Extension of universal cycles for globally identifying colorings of cycles

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LAAS-CNRS, France

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Identifying codes

Definition

An identifying code of a graph is a subset C of the vertices such that:

- Every vertex has a neighbor in C (Domination);
- Every vertex has a unique identifier (Identification).
  - Each vertex has a subset of C in its neighborhood and is the only one to have this set.

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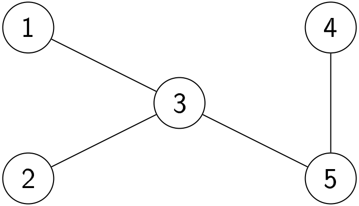

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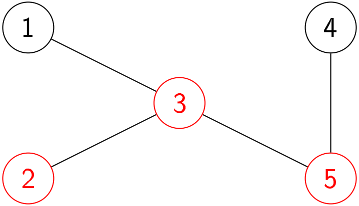

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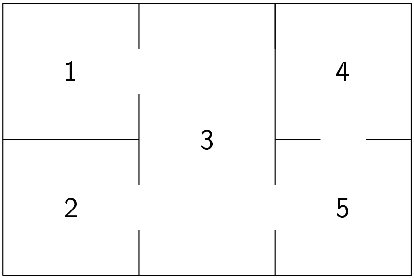

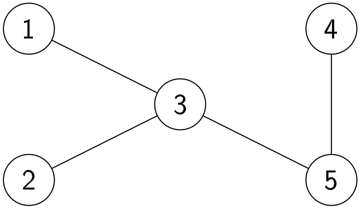

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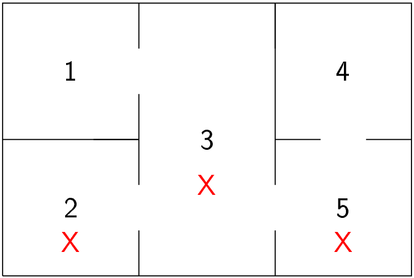

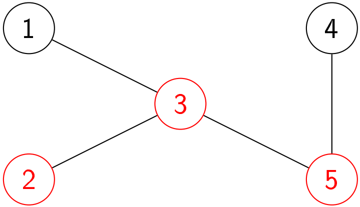

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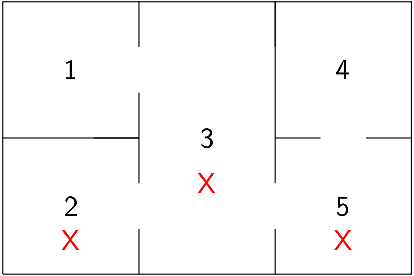

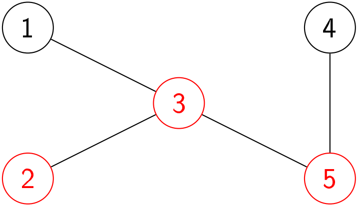

- Domination: each room is guarded.
- Identification: in case of fire, we know exactly where the fire is.

Variant: identifying coloring

- Each vertex has a color.
  - No need to have different colors for neighbors.
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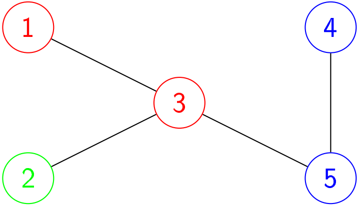

	Red	Green	Blue
1	X		
2	X	X	
3	X	X	X
4			X
5	X		X

Aim

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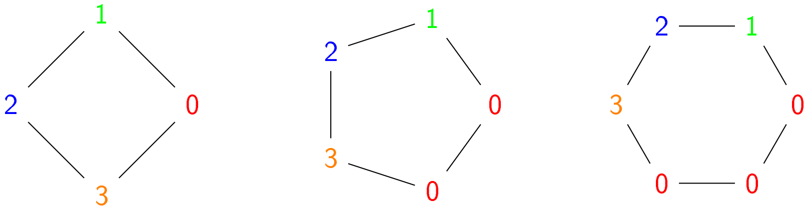

Jackson's universal cycles

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$$0 — 0 — 1 \qquad\qquad 0 — 1 — 1$$

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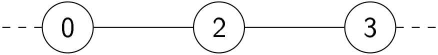
- The proof is constructive.

Differences

- A vertex: Up to 3 colors in its neighborhood.



