Series-parallelization of graphs

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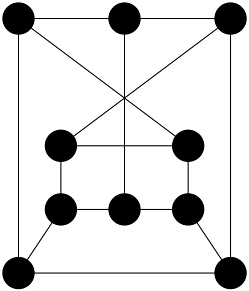
Joint work with Graham Farr (Monash University, Australia).

Planarization

MAXIMUM *INDUCED* PLANAR SUBGRAPH (MIPS)
Input: Graph G
Output: set $P \subseteq V(G)$ such that

- the *induced* subgraph $\langle P \rangle$ is planar,
- $|P|$ is maximum.



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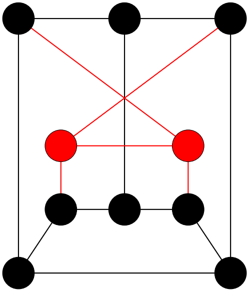

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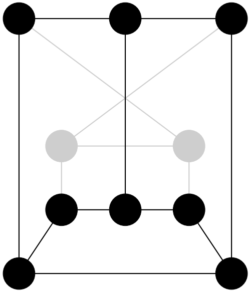


Planarization and Series-Parallelization

Equivalent to the Maximum Induced Planar Subgraph problem is the following:

Given a graph G, let $p(G)$ be the minimum number of vertices whose removal leaves a planar graph.

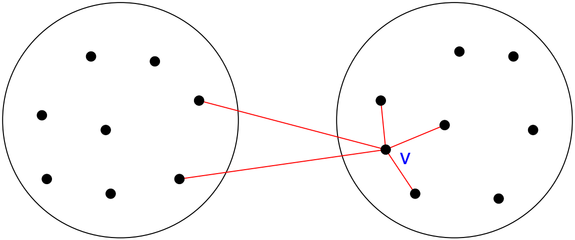
We may also consider $s(G)$, the minimum number of vertices whose removal leaves a series-parallel graph.

We will consider particularly graphs G with maximum or average degree at most d, and look for bounds of the form

$$p(G) \leq c_d|V(G)| \text{ or } s(G) \leq c_d|V(G)|$$

Simple argument for $d = 5$

Split vertices into two sets so that as many edges as possible cross the gap.



If a vertex v has degree 3 within one set, move it to other side.

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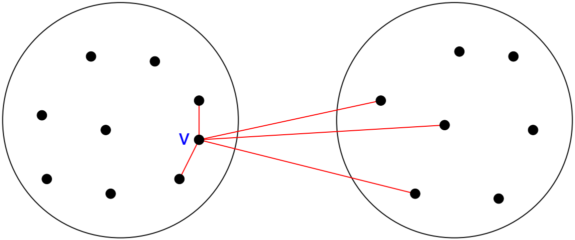

But then more edges cross the gap, which is impossible.

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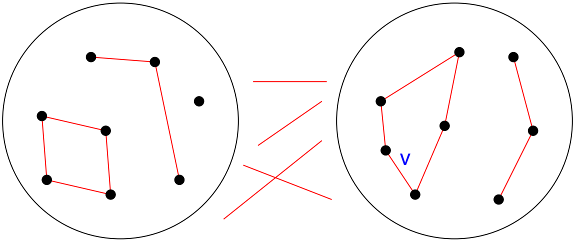

So within each set, every degree is at most two, so each set induces a series-parallel graph. Remove smaller set, so

$$s(G) \leq \frac{1}{2}|V| = \frac{d-2}{d+1}|V|.$$

Similar argument for $d = 8, 11, 14, \ldots$, i.e. $d \equiv 2 \pmod 3$.

