DOMINOES, DIMERS AND DETERMINANTS
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If you take two squares and glue them together, then you get a domino. If you take two atoms and join them by a bond, then you get a dimer. And if you take a linear algebra course and attend a tutorial, then you get to compute a determinant. In this talk, we’ll discuss some of the amazing mathematics connecting these objects.
Dominoes on a checkerboard

- Can you tile an 8 × 8 checkerboard with dominoes?
- Can you tile the checkerboard if one corner square is removed?
- Can you tile the checkerboard if opposite corner squares are removed?

- Can you always tile the checkerboard if one white square and one black square are removed?
Dominoes and marriage

When can you tile a subset of a checkerboard with dominoes?

Think of black squares as men and white squares as women.

We want to marry each man to one of his neighbours.

We need gender balance... but we also need each group of men to have enough neighbours.

Hall’s marriage theorem

In any set of men and women, we can marry each man to a woman he knows if and only if each group of men has enough acquaintances.
The problem of the calissons

A calisson is a parallelogram consisting of two equilateral triangles of side length 1. We want to tile a regular hexagon of side length $n$ with calissons. Must there be an equal number of calissons in each orientation?

Answer
The answer is “yes” and the proof requires no words!
A primer on dimers

Tiling with dominoes or with calissons are examples of dimer problems. A dimer covering of a graph is a set of edges that covers every vertex once.
Counting domino tilings

How many ways are there to tile a checkerboard with dominoes?

<table>
<thead>
<tr>
<th>$n$</th>
<th># tilings of an $n \times n$ checkerboard</th>
<th>prime factorisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$2^1$</td>
</tr>
<tr>
<td>4</td>
<td>36</td>
<td>$2^2.3^2$</td>
</tr>
<tr>
<td>6</td>
<td>6728</td>
<td>$2^3.29^2$</td>
</tr>
<tr>
<td>8</td>
<td>12988816</td>
<td>$2^4.17.53^2$</td>
</tr>
<tr>
<td>10</td>
<td>258584046368</td>
<td>$2^5.241.373^2$</td>
</tr>
<tr>
<td>12</td>
<td>530604775219600000</td>
<td>$2^6.5^4.31.53^2.701^2$</td>
</tr>
<tr>
<td>14</td>
<td>1122022087760361780000000</td>
<td>$2^7.3^{10}.5^6.19^2.29^4.61^2$</td>
</tr>
</tbody>
</table>

Surprisingly — or unsurprisingly if you read the title of this seminar — you need to know determinants to calculate the answer.
Discovering determinants

- The **determinant** of a matrix $M$ is the “volume” of the parallelepiped spanned by the rows of $M$.

- The determinant of $M$ can be calculated using the following formula.

  $$
  \det M = \sum_{\sigma \in S_n} \text{sign}(\sigma) \ M_{1,\sigma(1)} \ M_{2,\sigma(2)} \cdots \ M_{n,\sigma(n)}
  $$

- The determinant of $M$ is equal to the product of its **eigenvalues**. These are numbers $c$ for which there is a non-zero vector satisfying $Mx = cx$.
Kasteleyn’s method

For a subset $S$ of a checkerboard, construct the matrix $K$ whose rows are labelled by black squares and whose columns are labelled by white squares. Let the entry corresponding to a black square and a white square be

- 1 if the squares are horizontally adjacent;
- $i$ if the squares are vertically adjacent; and
- 0 if the squares are not adjacent.

Kasteleyn’s theorem

The number of domino tilings of $S$ is $|\det K|$.

Example

\[
K = \begin{bmatrix}
1 & 0 & i & 0 & 0 & 0 \\
1 & 1 & 0 & i & 0 & 0 \\
i & 0 & 1 & 1 & i & 0 \\
0 & i & 0 & 1 & 0 & i \\
0 & 0 & i & 0 & 1 & 0 \\
0 & 0 & 0 & i & 1 & 1 \\
\end{bmatrix}
\]

\[
\begin{array}{cccccc}
1 & 1 & 2 & 2 & & \\
3 & 3 & 4 & 4 & & \\
5 & 5 & 6 & 6 & & \\
\end{array}
\]

$|\det K| = 11$
Why does Kasteleyn’s method work?

- The permutation $\sigma$ is trying to “marry” black square 1 to white square $\sigma(1)$, black square 2 to white square $\sigma(2)$, and so on.

$$\det K = \sum_{\sigma \in S_6} \text{sign}(\sigma) K_{1,\sigma(1)} K_{2,\sigma(2)} \cdots K_{6,\sigma(6)}$$

- If this is possible, we get a non-zero contribution and if this is impossible, we get a zero contribution.

$$\det K = \sum_{\text{tilings}} \text{sign}(\sigma) K_{1,\sigma(1)} K_{2,\sigma(2)} \cdots K_{6,\sigma(6)}$$

- We need to check that $\text{sign}(\sigma) \ # \text{vertical dominoes}$ is the same for each tiling. This is handled by some careful accounting.

$$\det K = \sum_{\text{tilings}} \text{sign}(\sigma) \ i \ # \text{vertical dominoes}$$

Kasteleyn’s method can count dimer coverings on any planar graph — you just need to be clever about the choice of weight attached to each edge.
Kasteleyn on checkerboards

The number of domino tilings of an $m \times n$ checkerboard is $\sqrt{|D|}$.

$$D = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(2 \cos \frac{j\pi}{m+1} + 2i \cos \frac{k\pi}{n+1}\right)$$

Proof

The terms in the product above are the eigenvalues of $M = \begin{bmatrix} 0 & K \\ K^T & 0 \end{bmatrix}$.