Differences

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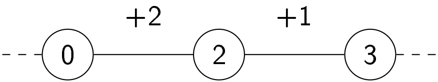
- 2 differences.



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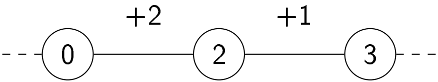

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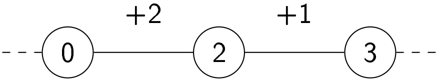

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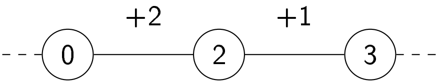

- A set of at most 3 colors: initial color + two differences.
- We arbitrary chose the order of the 3 colors.
    - For example, $\{0, 1\}$ will only appear with colors 0, 0, and 1, in this order.

Graph of differences

We define a new graph to work on these differences.

- Two consecutive differences a and b $\longrightarrow$ directed edge from vertex a to vertex b.

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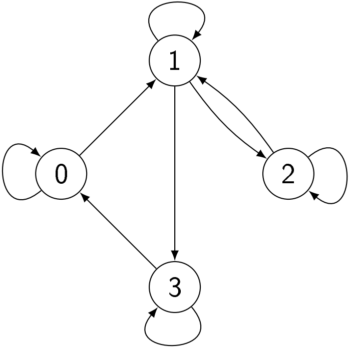

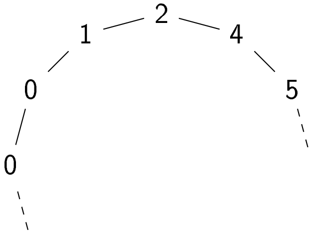

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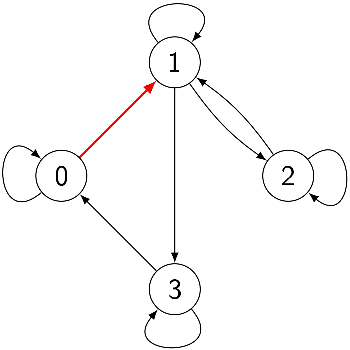

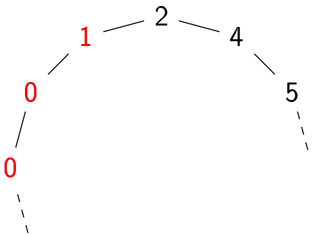

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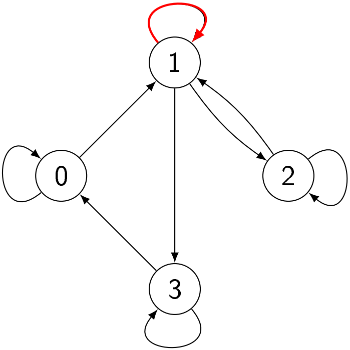

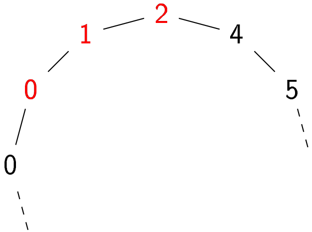

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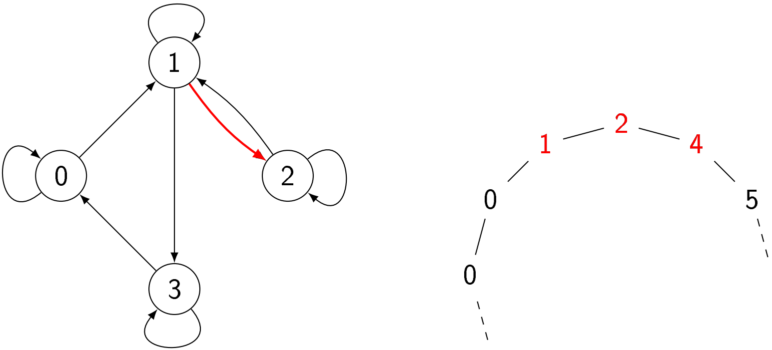

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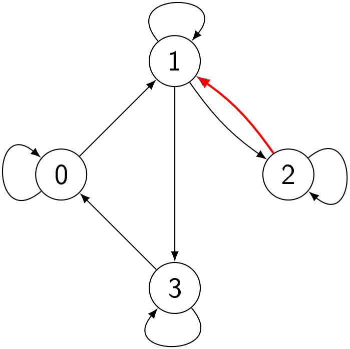

