Enumerating Eulerian Orientations.

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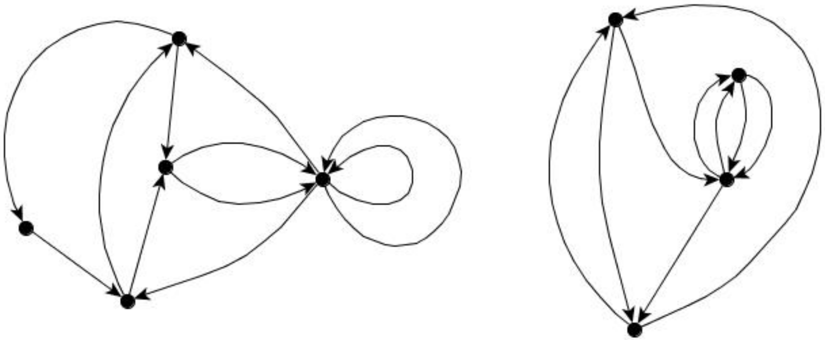
Joint work with Tony Guttmann and Mireille Bousquet-Melou

The University of Melbourne

20/11/2017

Rooted planar Eulerian orientations

ROOTED PLANAR EULERIAN ORIENTATIONS



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ONE EDGE ROOTED PLANAR EULERIAN ORIENTATIONS

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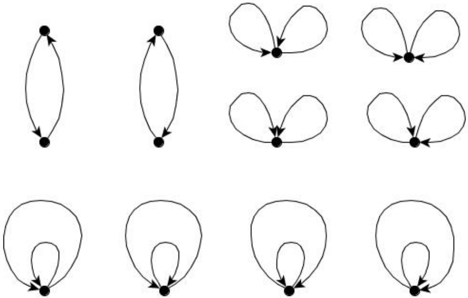
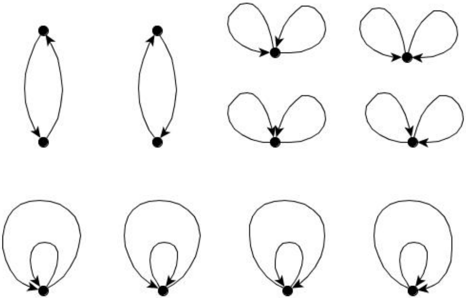
There are two planar rooted Eulerian orientations with one edge.

TWO EDGE ROOTED PLANAR EULERIAN ORIENTATIONS



Two edge rooted planar Eulerian orientations



There are 10 planar rooted Eulerian orientations with two edges.

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- Aim: Find a formula for a_n.

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- They computed the number a_n of these orientations for $n \leq 15$.
- They also proved that the growth rate

$$\mu = \lim_{n\to\infty} \sqrt[n]{a_n}$$

exists and lies in the interval $(11.56, 13.005)$

4-valent planar rooted Eulerian orientations

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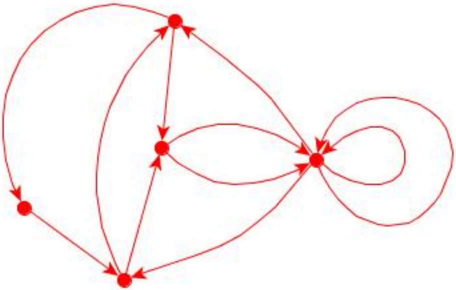
- It is also the sum of the Tutte polynomials $T_\Gamma(0,-2)$ over all 4-valent rooted planar maps Γ with n vertices.

