Series-parallelization of graphs

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Joint work with Graham Farr (Monash University, Australia).
Planarization

MAXIMUM INDUCED PLANAR SUBGRAPH (MIPS)
Input: Graph $G$
Output: set $P \subseteq V(G)$ such that
- the induced subgraph $\langle P \rangle$ is planar,
- $|P|$ is maximum.
Planarization

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Input: Graph $G$
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Planarization and Series-Parallelization

Equivalent to the Maximum Induced Planar Subgraph problem is the following:

Given a graph $G$, let $p(G)$ be the minimum number of vertices whose removal leaves a planar graph.

We may also consider $s(G)$, the minimum number of vertices whose removal leaves a series-parallel graph.

We will consider particularly graphs $G$ with maximum or average degree at most $d$, and look for bounds of the form

$$p(G) \leq c_d |V(G)| \text{ or } s(G) \leq c_d |V(G)|$$
Simple argument for $d = 5$

Split vertices into two sets so that as many edges as possible cross the gap.

If a vertex $v$ has degree 3 within one set, move it to the other side.
Simple argument for $d = 5$

Split vertices into two sets so that as many edges as possible cross the gap.

But then more edges cross the gap, which is impossible.
Simple argument for $d = 5$

Split vertices into two sets so that as many edges as possible cross the gap.

So within each set, every degree is at most two, so each set induces a series-parallel graph. Remove smaller set, so

$$s(G) \leq \frac{1}{2} |V| = \frac{d - 2}{d + 1} |V|.$$ 

Similar argument for $d = 8, 11, 14, \ldots$, i.e. $d \equiv 2 \pmod{3}$. 
Series-Parallel Reductions

Isolated vertex

delete
Series-Parallel Reductions

Isolated vertex
Series-Parallel Reductions

Isolated vertex

Leaf
Series-Parallel Reductions

Isolated vertex delete

Leaf delete
Series-Parallel Reductions

**Isolated vertex**
- Delete

**Leaf**
- Delete

**Degree 2**
- Delete
- Insert (if absent)
Series-Parallel Reductions

**Isolated vertex**
- Delete

**Leaf**
- Delete

**Degree 2**
- Delete
- Insert if absent
Series-Parallel Reductions

Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):
Series-Parallel Reductions

Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):

![Diagram showing series-parallel reduction with a series connection turned into a parallel connection.](image-url)
Series-Parallel Reductions

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Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):

Reduction

Delete vertices to planarize

Replace reduced vertex
Series-Parallel Reductions

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Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):

**Reduction**
- Delete vertices
- Replace reduced vertex to planarize

**Diagram:**
- Initial graph: Three connected vertices
- Reduction: Delete one vertex and replace it
- Planarization: Ensure graph is planar
Series-Parallel Reductions

Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):

Reduction

Delete vertices to planarize
Series-Parallel Reductions

Any set whose deletion makes new graph S-P (or planar) also makes original graph S-P (or planar):

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Replace reduced vertex
Series-Parallel Reductions

Reduced graph
Form reduced graph \( r(G) \) by applying the 3 reduction operations to \( G \) as many times as possible. (True but not obvious that \( r(G) \) is unique).

Properties

- \( r(G) \) has minimum degree at least 3.
- \( r(G) \) is empty if and only if \( G \) is S-P.
- Reducing does not change the minimum number of vertices which must be removed to make the graph S-P (or planar).

\[ s(G) = s(r(G)). \]
Series-Parallel Reductions

Example
Series-Parallel Reductions

Example
Series-Parallel Reductions

Example
Making a graph S-P: Upper bound

Theorem

If $G$ has minimum degree at least 3, then

$$s(G) \leq \sum_{v} \frac{d(v) - 2}{d(v) + 1}.$$ 

Very simple algorithm

$$X := \emptyset$$

while (graph is not empty)

    delete a vertex $w$ of maximum degree

    $$X := X \cup \{w\}$$

    reduce

end while

return $X$
Series-Parallelization Algorithm

Example

\[ R \]
Series-Parallelization Algorithm

Example
Series-Parallelization Algorithm

Example
Series-Parallelization Algorithm

Example
Series-Parallelization Algorithm

Example
Series-Parallelization Algorithm

Example
Series-Parallelization Algorithm

Example

![Graph Example]
Series-Parallelization Algorithm

Example
Series-Parallelization Algorithm

Example
Theorem

\[ s(G) \leq \sum_v \frac{d(v) - 2}{d(v) + 1}. \]

Proof

Induction on \( n = |V(G)| \).