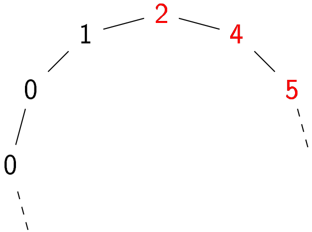

Restrictions

- Edges $(0, i)$ and $(i, 0)$ $\longrightarrow$ same color sets.

$$\cdots\; 0 \overset{+0}{\text{——}} 0 \overset{+i}{\text{——}} i \;\cdots \qquad \cdots\; 0 \overset{+i}{\text{——}} i \overset{+0}{\text{——}} i \;\cdots$$

Restrictions

- Edges $(0, i)$ and $(i, 0)$ $\longrightarrow$ same color sets.
- Edge $(\frac{L}{3}, \frac{L}{3})$ $\longrightarrow$ same color sets with different initial colors.

+0 +i

- - - 0 — 0 — i - - - -

+i +0

- - - 0 — i — i - - - -

+3 +3

- - - 0 — 3 — 6 - - -

+3 +3

- - - 3 — 6 — 0 - - -

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--- 0 — 0 — i ---

+i +0

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--- 0 — 3 — 6 ---

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+4 +4

--- 0 — 4 — 0 ---

+4 +4

--- 4 — 0 — 4 ---

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- Edge $(\frac{L}{3}, \frac{L}{3})$ $\longrightarrow$ same color sets with different initial colors.
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- The graph has to be Eulerian.

$$\cdots\ 0 \overset{+0}{\text{——}} 0 \overset{+i}{\text{——}} i\ \cdots \qquad \cdots\ 0 \overset{+i}{\text{——}} i \overset{+0}{\text{——}} i\ \cdots$$

$$\cdots\ 0 \overset{+3}{\text{——}} 3 \overset{+3}{\text{——}} 6\ \cdots \qquad \cdots\ 3 \overset{+3}{\text{——}} 6 \overset{+3}{\text{——}} 0\ \cdots$$

$$\cdots\ 0 \overset{+4}{\text{——}} 4 \overset{+4}{\text{——}} 0\ \cdots \qquad \cdots\ 4 \overset{+4}{\text{——}} 0 \overset{+4}{\text{——}} 4\ \cdots$$

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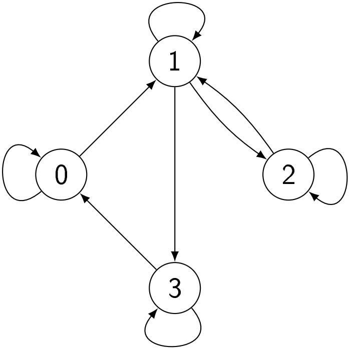

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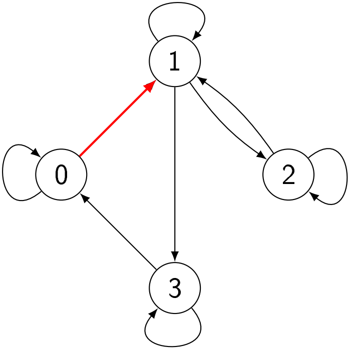

$+1$

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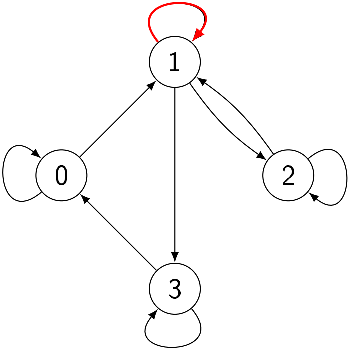

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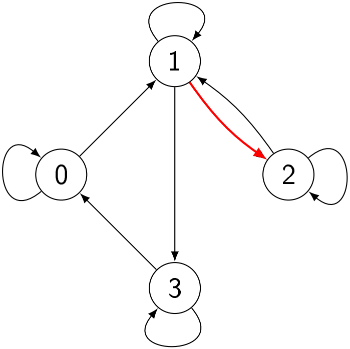