Series-Parallel Reductions

Isolated vertex

delete

Series-Parallel Reductions

Isolated vertex

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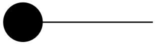
Series-Parallel Reductions

Isolated vertex

delete

Leaf

delete



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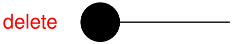
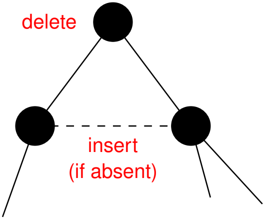


Degree 2



Series-Parallel Reductions

Isolated vertex

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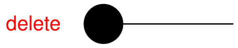

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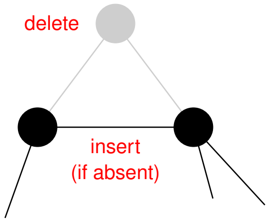
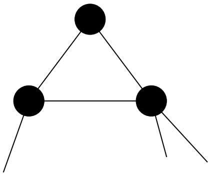


Series-Parallel Reductions

Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):



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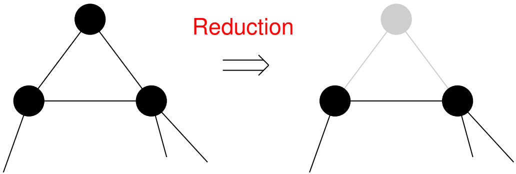

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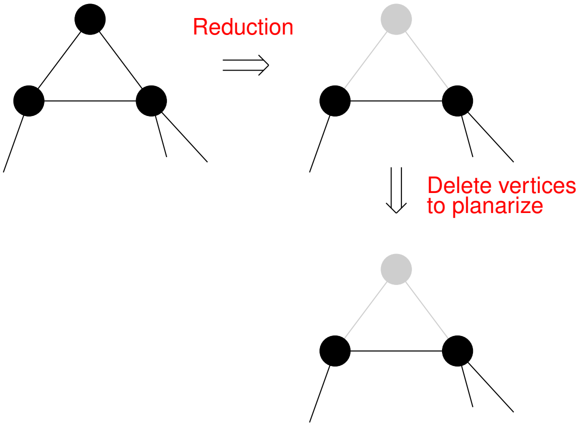

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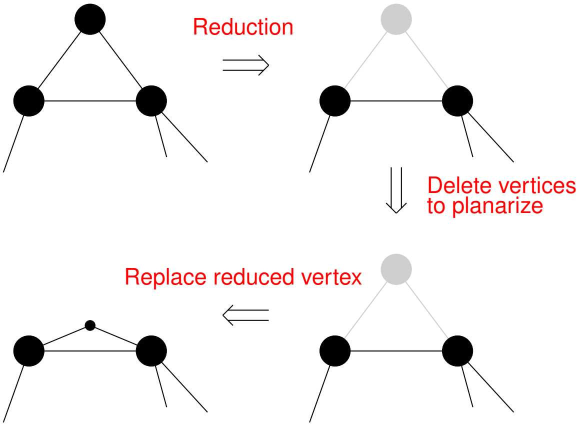

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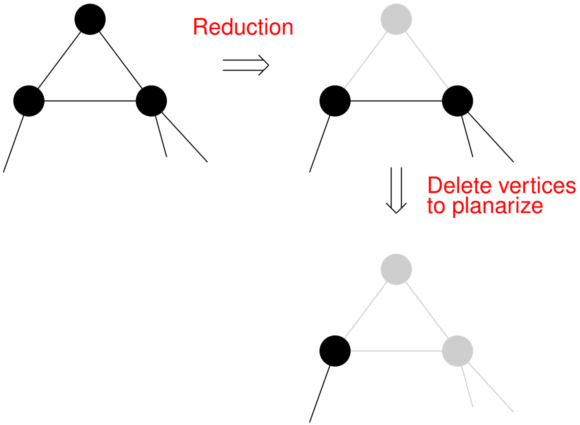

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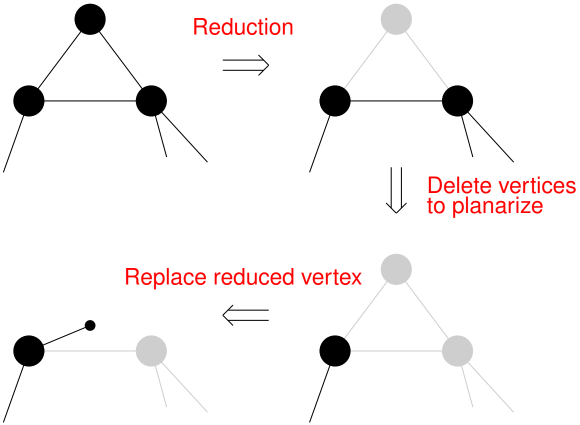