Inductive basis: empty graph, \( s(G) = 0 = \) empty sum.

Now let \( G \) be any non-empty graph with min degree \( \geq 3 \ldots \)
Making a graph S-P: Upper bound

Graph $G$

Delete vertex $w$ of maximum degree

$G' = G - w$

Reduce

$G^* = r(G')$
Making a graph S-P: Upper bound

\[ G' = G - w, \quad G^* = r(G') \]

\[
s(G) \leq 1 + s(G') = 1 + s(G^*) \leq 1 + \sum_{v \in V(G^*)} \frac{d^*(v) - 2}{d^*(v) + 1} \text{ (induction)}
\]

\[
\leq 1 + \sum_{v \in V(G')} d''(v) - 2 \quad \frac{d''(v) - 2}{d''(v) + 1}
\]

\[
= 1 + \sum_{v \in V(G'), v \sim w} \frac{d''(v) - 2}{d''(v) + 1} + \sum_{v \in V(G'), v \not\sim w} \frac{d''(v) - 2}{d''(v) + 1}
\]

\[
= 1 + \sum_{v \in V(G'), v \sim w} \frac{d(v) - 3}{d(v)} + \sum_{v \in V(G'), v \not\sim w} \frac{d(v) - 2}{d(v) + 1}
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\text{s}(G) & \leq 1 + \text{s}(G') \\
& = 1 + \text{s}(G^*) \\
& \leq 1 + \sum_{v \in V(G^*)} \frac{d^*(v) - 2}{d^*(v) + 1} \quad \text{(induction)} \\
& \leq 1 + \sum_{v \in V(G')} \frac{d'(v) - 2}{d'(v) + 1} \\
& = 1 + \sum_{v \in V(G'), \, v \sim w} \frac{d'(v) - 2}{d'(v) + 1} + \sum_{v \in V(G'), \, v \not\sim w} \frac{d'(v) - 2}{d'(v) + 1} \\
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\[
\leq 1 + \sum_{v \in V(G^*)} \frac{d^*(v) - 2}{d^*(v) + 1} \quad \text{(induction)}
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\[
\leq 1 + \sum_{v \in V(G')} \frac{d'(v) - 2}{d'(v) + 1}
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= 1 + \sum_{v \in V(G'), v \sim w} \left( \frac{d(v) - 3}{d(v)} - \frac{d(v) - 2}{d(v) + 1} \right) + \sum_{v \in V(G')} \frac{d(v) - 2}{d(v) + 1}
\]

\[
= 1 - \sum_{v \in V(G'), v \sim w} \frac{3}{d(v)(d(v) + 1)} + \sum_{v \in V(G')} \frac{d(v) - 2}{d(v) + 1}
\]

\[
\leq 1 - \sum_{v \in V(G'), v \sim w} \frac{3}{d(w)(d(w) + 1)} + \sum_{v \in V(G')} \frac{d(v) - 2}{d(v) + 1}
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Making a graph S-P: Upper bound

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\end{align*} \]
Making a graph S-P: Upper bound

From this we obtain a result for average degree $d \geq 2$:

**Theorem**

*Let $G$ be a connected graph of average degree at most $d$. Then*

$$s(G) \leq \frac{d - 2}{d + 1} |V(G)|.$$

For series-parallelization, this is best possible, because $K_{d+1}$ is regular of degree $d$ and we have to remove $d - 2$ vertices to avoid a $K_4$ subgraph.
From this we obtain a result for average degree $d \geq 2$:

**Theorem**

*Let $G$ be a connected graph of average degree at most $d$. Then*

$$s(G) \leq \frac{d - 2}{d + 1} |V(G)|.$$

**Connectedness is necessary:**

13 vertices, 14 edges, average degree $\frac{28}{13}$.

$$\frac{d - 2}{d + 1} |V| = \frac{26}{41} < 1,$$ but $p(G) = 1$, $s(G) = 2$. 
Fragmentability

...is about removing few vertices so as to break graphs into small pieces.
Fragmentability

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Fragmentability

remove few vertices: \( \leq \varepsilon \) of the vertices of the graph,

...to leave small pieces: \( \leq C \) vertices in each component

A graph is \((C, \varepsilon)\)-fragmentable if, by removing some fraction \( \leq \varepsilon \) of its vertices, you can leave components all of size \( \leq C \).