+1 +1 +2

Circuit

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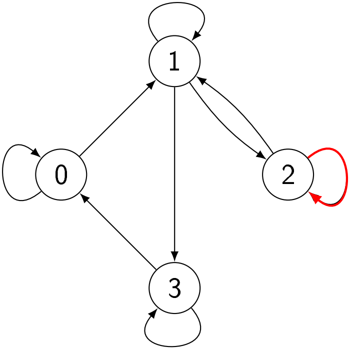

+1 +1 +2 +2

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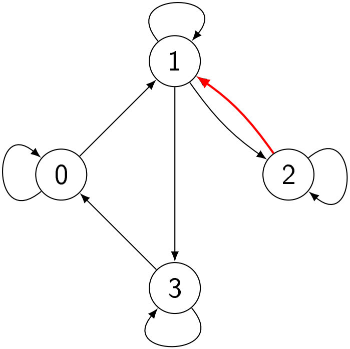

$+1 \; +1 \; +2 \; +2 \; +1$

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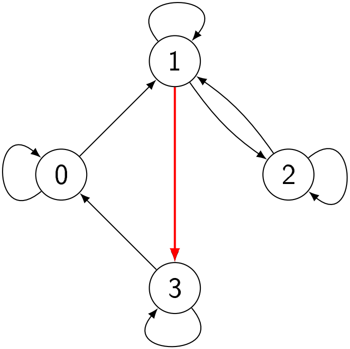

+1 +1 +2 +2 +1 +3

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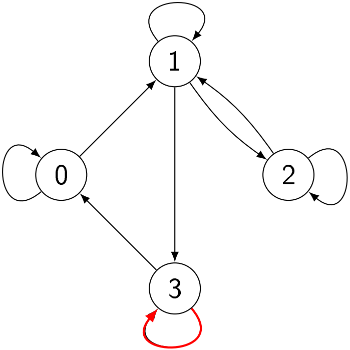

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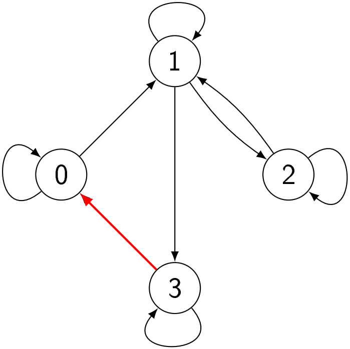

+1 +1 +2 +2 +1 +3 +3 +0

Circuit

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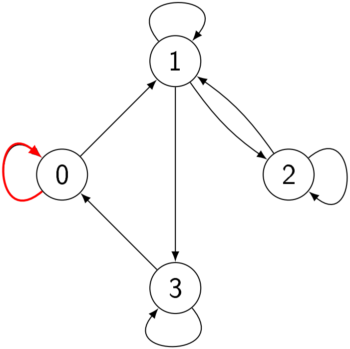

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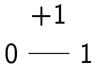

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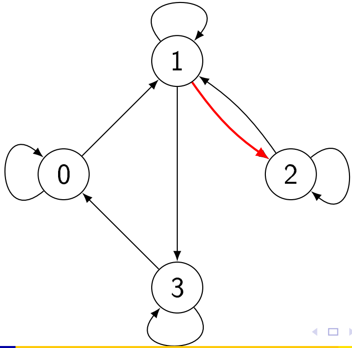
$$0 \overset{+1}{——} 1 \overset{+1}{——} 2 \overset{+2}{——} 4$$



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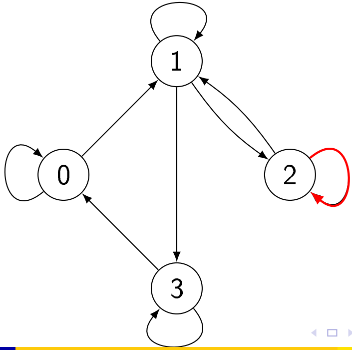
$$0 \overset{+1}{\text{——}} 1 \overset{+1}{\text{——}} 2 \overset{+2}{\text{——}} 4 \overset{+2}{\text{——}} 6$$



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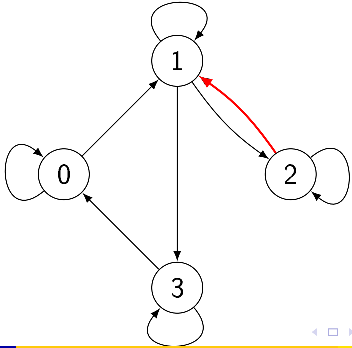
$$\begin{array}{ccccccccccc} & +1 & & +1 & & +2 & & +2 & & +1 & \\ 0 & — & 1 & — & 2 & — & 4 & — & 6 & — & 7 \end{array}$$