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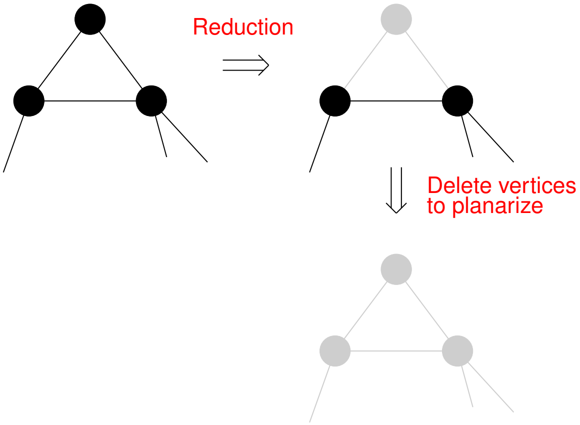

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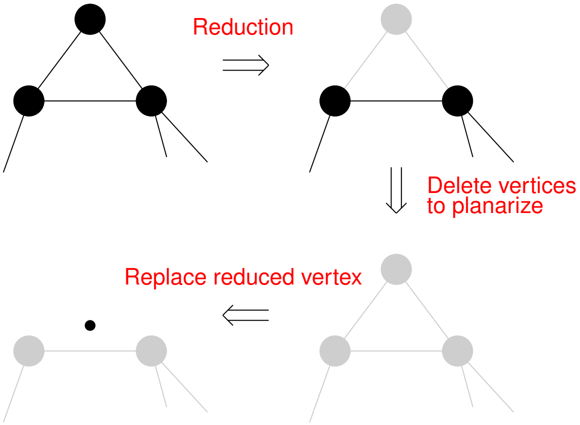


Series-Parallel Reductions

Reduced graph

Form reduced graph $r(G)$ by applying the 3 reduction operations to G as many times as possible. (True but not obvious that $r(G)$ is unique).

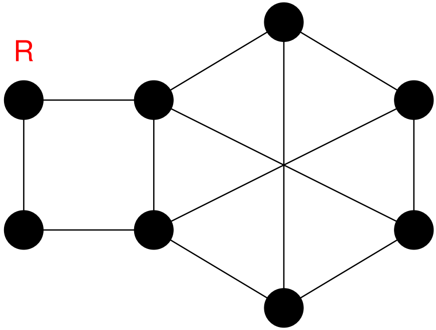
Properties

- $r(G)$ has minimum degree at least 3.
- $r(G)$ is empty if and only if G is S-P.
- Reducing does not change the minimum number of vertices which must be removed to make the graph S-P (or planar).

$$s(G) = s(r(G)).$$

Series-Parallel Reductions

Example



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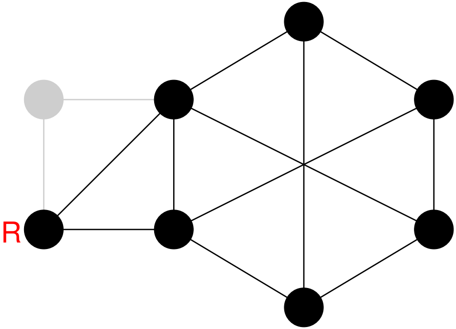

Series-Parallel Reductions

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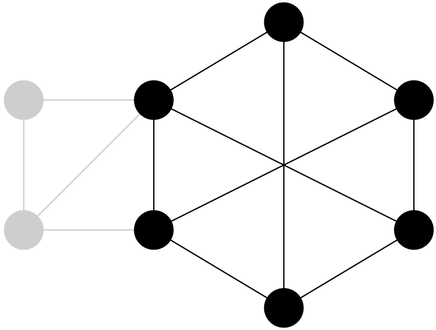


Making a graph S-P: Upper bound

Theorem

If G has minimum degree at least 3, then

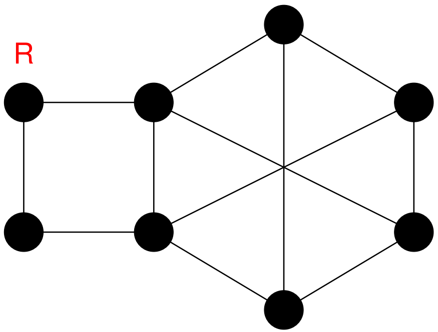
$$s(G) \le \sum_v \frac{d(v) - 2}{d(v) + 1}.$$