Example

Consider the calculation for the $8 \times 8$ checkerboard.

$$\begin{pmatrix} 2 \cos \frac{\pi}{9} + 2i \cos \frac{\pi}{9} & 2 \cos \frac{\pi}{9} + 2i \cos \frac{2\pi}{9} & \cdots & 2 \cos \frac{\pi}{9} + 2i \cos \frac{8\pi}{9} \\ 2 \cos \frac{2\pi}{9} + 2i \cos \frac{\pi}{9} & 2 \cos \frac{2\pi}{9} + 2i \cos \frac{2\pi}{9} & \cdots & 2 \cos \frac{2\pi}{9} + 2i \cos \frac{8\pi}{9} \\ \vdots & \vdots & \ddots & \vdots \\ 2 \cos \frac{8\pi}{9} + 2i \cos \frac{\pi}{9} & 2 \cos \frac{8\pi}{9} + 2i \cos \frac{2\pi}{9} & \cdots & 2 \cos \frac{8\pi}{9} + 2i \cos \frac{8\pi}{9} \end{pmatrix}$$

\[12\,988\,816^2\]
Consequences of Kasteleyn

- Place \( k \) dominoes \( B_1 W_1, B_2 W_2, \ldots , B_k W_k \) on a checkerboard. The probability that these occur in a uniformly chosen random tiling is

\[
\prod_{i=1}^{k} K(B_i, W_i) \times \text{det} \left[ K^{-1}(W_i, B_j) \right]_{1 \leq i, j \leq k}.
\]

- The \( mn^{th} \) root of the number of domino tilings of an \( m \times n \) checkerboard limits to \( e^{G/\pi} \), where \( G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \) is Catalan’s constant.
Partitions

A partition is a way to
- write $n$ as a sum of positive integers where order doesn’t matter; or
- push $n$ square boxes into the corner of a 2-D room.

\[ 19 = 7 + 4 + 3 + 2 + 2 + 1 \]

Theorem

The number of partitions of $n$ is the coefficient of $x^n$ in \( \prod_{k=1}^{\infty} \frac{1}{1-x^k} \).

Proof

Write each term in the product as a geometric series and expand.

\[
\begin{align*}
(1 + x^1 + x^2 + x^3 + x^4 + \cdots) \\
\times (1 + x^2 + x^4 + x^6 + x^8 + \cdots) \\
\times (1 + x^3 + x^6 + x^9 + x^{12} + \cdots) \\
\times (\text{and so on})
\end{align*}
\]
Plane partitions

A plane partition is a way to push $n$ cubic boxes into the corner of a 3-D room.

MacMahon’s formula

- The number of plane partitions of $n$ that fit inside an $a \times b \times c$ box is the coefficient of $x^n$ in

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - x^{i+j+k-1}}{1 - x^{i+j+k-2}}.$$  

- The number of plane partitions of $n$ is the coefficient of $x^n$ in

$$\prod_{k=1}^{\infty} \frac{1}{(1 - x^k)^k}.$$
The Lindström–Gessel–Viennot trick

- Let $G$ be a directed acyclic graph with weighted edges.
- Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be sets of vertices.
- Let $M_{ij}$ be the sum of the weights of paths from $A_i$ to $B_j$.

**Theorem**

The determinant of the matrix $M$ has the following combinatorial meaning.

$$\det M = \sum_{\sigma} \sum_P \text{sign}(\sigma) \omega(P)$$

The sum is over permutations $\sigma \in S_n$ and tuples $P = (P_1, P_2, \ldots, P_n)$ of non-intersecting paths, where $P_k$ is a path from $A_k$ to $B_{\sigma(k)}$. 

Aztec diamonds

The Aztec diamond $AZ(n)$ is the $n$th term in the following sequence of shapes.

Theorem (Elkies–Kuperberg–Larsen–Propp, 1992)
The number of ways to tile $AZ(n)$ with dominoes is $2^{n(n+1)/2}$. 

From Aztec diamonds to arctic circles

Colour the dominoes in a tiling of $AZ(n)$ according to

- whether the domino is horizontal or vertical; and
- whether its left/upper square is on a black or white square.

Arctic circle theorem (Jockusch–Propp–Shor, 1998)

The frozen boundary of almost all domino tilings of $AZ(n)$ approaches a circle as $n$ approaches infinity.
Limit shape theorem (Cohn–Kenyon–Propp, 2001)

Take a polygon that can be tiled with calissons and let the tile size approach zero. In the limit, a frozen boundary occurs in the shape of an algebraic curve.

- One idea in the proof is to treat a tiling as a “random surface”.

- Kenyon and Okounkov described the limit shape using ideas from mathematical physics.
Thanks

If you would like more information, you can

- find the slides at http://users.monash.edu/~normd
- email me at normdo@gmail.com
- speak to me