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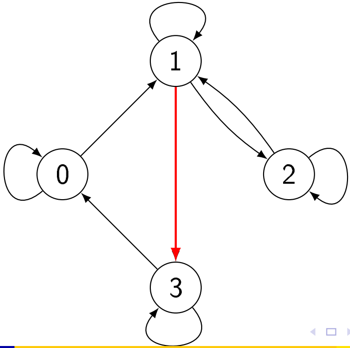
$$0 \overset{+1}{\text{———}} 1 \overset{+1}{\text{———}} 2 \overset{+2}{\text{———}} 4 \overset{+2}{\text{———}} 6 \overset{+1}{\text{———}} 7 \overset{+3}{\text{———}} 2$$



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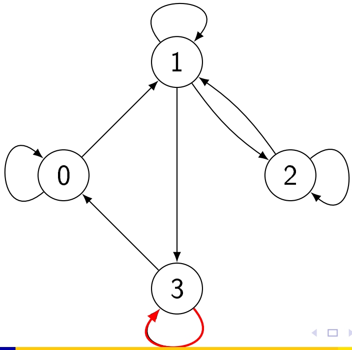

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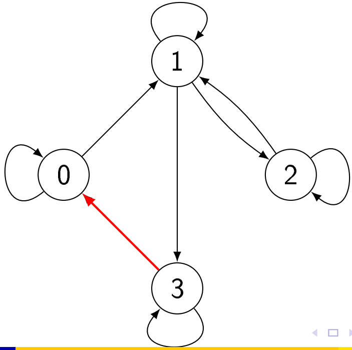
$$\begin{matrix} & +1 & & +1 & & +2 & & +2 & & +1 & & +3 & & +3 & & +0 & \\ 0 & — & 1 & — & 2 & — & 4 & — & 6 & — & 7 & — & 2 & — & 5 & — & 5 \end{matrix}$$



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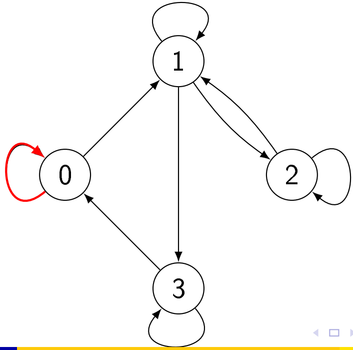

Loop

- Repetition of the process from the previous last vertex...

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    - Merging consecutive paths

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+1 +1 +2 +2 +1 +3 +3 +0 +0

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	+1		+1		+2		+2		+1		+3		+3		+0		+0	
0		1		2		4		6		7		2		5		5		5

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	+1	+1	+2	+2	+1	+3	+3	+0	+0
0	1	2	4	6	7	2	5	5	5
5	6	7	1	3	4	7	2	2	2

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+1	+1	+2	+2	+1	+3	+3	+0	+0	
0	1	2	4	6	7	2	5	5	5
5	6	7	1	3	4	7	2	2	2
2	3	4	6	0	1	4	7	7	7

Loop

- Repetition of the process from the previous last vertex...
  - Merging consecutive paths
- ... until the cycle is complete (color of the last vertex = color of the first vertex = 0).

	+1	+1	+2	+2	+1	+3	+3	+0	+0
0	1	2	4	6	7	2	5	5	5
5	6	7	1	3	4	7	2	2	2
2	3	4	6	0	1	4	7	7	7
7	0	1	3	5	6	1	4	4	4

Loop

- Repetition of the process from the previous last vertex...
  - Merging consecutive paths
- ... until the cycle is complete (color of the last vertex = color of the first vertex = 0).

	+1	+1	+2	+2	+1	+3	+3	+0	+0
0	1	2	4	6	7	2	5	5	5
5	6	7	1	3	4	7	2	2	2
2	3	4	6	0	1	4	7	7	7
7	0	1	3	5	6	1	4	4	4
4	5	6	0	2	3	6	1	1	1

Loop

- Repetition of the process from the previous last vertex...
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2	3	4	6	0	1	4	7	7	7
7	0	1	3	5	6	1	4	4	4
4	5	6	0	2	3	6	1	1	1
1	2	3	5	7	0	3	6	6	6

Loop

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7	0	1	3	5	6	1	4	4	4
4	5	6	0	2	3	6	1	1	1
1	2	3	5	7	0	3	6	6	6
6	7	0	2	4	5	0	3	3	3

Loop

- Repetition of the process from the previous last vertex...
  - Merging consecutive paths
- ... until the cycle is complete (color of the last vertex = color of the first vertex = 0).