Very simple algorithm

$X := \emptyset$
while (graph is not empty)
  delete a vertex w of maximum degree
  $X := X \cup \{w\}$
  reduce
end while
return X

Series-Parallelization Algorithm

Example



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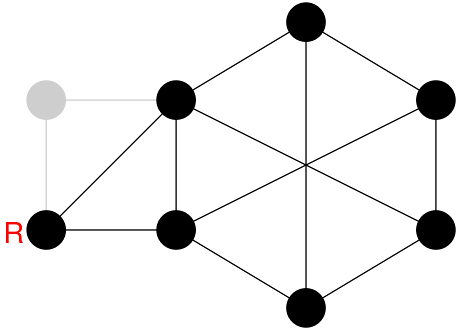

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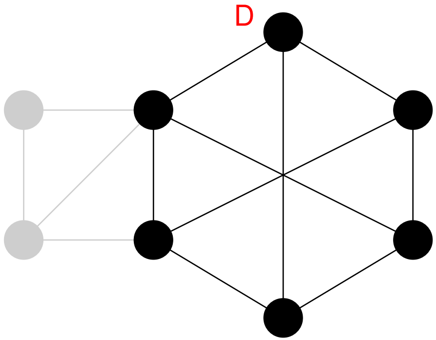

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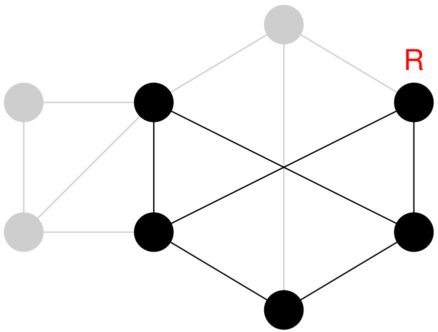

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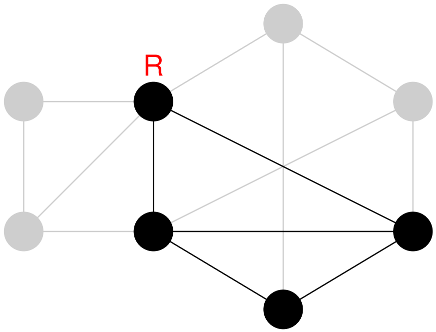

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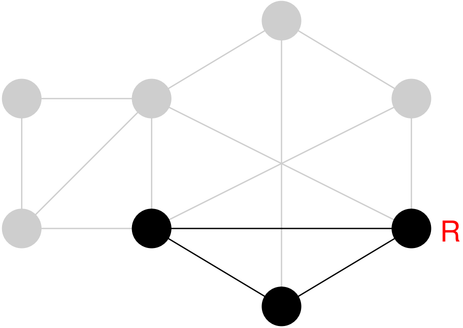

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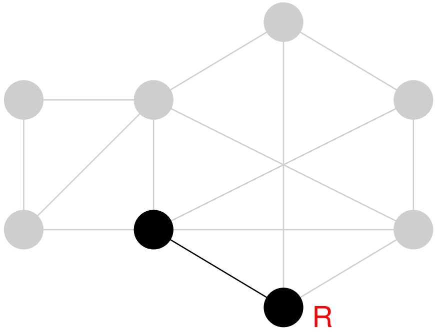

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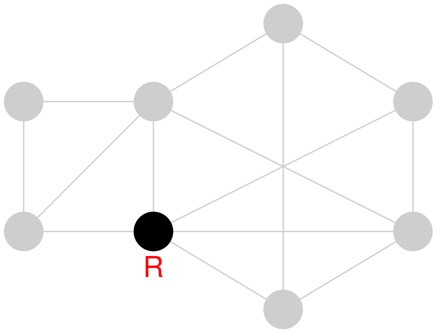


