and work in progress

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$$\hat{r}(G_d^2) = O(n^{3+o(1)})$$
Open problems and work in progress

Problem 1 (What about grids?): Let $G_{k,n}$ be the grid graph of size $k \times n$, i.e. the cartesian product of the paths $P_k$ and $P_n$. If $k$ is a fixed constant, then the previous result implies

$$\hat{r}(G_{k,n}) = O(n).$$

Now, let $G^d_n$ be the $d$-dimensional grid, obtained by taking the cartesian product of $d$ copies of $P_n$.

Question (CJKMMRR; 2017+)

For any $d \geq 2$, is it true that $\hat{r}(G^d_n) = O(n^d)$?

Theorem (C., Miralaei, Reding, Schacht, Taraz; 2018+)

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Problem 2 (What about more than 2 colours?):
Denote with $\hat{r}_q(F)$ the size Ramsey number in case of colourings with $q$ colours. It is still true that

$\hat{r}_q(P_n^k) = O(n)$?
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Problem 3 (What about hypergraphs?):
How large is the size Ramsey number of tight paths and tight cycles?
Thanks for your attention!