	+1	+1	+2	+2	+1	+3	+3	+0	+0
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5	6	7	1	3	4	7	2	2	2
2	3	4	6	0	1	4	7	7	7
7	0	1	3	5	6	1	4	4	4
4	5	6	0	2	3	6	1	1	1
1	2	3	5	7	0	3	6	6	6
6	7	0	2	4	5	0	3	3	3
3	4	5	7	1	2	5	0	0	0

Cycle with 8 colors

1 0 6 4 3 2 2 2 7 4 3 1 7 6 5 5 5 2 7 6 4 2 1 0 0 0 5 2 1 7 5 4 3 3 3 0 5 4 2 0 7 6 6 6 3 0 7 5 3 2 1 1 1 6 3 2 0 6 5 4 4 4 1 6 5 3 1 0 7 7 7 4

Occasional problem

- What if we obtain 0 too soon ?

$$0 \overset{+1}{\text{——}} 1 \overset{+1}{\text{——}} 2 \overset{+2}{\text{——}} 4 \overset{+2}{\text{——}} 6 \overset{+1}{\text{——}} 7 \text{——} \cdots \text{——} 0$$

Occasional problem

- What if we obtain 0 too soon ?
- $\rightarrow$ Consider a smaller path without the first difference.

$$0 \overset{+1}{\text{——}} 1 \overset{+1}{\text{——}} 2 \overset{+2}{\text{——}} 4 \overset{+2}{\text{——}} 6 \overset{+1}{\text{——}} 7 \text{ —— } \cdots \text{ —— } 0$$

$$0 \text{ ———————— } 1 \overset{+1}{\text{——}} 3 \overset{+2}{\text{——}} 5 \overset{+2}{\text{——}} 6 \overset{+1}{\text{——}} \cdots \text{ — } L\text{-}1$$

$$L\text{-}1 \overset{+1}{\text{—}} 0 \overset{+1}{\text{——}} 1 \overset{+2}{\text{——}} 3 \overset{+2}{\text{——}} 5 \overset{+1}{\text{——}} 6 \text{ —— } \cdots \text{ — } L\text{-}1$$

Occasional problem

- What if we obtain 0 too soon ?
- $\rightarrow$ Consider a smaller path without the first difference.
  - The new path is a shift of the first and is therefore different.

$$0 \overset{+1}{—} 1 \overset{+1}{—} 2 \overset{+2}{—} 4 \overset{+2}{—} 6 \overset{+1}{—} 7 — \cdots — 0$$

$$0 \overset{+1}{———} 1 \overset{+2}{—} 3 \overset{+2}{—} 5 \overset{+1}{—} 6 — \cdots — L\text{-}1$$

$$L\text{-}1 \overset{+1}{—} 0 \overset{+1}{—} 1 \overset{+2}{—} 3 \overset{+2}{—} 5 \overset{+1}{—} 6 — \cdots — L\text{-}1$$

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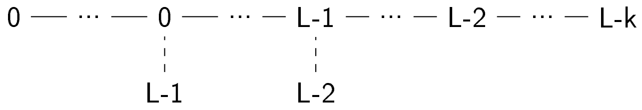
$$0 \overset{+1}{———} 1 \overset{+2}{—} 3 \overset{+2}{—} 5 \overset{+1}{—} 6 — \cdots — \text{L-1}$$

$$\text{L-1} \overset{+1}{—} 0 \overset{+1}{—} 1 \overset{+2}{—} 3 \overset{+2}{—} 5 \overset{+1}{—} 6 — \cdots — \text{L-1}$$

- Lose vertex with initial color $L-1$ and differences $+1$, $+1$.

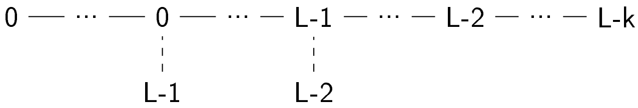
Occasional problem

- Repeat k times, until all L paths (for all initial colors) have been used.