Series-Parallelization Algorithm

Example

Making a graph S-P: Upper bound

Theorem

$$s(G) \leq \sum_{v} \frac{d(v) - 2}{d(v) + 1}.$$

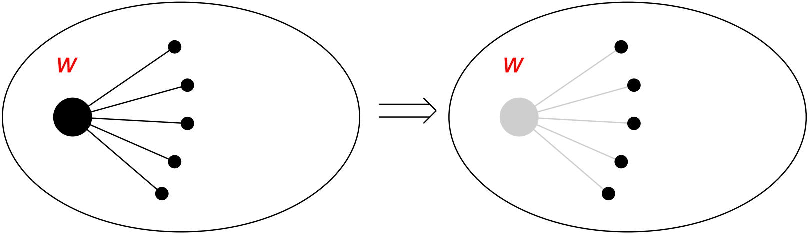
Proof

Induction on $n = |V(G)|$.

Inductive basis: empty graph, $s(G) = 0 =$ empty sum.

Now let G be any non-empty graph with min degree $\geq 3 \ldots$

Making a graph S-P: Upper bound



Graph G

$G' = G - w$

Delete vertex w of maximum degree

$\Downarrow$ Reduce

$G^* = r(G')$

Making a graph S-P: Upper bound

$G' = G - w,\ G^* = r(G')$

$$
\begin{aligned}
s(G) &\le 1 + s(G') \\
&= 1 + s(G^*) \\
&\le 1 + \sum_{v \in V(G^*)} \frac{d^*(v) - 2}{d^*(v) + 1} \quad \text{(induction)} \\
&\le 1 + \sum_{v \in V(G')} \frac{d'(v) - 2}{d'(v) + 1} \\
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$G' = G - w$, $G^* = r(G')$

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&= 1 - \sum_{v \in V(G'), v \sim w} \frac{3}{d(v)(d(v) + 1)} + \sum_{v \in V(G')} \frac{d(v) - 2}{d(v) + 1} \\
&\leq 1 - \sum_{v \in V(G'), v \sim w} \frac{3}{d(w)(d(w) + 1)} + \sum_{v \in V(G')} \frac{d(v) - 2}{d(v) + 1} \\
&= 1 - \frac{3}{d(w) + 1} + \sum_{v \in V(G')} \frac{d(v) - 2}{d(v) + 1} \\
&= \sum_{v \in V(G)} \frac{d(v) - 2}{d(v) + 1}.
\end{aligned}
$$

Making a graph S-P: Upper bound

$$
\begin{aligned}
&= 1 + \sum_{v \in V(G'), v \sim w} \frac{d(v)-3}{d(v)} + \sum_{v \in V(G'), v \not\sim w} \frac{d(v)-2}{d(v)+1} \\
&= 1 + \sum_{v \in V(G'), v \sim w} \left(\frac{d(v)-3}{d(v)} - \frac{d(v)-2}{d(v)+1} \right) + \sum_{v \in V(G')} \frac{d(v)-2}{d(v)+1} \\
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&\leq 1 - \sum_{v \in V(G'), v \sim w} \frac{3}{d(w)(d(w)+1)} + \sum_{v \in V(G')} \frac{d(v)-2}{d(v)+1} \\
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&= \sum_{v \in V(G)} \frac{d(v)-2}{d(v)+1}.
\end{aligned}
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Making a graph S-P: Upper bound

From this we obtain a result for average degree $d \geq 2$:

Theorem

Let G be a connected graph of average degree at most d. Then

$$s(G) \leq \frac{d-2}{d+1}|V(G)|.$$

For series-parallelization, this is best possible, because K_{d+1} is regular of degree d and we have to remove $d-2$ vertices to avoid a K_4 subgraph.