BIJECTION TO NUMBERED MAPS (N-MAPS)

Bijection to numbered maps (N-maps)



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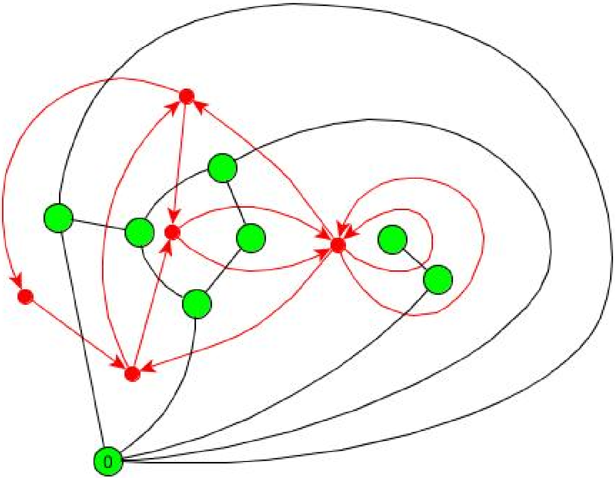

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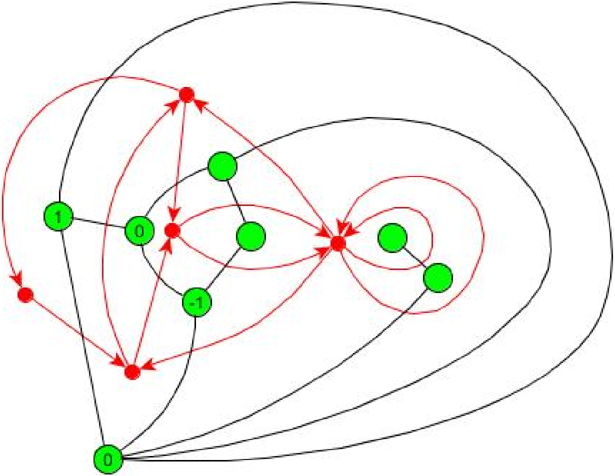

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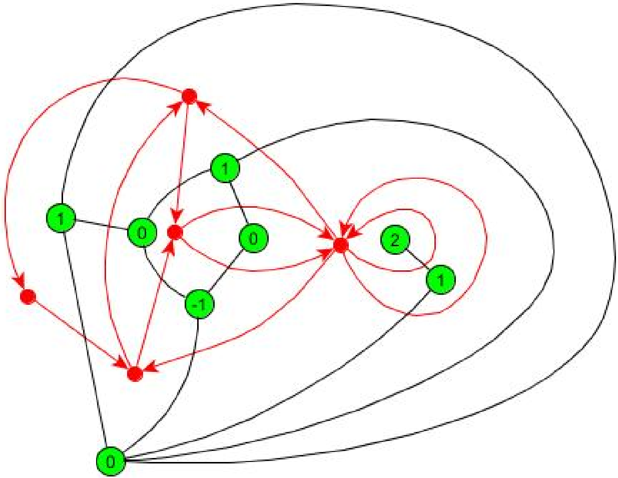

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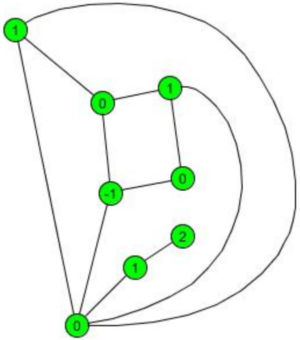

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So, b_n is the number of numbered quadrangulations (N-quadrangulations) with n faces.

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- As I mentioned, this is equivalent to enumerating 4-valent rooted planar Eulerian orientations.
- We want to find a way to decompose all large N-quadrangulations into smaller N-quadrangulations.
- That will hopefully lead to a recursive formula for calculating the numbers b_n.

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- Contract τ to a single vertex.

