DOMINOES, DIMERS AND DETERMINANTS

Discrete Mathematics Seminar — 28 August 2012

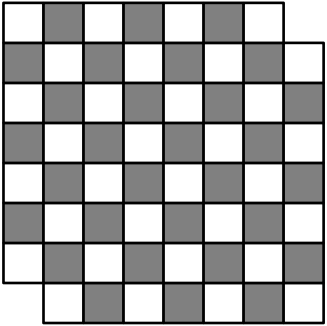
Norm Do

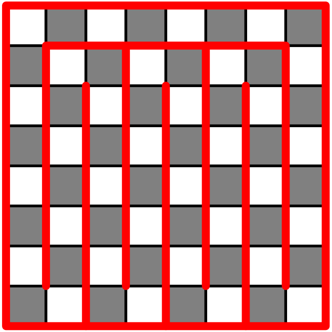
Monash University

If you take two squares and glue them together, then you get a domino. If you take two atoms and join them by a bond, then you get a dimer. And if you take a linear algebra course and attend a tutorial, then you get to compute a determinant. In this talk, we'll discuss some of the amazing mathematics connecting these objects.

Dominoes on a checkerboard

- Can you tile an 8×8 checkerboard with dominoes?
- Can you tile the checkerboard if one corner square is removed?
- Can you tile the checkerboard if opposite corner squares are removed?

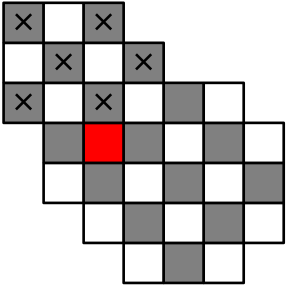




- Can you always tile the checkerboard if one white square and one black square are removed?

Dominoes and marriage

When can you tile a subset of a checkerboard with dominoes?



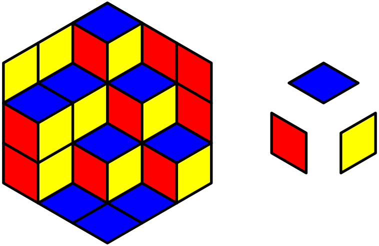
- Think of black squares as men and white squares as women.
- We want to marry each man to one of his neighbours.
- We need gender balance. . . but we also need each group of men to have enough neighbours.

Hall’s marriage theorem

In any set of men and women, we can marry each man to a woman he knows if and only if each group of men has enough acquaintances.

The problem of the calissons

A calisson is a parallelogram consisting of two equilateral triangles of side length 1. We want to tile a regular hexagon of side length n with calissons. Must there be an equal number of calissons in each orientation?

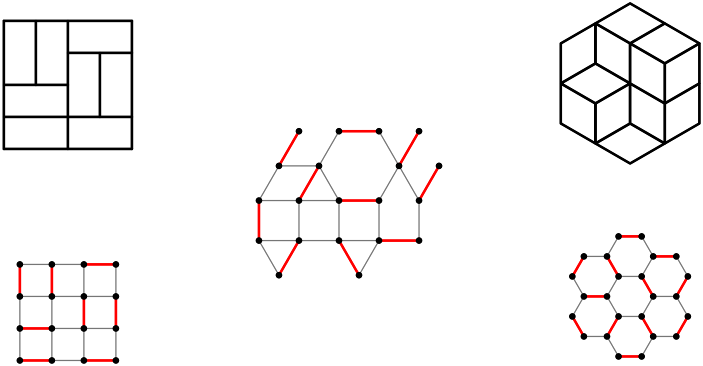


Answer

The answer is “yes” and the proof requires no words!

A primer on dimers

Tiling with dominoes or with calissons are examples of dimer problems. A dimer covering of a graph is a set of edges that covers every vertex once.



Counting domino tilings

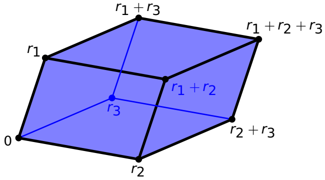
How many ways are there to tile a checkerboard with dominoes?

n	# tilings of an $n \times n$ checkerboard	prime factorisation
0	1	1
2	2	2^1
4	36	$2^2.3^2$
6	6728	$2^3.29^2$
8	12988816	$2^4.17^2.53^2$
10	258584046368	$2^5.241^2.373^2$
12	53060477521960000	$2^6.5^4.31^2.53^2.701^2$
14	112202208776036178000000	$2^7.3^{10}.5^6.19^2.29^4.61^2$

Surprisingly — or unsurprisingly if you read the title of this seminar — you need to know determinants to calculate the answer.

Discovering determinants

- The determinant of a matrix M is the “volume” of the parallelepiped spanned by the rows of M.