Making a graph S-P: Upper bound

From this we obtain a result for average degree $d \geq 2$:

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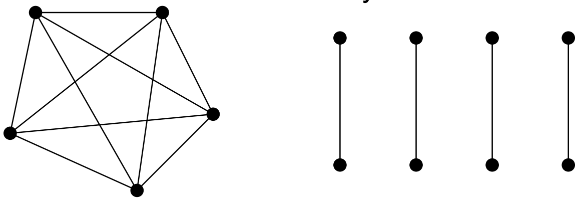
Connectedness is necessary:

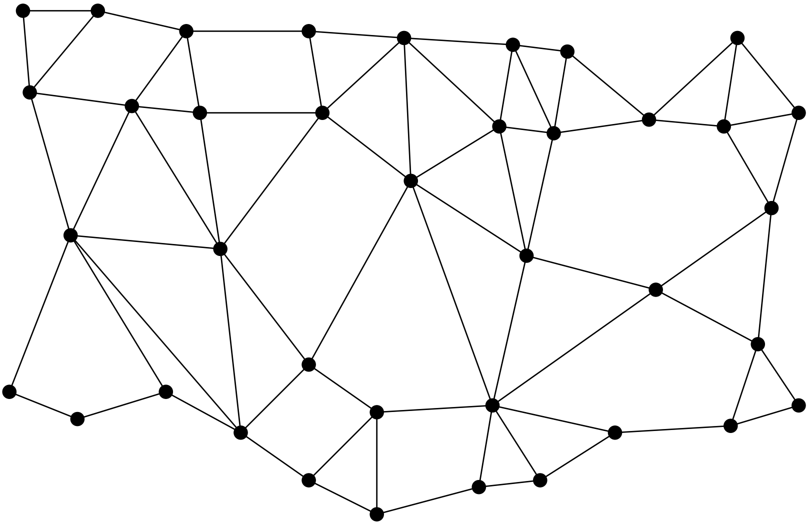


13 vertices, 14 edges, average degree $\frac{28}{13}$.

$\frac{d-2}{d+1}|V| = \frac{26}{41} < 1$, but $p(G) = 1, s(G) = 2$.

Fragmentability

. . . is about removing few vertices so as to break graphs into small pieces.



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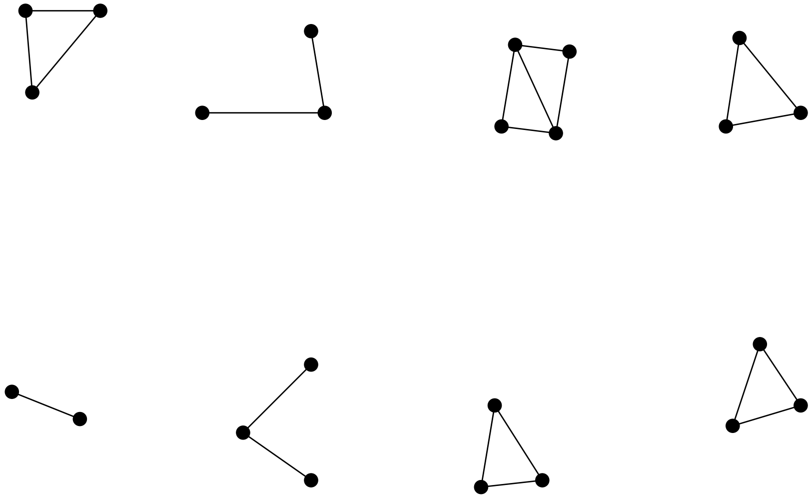

Fragmentability

remove few vertices: $\leq \varepsilon$ of the vertices of the graph,

...to leave small pieces: $\leq C$ vertices in each component

A graph is (C, ε)-*fragmentable* if, by removing some fraction $\leq \varepsilon$ of its vertices, you can leave components all of size $\leq C$.

A *class* of graphs is ε-fragmentable if there is a constant C so that every graph in the class is (C, ε)-fragmentable.

The lowest (infimum) possible ε is the *coefficient of fragmentability* of the class.

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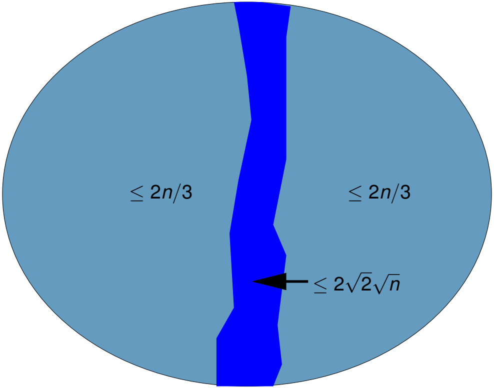
Series-parallelization and fragmentability

Series-parallelization is useful for breaking graphs into small pieces.
Given G, with max/ave degree $\leq d$:

1. remove vertices from G to leave induced series-parallel subgraph $\langle P \rangle$;
2. remove $o(n)$ vertices from $\langle P \rangle$ to leave bounded size pieces (e.g., apply Planar Separator Theorem (Lipton & Tarjan) recursively).