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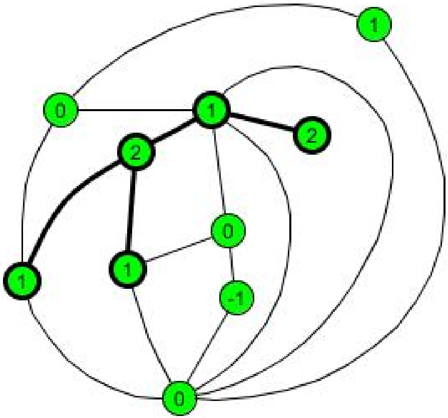

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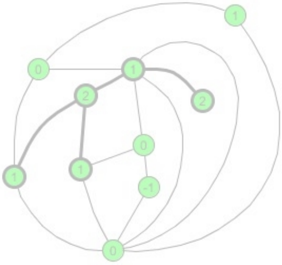

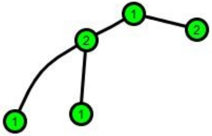

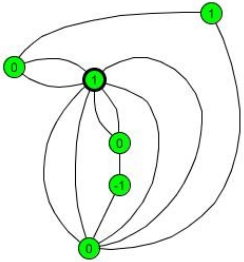

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To use this, we need to enumerate patches. In patches the outer face may have any (even) degree.

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- Every *inner* face has degree 4.
- All vertices adjacent to the root vertex v_0 are numbered 1.
- The vertices around the outer are alternately numbered 0 and 1.

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We count the T-maps using the generating function

$$T(t, a, b) = \sum_{\Gamma} t^{|V(\Gamma)|} a^{d(v_0)} b^{f(\Gamma)},$$

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Now we need a way to decompose T-maps into smaller maps.

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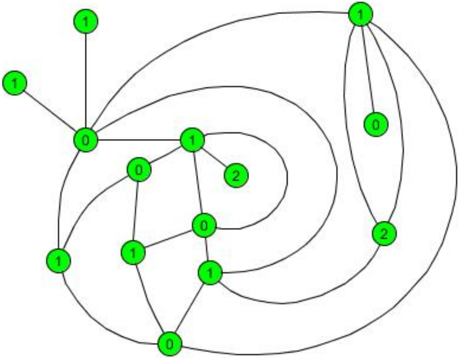

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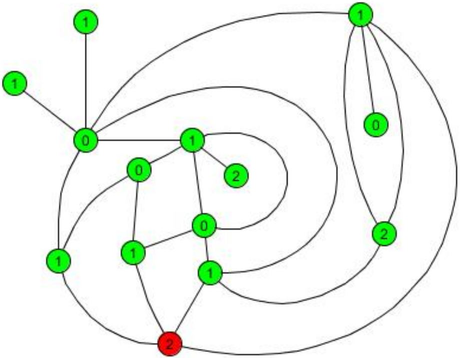

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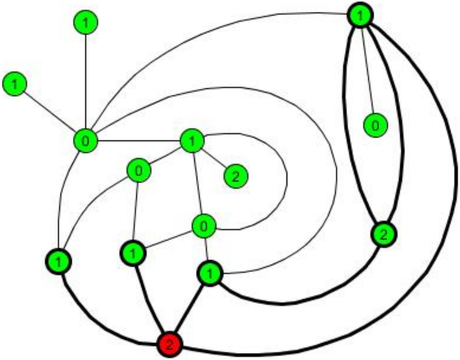

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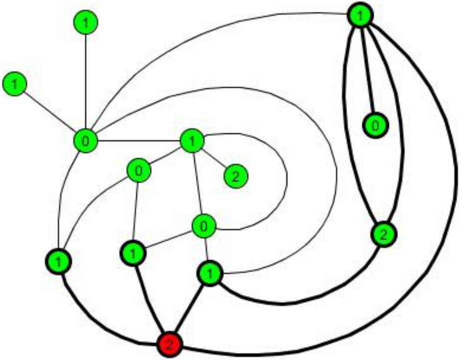