A class of graphs is \( \varepsilon \)-fragmentable if there is a constant \( C \) so that every graph in the class is \((C, \varepsilon)\)-fragmentable.

The lowest (infimum) possible \( \varepsilon \) is the coefficient of fragmentability of the class.
Fragmentability

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\ldots \text{to leave small pieces:} \quad \leq C \text{ vertices in each component}

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remove few vertices: $\leq \varepsilon$ of the vertices of the graph,

\ldots to leave small pieces: $\leq C$ vertices in each component

A graph is $(C, \varepsilon)$-fragmentable if, by removing some fraction $\leq \varepsilon$ of its vertices, you can leave components all of size $\leq C$.

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The lowest (infimum) possible \( \varepsilon \) is the coefficient of fragmentability of the class.
Series-parallelization and fragmentability

Series-parallelization is useful for breaking graphs into small pieces.
Given $G$, with max/ave degree $\leq d$:

1. remove vertices from $G$ to leave induced series-parallel subgraph $\langle P \rangle$;
2. remove $o(n)$ vertices from $\langle P \rangle$ to leave bounded size pieces (e.g., apply Planar Separator Theorem (Lipton & Tarjan) recursively).
Series-parallelization and fragmentability

Lipton-Tarjan separator theorem: A planar graph with $n$ vertices can be broken into 2 pieces with at most $2n/3$ vertices each by removing at most $2\sqrt{2} \sqrt{n}$ vertices.
Series-parallelization and fragmentability

Lipton-Tarjan separator theorem: A planar graph with \( n \) vertices can be broken into 2 pieces with at most \( 2n/3 \) vertices each by removing at most \( 2\sqrt{2}\sqrt{n} \) vertices.

By repeating the process, we can break up the graph into small (\( \leq C \) vertices) pieces.
Series-parallelization and fragmentability

Lipton-Tarjan separator theorem: A planar graph with \( n \) vertices can be broken into 2 pieces with at most \( 2n/3 \) vertices each by removing at most \( 2\sqrt{2}\sqrt{n} \) vertices.

Conclusion: For any \( \varepsilon > 0 \), we can remove a proportion \( \varepsilon \) of the vertices from any planar graph, and ensure no fragment has more than \( 535/\varepsilon^2 \) vertices.
For series-parallel graphs, the coefficient of fragmentability is 0.

Hence, for the class of graphs with maximum or average degree at most \( d \), the coefficient of fragmentability is at most \( \frac{d-2}{d+1} \).

The best lower bound (due to Haxell, Pikhurko and Thomason) is:

\[
\frac{d - 2}{d + 2} \quad \text{for even } d \geq 4, \quad \text{and} \quad \frac{d^2 - 5}{(d + 1)(d + 3)} \quad \text{for odd } d \geq 5.
\]

Note that lower bounds for fragmentability are also lower bounds for series-parallelization.
For $d = 3$, we have $p(G) \leq \frac{1}{4} |V(G)|$ and the fraction $\frac{1}{4}$ is best possible (from fragmentability bounds).

But for $d \geq 4$, there is a gap between upper and lower bounds:

<table>
<thead>
<tr>
<th>$d$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d-2}{d+1}$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{4}{7}$</td>
</tr>
<tr>
<td>Lower bound</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{5}{12}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
Consider a graph of maximum degree 4. Suppose there is a vertex $v$ of degree 4 adjacent to a vertex $w$ of degree 3.

Delete the vertex $v$. 
Better Planarization

Consider a graph of maximum degree 4. Suppose there is a vertex \( v \) of degree 4 adjacent to a vertex \( w \) of degree 3.

Vertex \( w \) is now degree 2, so is removed by reduction.
Better Planarization

Consider a graph of maximum degree 4. Suppose there is a vertex $v$ of degree 4 adjacent to a vertex $w$ of degree 3.