Series-parallelization and fragmentability

Lipton-Tarjan separator theorem: A planar graph with n vertices can be broken into 2 pieces with at most $2n/3$ vertices each by removing at most $2\sqrt{2}\sqrt{n}$ vertices.



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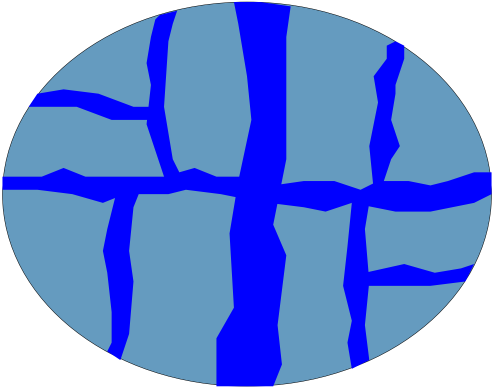
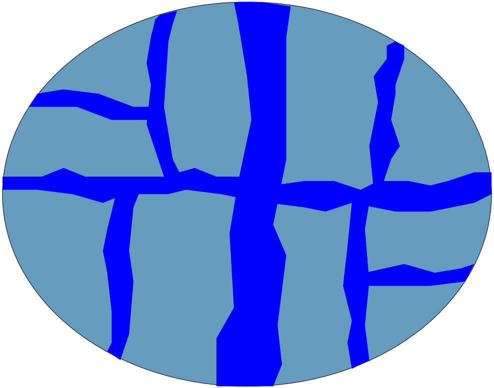


By repeating the process, we can break up the graph into small ($\leq C$ vertices) pieces.

Series-parallelization and fragmentability

Lipton-Tarjan separator theorem: A planar graph with n vertices can be broken into 2 pieces with at most $2n/3$ vertices each by removing at most $2\sqrt{2}\sqrt{n}$ vertices.



Conclusion: For any $\varepsilon > 0$, we can remove a proportion ε of the vertices from any planar graph, and ensure no fragment has more than $535/\varepsilon^2$ vertices.

Series-parallelization and fragmentability

For series-parallel graphs, the coefficient of fragmentability is 0.

Hence, for the class of graphs with maximum or average degree at most d, the coefficient of fragmentability is at most $\frac{d-2}{d+1}$.

The best lower bound (due to Haxell, Pikhurko and Thomason) is:

$$\frac{d-2}{d+2} \text{ for even } d \geq 4, \text{ and } \frac{d^2-5}{(d+1)(d+3)} \text{ for odd } d \geq 5.$$

Note that lower bounds for fragmentability are also lower bounds for series-parallelization.

Back to planarization

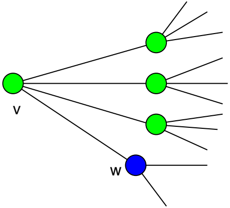
For $d = 3$, we have $p(G) \leq \frac{1}{4}|V(G)|$ and the fraction $\frac{1}{4}$ is best possible (from fragmentability bounds).

But for $d \geq 4$, there is a gap between upper and lower bounds:

d	2	3	4	5	6
$\frac{d-2}{d+1}$	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{7}$
Lower bound	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$

Better Planarization

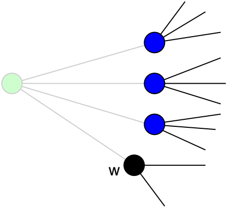
Consider a graph of maximum degree 4. Suppose there is a vertex *v* of degree 4 adjacent to a vertex *w* of degree 3.



Delete the vertex *v*.

Better Planarization

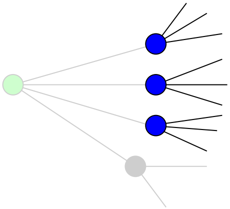
Consider a graph of maximum degree 4. Suppose there is a vertex *v* of degree 4 adjacent to a vertex *w* of degree 3.