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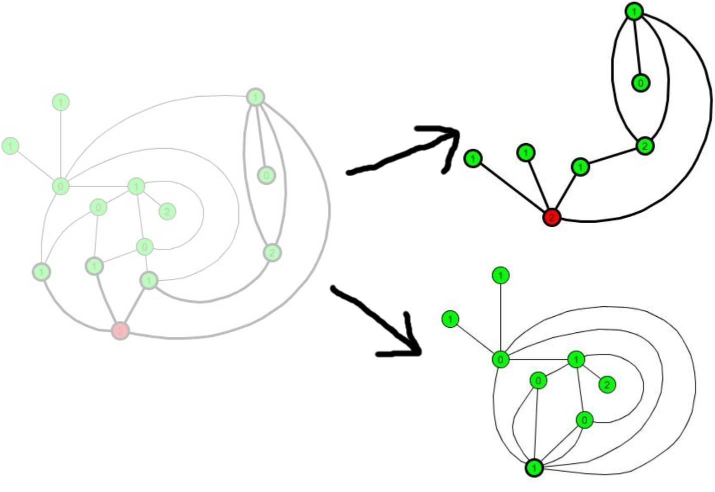

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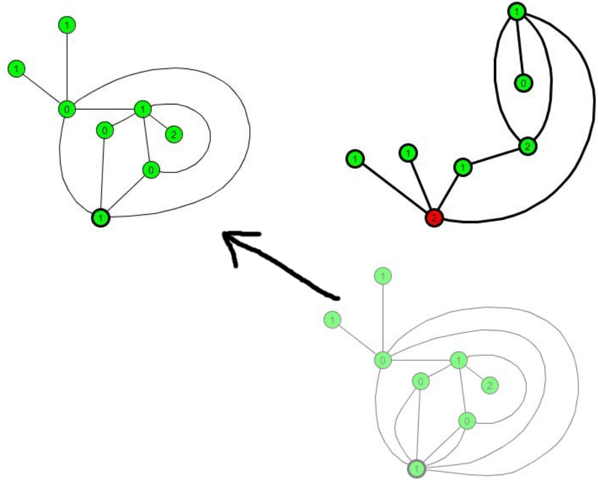

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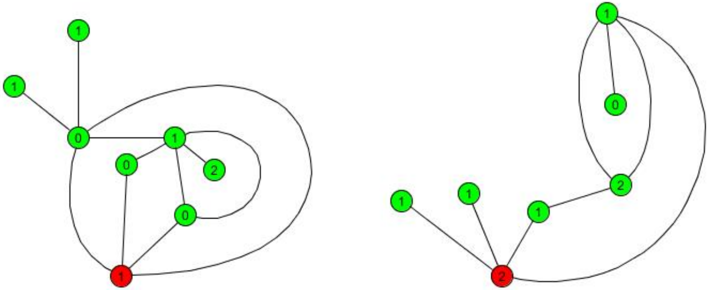

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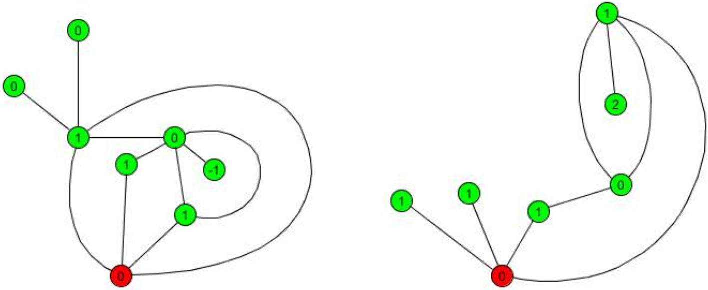

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- Moreover, This allows us to calculate the coefficients of T in polynomial time.

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The algorithm

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- Using a similar algorithm, we computed a_n for $n < 90$.

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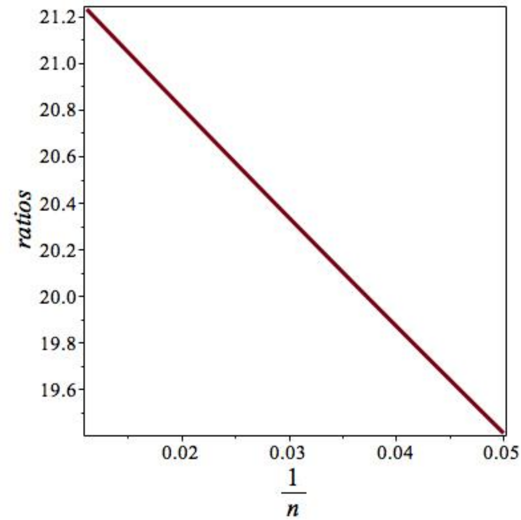
PLOT OF RATIOS b_n/b_{n-1}



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- First, we approximately extend the series.