- The determinant of M can be calculated using the following formula.

$$\det M = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, M_{1,\sigma(1)}\, M_{2,\sigma(2)} \cdots M_{n,\sigma(n)}$$

- The determinant of M is equal to the product of its eigenvalues. These are numbers c for which there is a non-zero vector satisfying $Mx = cx$.

Kasteleyn's method

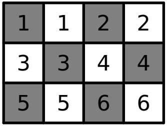
For a subset S of a checkerboard, construct the matrix K whose rows are labelled by black squares and whose columns are labelled by white squares. Let the entry corresponding to a black square and a white square be

- 1 if the squares are horizontally adjacent;
- i if the squares are vertically adjacent; and
- 0 if the squares are not adjacent.

Kasteleyn's theorem

The number of domino tilings of S is $|\det K|$.

Example



$$|\det K| = 11$$

$$K = \begin{bmatrix} 1 & 0 & i & 0 & 0 & 0 \\ 1 & 1 & 0 & i & 0 & 0 \\ i & 0 & 1 & 1 & i & 0 \\ 0 & i & 0 & 1 & 0 & i \\ 0 & 0 & i & 0 & 1 & 0 \\ 0 & 0 & 0 & i & 1 & 1 \end{bmatrix}$$

Why does Kasteleyn's method work?

- The permutation σ is trying to "marry" black square 1 to white square $\sigma(1)$, black square 2 to white square $\sigma(2)$, and so on.

$$\det K = \sum_{\sigma \in S_6} \text{sign}(\sigma)\, K_{1,\sigma(1)}\, K_{2,\sigma(2)} \cdots K_{6,\sigma(6)}$$

- If this is possible, we get a non-zero contribution and if this is impossible, we get a zero contribution.

$$\det K = \sum_{\text{tilings}} \text{sign}(\sigma)\, K_{1,\sigma(1)}\, K_{2,\sigma(2)} \cdots K_{6,\sigma(6)}$$

- We need to check that $\text{sign}(\sigma)\, i^{\#\text{ vertical dominoes}}$ is the same for each tiling. This is handled by some careful accounting.

$$\det K = \sum_{\text{tilings}} \text{sign}(\sigma)\, i^{\#\text{ vertical dominoes}}$$

Kasteleyn's method can count dimer coverings on any planar graph — you just need to be clever about the choice of weight attached to each edge.

Kasteleyn on checkerboards

The number of domino tilings of an $m \times n$ checkerboard is $\sqrt{|D|}$.

$$D = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(2\cos\frac{j\pi}{m+1} + 2i\cos\frac{k\pi}{n+1} \right)$$

Proof

The terms in the product above are the eigenvalues of $M = \left[\begin{array}{c|c} 0 & K \\ \hline K^T & 0 \end{array} \right]$.

Example

Consider the calculation for the 8×8 checkerboard.

$$\begin{array}{cccc}
\left(2\cos\frac{\pi}{9} + 2i\cos\frac{\pi}{9}\right) & \left(2\cos\frac{\pi}{9} + 2i\cos\frac{2\pi}{9}\right) & \cdots & \left(2\cos\frac{\pi}{9} + 2i\cos\frac{8\pi}{9}\right) \\
\left(2\cos\frac{2\pi}{9} + 2i\cos\frac{\pi}{9}\right) & \left(2\cos\frac{2\pi}{9} + 2i\cos\frac{2\pi}{9}\right) & \cdots & \left(2\cos\frac{2\pi}{9} + 2i\cos\frac{8\pi}{9}\right) \\
\vdots & \vdots & & \vdots \\
\left(2\cos\frac{8\pi}{9} + 2i\cos\frac{\pi}{9}\right) & \left(2\cos\frac{8\pi}{9} + 2i\cos\frac{2\pi}{9}\right) & \cdots & \left(2\cos\frac{8\pi}{9} + 2i\cos\frac{8\pi}{9}\right) \\
\hline
\end{array}$$

$$12\,988\,816^2$$

Consequences of Kasteleyn

- Place k dominoes $B_1W_1, B_2W_2, \ldots, B_kW_k$ on a checkerboard. The probability that these occur in a uniformly chosen random tiling is

$$\prod_{i=1}^{k} K(B_i, W_i) \times \det \left[K^{-1}(W_i, B_j)\right]_{1 \leq i,j \leq k}.$$

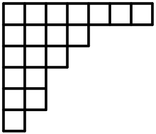
- The mn^{th} root of the number of domino tilings of an $m \times n$ checkerboard limits to $e^{G/\pi}$, where $G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$ is Catalan's constant.