Vertex *w* is now degree 2, so is removed by reduction.

Better Planarization

Consider a graph of maximum degree 4. Suppose there is a vertex v of degree 4 adjacent to a vertex w of degree 3.



Overall effect: $v_4 \longrightarrow v_4 - 4$, $v_3 \longrightarrow v_3 + 2$. After (roughly) $v_4/4$ such steps, we get graph G' which is 3-regular with $v_3 + v_4/2$ vertices. Then $p(G') \leq v_3/4 + v_4/8$, so

$$p(G) \leq v_4/4 + v_3/4 + v_4/8 \leq 3|V|/8.$$

Back to planarization

This argument can be made precise and extended to general (average) degree *d*. We get an upper bound of the form

$$\frac{d - 9/4}{d + 1} + O(1/d^3).$$

d	2	3	4	5	6
$\frac{d-2}{d+1}$	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{7}$
New upper bound	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{19}{40}$	$\frac{131}{240}$
Lower bound	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$

Better series-parallelization

In a sense, the upper bound of $\frac{d-2}{d+1}$ for series parallelization is best possible, since for any graph in which every component is K_{d+1}, we must remove $d-2$ vertices from each component to make it series-parallel.

However, for connected graphs, we might hope to get

$$s(G) \leq j(d)n + o(n)$$

where $j(d) < \frac{d-2}{d+1}$.

Better series-parallelization

However, for connected graphs, we might hope to get

$$s(G) \leq j(d)n + o(n)$$

where $j(d) < \frac{d-2}{d+1}$.
For $d = 3$, there can be no improvement. But for maximum degree $d = 4, 5, 6$ we can get

$$s(G) \leq j(d)n + C_d$$

d	2	3	4	5	6
$\frac{d-2}{d+1}$	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{7}$
Planarization u.b.	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{19}{40}$	$\frac{131}{240}$
j(d)	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{19}{40}$	$\frac{11}{20}$

Better series-parallelization

For maximum degree $d \leq 6$ we can get

$$s(G) \leq j(d)n + C_d$$

d	2	3	4	5	6
$\frac{d-2}{d+1}$	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{7}$
j(d)	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{19}{40}$	$\frac{11}{20}$

In fact for maximum degree ≤ 6 we can get the equivalent "vertex-wise" result:

$$s(G) \leq \sum_{v} j(d(v)) + C_d.$$

Better series-parallelization

For maximum degree $d \leq 6$ we have

$$s(G) \leq \sum_v j(d(v)) + C_d.$$

It seems natural to want to extend this to all d. But it turns out that this cannot be done while keeping $j(d) \leq \frac{d-2}{d+1}$ for all d.

d	2	3	4	5	6	7	8
$\frac{d-2}{d+1}$	0	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{6}{9}$
j(d)	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{19}{40}$	$\frac{11}{20}$	?	?

More generally

For a set S of graphs, define $\mu(S, \Gamma)$ to be the minimum number μ such that any graph in Γ with n vertices can be made S-minor-free by removing at most $(\mu + o(1))n$ vertices.

So we have been considering $\mu(\{K_5, K_{3,3}\}, \Gamma_d^c)$ and $\mu(\{K_4\}, \Gamma_d^c)$.

More generally

What do we know about $\mu(\{K_r\}, \Gamma_d^c)$?

r\ d	2	3	4	5	6	7	…	13
2	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$		$\frac{12}{13}$
3	0	$\frac{1}{3}$?	?	?	?	?		?
4	0	$\frac{1}{4}$	$\leq \frac{3}{8}$	$\leq \frac{19}{40}$	$\leq \frac{11}{20}$	$\leq \frac{5}{8}$		$\geq \frac{10}{13}$
5	0	$\frac{1}{4}$	$\leq \frac{3}{8}$	$\leq \frac{19}{40}$	$\leq \frac{131}{240}$	$\leq \frac{1009}{1680}$		$< \frac{10}{13}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$c_f(\Gamma_d^c)$	0	$\frac{1}{4}$	$\geq \frac{1}{3}$	$\geq \frac{5}{12}$	$\geq \frac{1}{2}$	$\geq \frac{21}{40}$		