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- Choose a random sequence of positive integers $L, M, d_0, \dots, d_M$ which sum to 100 (where $M = 2$ or 3 and no two values of d_i differ by more than 2).
- Calculate the unique polynomials $P, Q_0, Q_1, \dots, Q_M$ (up to scaling) of degrees $L, M, d_0, \dots, d_M$ such that the first 100 coefficients of

$$P(t) - \sum_{k=0}^{M} Q_k(t) \left(t \frac{d}{dt} \right)^k B(t)$$

are all 0.

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Surprisingly, these estimates generally seem to be very accurate.

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- so now we have a sequence of ratios and approximate ratios $r_1, r_2, \ldots, r_{1100}$.

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- so now we have a sequence of ratios and approximate ratios $r_1, r_2, \ldots, r_{1100}$.
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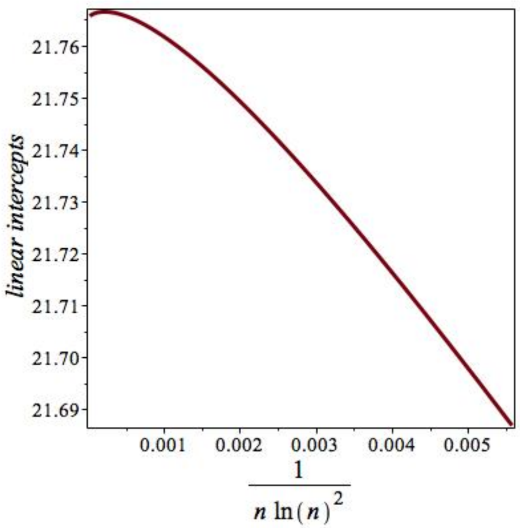
- Using the line between adjacent points in this plot and taking their intercept with the y-axis gives better approximations for the growth rate μ.
- Now we plot these approximations.

PLOT OF RATIOS APPROXIMATIONS FOR μ



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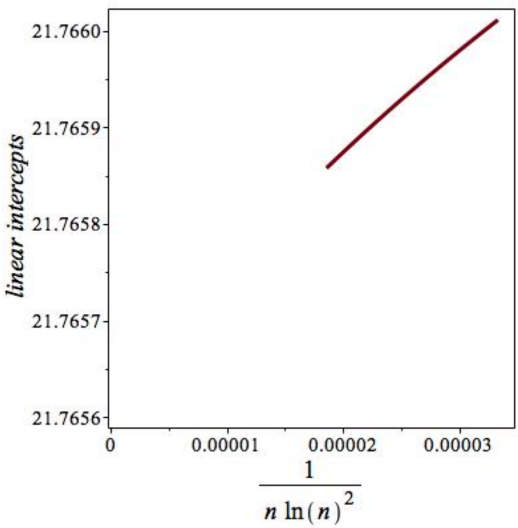

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- In this case we find that the growth rate is approximately 4π.
- We conjecture that 4π and $4\sqrt{3}\pi$ are the exact growth rates.

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  which agrees with the 100 terms that we have computed exactly.
- Assuming that this conjectures are correct, this solution is D-algebraic but not D-finite.
- Using this we can produce thousands of conjectured terms b_n. It turn out that our approximate ratios were all correct to 30 significant digits!

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- Collaborating with Mireille Bousquet-Melou, we have proven this.
- We are still working on the conjecture for $b_0, b_1, \ldots$

Further questions

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for other specific values of x, y?
- For all x, y??

Thank You!