Partitions

A partition is a way to

- write n as a sum of positive integers where order doesn't matter; or
- push n square boxes into the corner of a 2-D room.

$$19 = 7 + 4 + 3 + 2 + 2 + 1 \quad \longleftrightarrow$$



Theorem

The number of partitions of n is the coefficient of x^n in $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

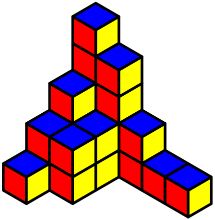
Proof

Write each term in the product as a geometric series and expand.

$$\begin{array}{cccccccccccccc}
 & (1 & + & x^1 & + & x^2 & + & x^3 & + & x^4 & + & \cdots) \\
\times & (1 & + & x^2 & + & x^4 & + & x^6 & + & x^8 & + & \cdots) \\
\times & (1 & + & x^3 & + & x^6 & + & x^9 & + & x^{12} & + & \cdots) \\
\times & (& & & & \text{and} & \text{so} & \text{on} & & & &)
\end{array}$$

Plane partitions

A plane partition is a way to push n cubic boxes into the corner of a 3-D room.



$\longleftrightarrow$

5	4	2	1	1
3	2			
2	2			
1				

MacMahon's formula

- The number of plane partitions of n that fit inside an $a \times b \times c$ box is the coefficient of x^n in

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{1-x^{i+j+k-1}}{1-x^{i+j+k-2}}.$$

- The number of plane partitions of n is the coefficient of x^n in

$$\prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}.$$

The Lindström–Gessel–Viennot trick

- Let G be a directed acyclic graph with weighted edges.
- Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be sets of vertices.
- Let M_{ij} be the sum of the weights of paths from A_i to B_j.

Theorem

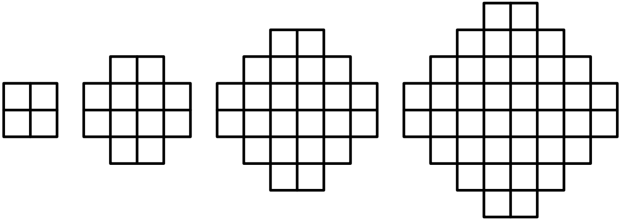
The determinant of the matrix M has the following combinatorial meaning.

$$\det M = \sum_{\sigma} \sum_{P} \operatorname{sign}(\sigma)\ \omega(P)$$

The sum is over permutations $\sigma \in S_n$ and tuples $P = (P_1, P_2, \ldots, P_n)$ of non-intersecting paths, where P_k is a path from A_k to $B_{\sigma(k)}$.

Aztec diamonds

The Aztec diamond $AZ(n)$ is the nth term in the following sequence of shapes.



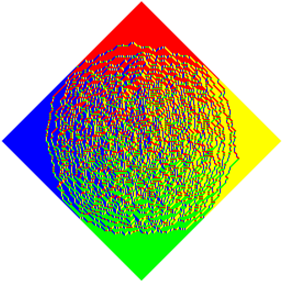
Theorem (Elkies–Kuperberg–Larsen–Propp, 1992)

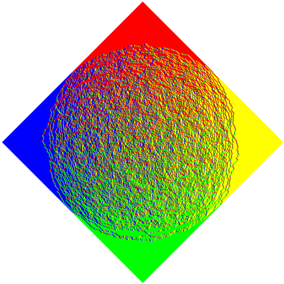
The number of ways to tile $AZ(n)$ with dominoes is $2^{n(n+1)/2}$.

From Aztec diamonds to arctic circles

Colour the dominoes in a tiling of $AZ(n)$ according to

- whether the domino is horizontal or vertical; and
- whether its left/upper square is on a black or white square.



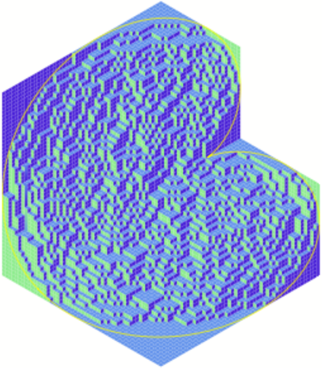


Arctic circle theorem (Jockusch–Propp–Shor, 1998)

The frozen boundary of almost all domino tilings of $AZ(n)$ approaches a circle as n approaches infinity.

Limit shape theorem (Cohn–Kenyon–Propp, 2001)

Take a polygon that can be tiled with calissons and let the tile size approach zero. In the limit, a frozen boundary occurs in the shape of an algebraic curve.



- One idea in the proof is to treat a tiling as a “random surface”.
- Kenyon and Okounkov described the limit shape using ideas from mathematical physics.

Thanks

If you would like more information, you can

- find the slides at `http://users.monash.edu/~normd`
- email me at `normdo@gmail.com`
- speak to me