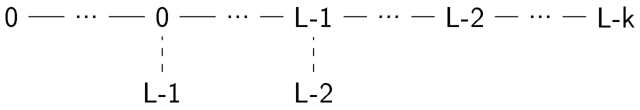
Occasional problem

- Repeat k times, until all L paths (for all initial colors) have been used.
- Path starting with 0, ending with $L-k$.



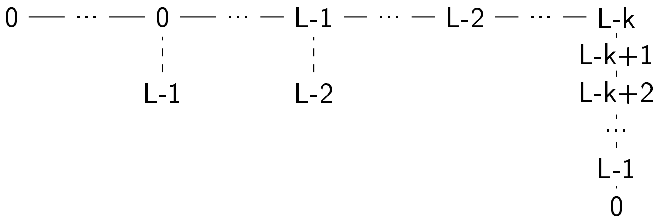
Occasional problem

- Repeat k times, until all L paths (for all initial colors) have been used.
- Path starting with 0, ending with $L-k$.
- Missing vertices with differences $(+1, +1)$ and initial colors $L-1$, ..., $L-k$.



Occasional problem

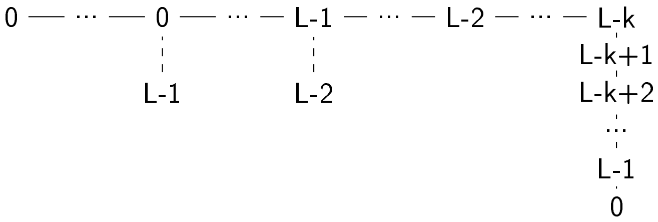
- Repeat k times, until all L paths (for all initial colors) have been used.
- Path starting with 0, ending with $L - k$.
- Missing vertices with differences $(+1, +1)$ and initial colors $L - 1$, ..., $L - k$.



- Add them at the end!

Occasional problem

- Repeat k times, until all L paths (for all initial colors) have been used.
- Path starting with 0, ending with $L - k$.
- Missing vertices with differences $(+1, +1)$ and initial colors $L - 1$, ..., $L - k$.



- Add them at the end!
  - No missing $(+1, +1)$ differences

Occasional problem

- Repeat k times, until all L paths (for all initial colors) have been used.
- Path starting with 0, ending with $L-k$.
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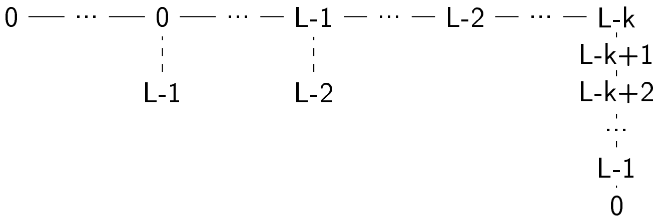

- Add them at the end!
  - No missing $(+1, +1)$ differences
  - Path starting and ending with 0 $\approx$ cycle

Improvement

- When L odd, we can add some vertices with differences $(0, \frac{L-1}{2})$, $(\frac{L-1}{2}, \frac{L-1}{2})$ and $(\frac{L-1}{2}, 0)$.
- Path starting and ending with two 0s.

$$0 — 0 — 4 — 8 — 3 — 7 — 2 — 6 — 1 — 5 — 0 — 0$$

- Easy to insert in the previous cycle.

Results

With L colors, cycles build with this method have length:

$$\frac{L^3+5L}{6} - \left\{ \begin{array}{ll} \frac{L+2}{4}L & \text{if } L \equiv 0[4] \\ \frac{L+3}{4}L-1 & \text{if } L \equiv 1[4] \\ \frac{L^2}{4} & \text{if } L \equiv 2[4] \\ \frac{L+1}{4}L-1 & \text{if } L \equiv 3[4] \end{array} \right\} - \left\{ \begin{array}{l} \frac{L}{3} \text{ if } L \equiv 0[3] \\ 0 \text{ else} \end{array} \right\}$$

Not optimal

- For $L = 5$, we can build a cycle with $\frac{L^3+5L}{6} = 25$ vertices.
  - Choosing differences (+1,+2) instead of (+2,+2).
- Not for $L = 4$ (optimal $= 10 < 14$).

Thank you (in French)