Overall effect: $v_4 \rightarrow v_4 - 4$, $v_3 \rightarrow v_3 + 2$. After (roughly) $v_4/4$ such steps, we get graph $G'$ which is 3-regular with $v_3 + v_4/2$ vertices. Then $p(G') \leq v_3/4 + v_4/8$, so

$$p(G) \leq v_4/4 + v_3/4 + v_4/8 \leq 3|V|/8.$$
This argument can be made precise and extended to general (average) degree $d$. We get an upper bound of the form

$$\frac{d - 9/4}{d + 1} + O\left(1/d^3\right).$$

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<tbody>
<tr>
<td>$\frac{d-2}{d+1}$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{4}{7}$</td>
</tr>
<tr>
<td>New upper bound</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{19}{40}$</td>
<td>$\frac{131}{240}$</td>
</tr>
<tr>
<td>Lower bound</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{5}{12}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
Better series-parallelization

In a sense, the upper bound of $\frac{d-2}{d+1}$ for series parallelization is best possible, since for any graph in which every component is $K_{d+1}$, we must remove $d - 2$ vertices from each component to make it series-parallel.

However, for connected graphs, we might hope to get

$$s(G) \leq j(d)n + o(n)$$

where $j(d) < \frac{d-2}{d+1}$. 
Better series-parallelization

However, for connected graphs, we might hope to get

\[ s(G) \leq j(d)n + o(n) \]

where \( j(d) < \frac{d-2}{d+1} \).

For \( d = 3 \), there can be no improvement. But for maximum degree \( d = 4, 5, 6 \) we can get

\[ s(G) \leq j(d)n + C_d \]

<table>
<thead>
<tr>
<th>d</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d-2}{d+1} )</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{2}{5} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{4}{7} )</td>
</tr>
<tr>
<td>Planarization u.b.</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{19}{40} )</td>
<td>( \frac{131}{240} )</td>
</tr>
<tr>
<td>j(d)</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{19}{40} )</td>
<td>( \frac{11}{20} )</td>
</tr>
</tbody>
</table>
Better series-parallelization

For maximum degree $d \leq 6$ we can get

$$s(G) \leq j(d)n + C_d$$

<table>
<thead>
<tr>
<th>$d$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d-2}{d+1}$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{4}{7}$</td>
</tr>
<tr>
<td>$j(d)$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{19}{40}$</td>
<td>$\frac{11}{20}$</td>
</tr>
</tbody>
</table>

In fact for maximum degree $\leq 6$ we can get the equivalent “vertex-wise” result:

$$s(G) \leq \sum_{v} j(d(v)) + C_d.$$
Better series-parallelization

For maximum degree $d \leq 6$ we have

$$s(G) \leq \sum_v j(d(v)) + C_d.$$ 

It seems natural to want to extend this to all $d$. But it turns out that this cannot be done while keeping $j(d) \leq \frac{d-2}{d+1}$ for all $d$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d-2}{d+1}$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{4}{7}$</td>
<td>$\frac{5}{8}$</td>
<td>$\frac{6}{9}$</td>
</tr>
<tr>
<td>$j(d)$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{19}{40}$</td>
<td>$\frac{11}{20}$</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
More generally

For a set $S$ of graphs, define $\mu(S, \Gamma)$ to be the minimum number $\mu$ such that any graph in $\Gamma$ with $n$ vertices can be made $S$-minor-free by removing at most $(\mu + o(1))n$ vertices.

So we have been considering $\mu(\{K_5, K_{3,3}\}, \Gamma_d^c)$ and $\mu(\{K_4\}, \Gamma_d^c)$. 
More generally

What do we know about $\mu(\{K_r\}, \Gamma^c_d)$?

<table>
<thead>
<tr>
<th>$r \setminus d$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>$\ldots$</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{4}{5}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{6}{7}$</td>
<td>$\frac{12}{13}$</td>
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</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\leq \frac{3}{8}$</td>
<td>$\leq \frac{19}{40}$</td>
<td>$\leq \frac{11}{20}$</td>
<td>$\leq \frac{5}{8}$</td>
<td>$\geq \frac{10}{13}$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\leq \frac{3}{8}$</td>
<td>$\leq \frac{19}{40}$</td>
<td>$\leq \frac{131}{240}$</td>
<td>$\leq \frac{1009}{1680}$</td>
<td>$&lt; \frac{10}{13}$</td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$c_f(\Gamma^c_d)$</td>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\geq \frac{1}{3}$</td>
<td>$\geq \frac{5}{12}$</td>
<td>$\geq \frac{1}{2}$</td>
<td>$\geq \frac{21}{40}$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>