An $n$–component face-cubic model on the complete graph

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Outline

Brief introduction to lattice models and phase transitions

Large deviations theory

An $n$-component face-cubic model

Limit theorems for the face-cubic model on the complete graph

Conclusion
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Ising model

- **Graph** $G = (V, E)$
- Assign a random variable $W_i$ on $i$, for $i \in V$
- $W_i$ takes values in a state space $\Sigma = \{1, -1\}$
- Configuration $\omega = \{W_1 = \omega_1, W_2 = \omega_2, \cdots, W_N = \omega_N\} \in \Sigma^N$, where $N = |V|$.
- The Ising model is defined by choosing configurations $\omega$ randomly via Gibbs measure

$$\pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)} , \ \omega \in \Sigma^N$$

- **Hamiltonian (energy)** $H(\omega)$

$$H(\omega) = - \sum_{ij \in E} \omega_i \cdot \omega_j$$

- **Partition sum** $Z_N(T)$

$$Z_N(T) = \sum_{\omega \in \Sigma^N} e^{-H(\omega)/T}$$
High and low temperature phases

- Recall Gibbs measure

\[ \pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)}, \quad H(\omega) = - \sum_{i,j \in E} \omega_i \cdot \omega_j \]
High and low temperature phases

- Recall Gibbs measure

\[ \pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)}, \quad H(\omega) = -\sum_{i,j \in E} \omega_i \cdot \omega_j \]

- Relative weight for two configurations \( \omega, \omega' \)

\[ \frac{\pi(\omega)}{\pi(\omega')} = e^{-\left( H(\omega) - H(\omega') \right)/T} \]
High and low temperature phases

- Recall Gibbs measure
  \[ \pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)}, \quad H(\omega) = - \sum_{i,j \in E} \omega_i \cdot \omega_j \]

- Relative weight for two configurations \( \omega, \omega' \)
  \[ \frac{\pi(\omega)}{\pi(\omega')} = e^{-(H(\omega) - H(\omega'))/T} \]

- If \( T \) is low, spins prefer to like their neighbours, which is called ordered phase or low temperature phase.
- If \( T \) is high, spins are independent of each other, which is called disordered phase or high temperature phase.
- A critical point at \( T = T_c \).
Order parameter

- Order parameter is used to quantitatively characterise phase transitions.
- For Ising model, the order parameter is the magnetisation,

\[ M = \left\langle \left| \frac{\sum_{i=1}^{N} W_i}{N} \right| \right\rangle \]

- Critical behaviors
  - If \( T \geq T_c \), \( M = 0 \)
  - If \( T \to T_c^- \), \( M \sim (T_c - T)^\beta \)
- The other independent critical exponent is defined from correlation length \( \xi \sim |T - T_c|^{-\nu} \)
Phase transitions classification

- Phase transitions are classified by the continuity of the order parameter.
- First order phase transition (discontinuous): ice-liquid-gas transition, phase coexistence.
- Continuous phase transition: ferromagnetic-paramagnetic transition, superconducting transition, Kosterlitz-Thouless transition.
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Other Important concepts

- Phase transitions happen only in thermodynamic limit
- Ensemble hypothesis: approximate time average by ensemble average
- Universality class: various continuous phase transitions fall into several universality class, in which all models have the same critical phenomena, and share same critical exponents.
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Cramér’s theorem

- Consider a sequence of identically and independently distributed random variables:
  \[ X_1, X_2, \cdots, X_N \]

- State space \( \Sigma = \{a_1, a_2, \cdots, a_m\}, a_i \in \mathbb{R}^d, d \in \mathbb{N}^+ \)

- \( X_i \) is distributed according to a law \( \mu \) and \( \mathbb{E}(X_i) = \bar{X} \).

- Sample mean \( S_N = \frac{1}{N} \sum_{i=1}^{N} X_i \)

- Law of large numbers tells \( S_N \to \bar{X} \) as \( N \to +\infty \).

- What’s the probability that \( S_N = x \) with \( x \) deviating far from \( \bar{X} \)?
Cramér’s theorem

\[ P_N(S_N = x) \sim e^{-NI(x)}, \text{ as } N \to +\infty \]

Logarithmic generating function \( \lambda(k) \), for any \( k \in \mathbb{R}^d \),

\[ \lambda(k) = \log \mathbb{E}[e^{k \cdot S_N}] \]

Rate function from Legendre-Fenchel transform

\[ I(x) = \sup_{k \in \mathbb{R}^d} \{ \langle k, x \rangle - \lambda(k) \} \]

\( I(x) \) is convex, non-negative and \( \min_x I(x) = 0 \)

Set \( \{ x : I(x) = 0 \} \) is called the most probable macroscopic states
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Face-cubic model

- Given $G = (V, E)$.
- Assign a random variable $W_i$ on $i$, for $i \in V$.
- $W_i$ takes values in a state space $\Sigma$.
- State space

$$\Sigma = \{ (\pm 1, 0, 0, \cdots, 0) , \\ (0, \pm 1, 0, \cdots, 0) , \\ \vdots \\ (0, 0, \cdots, 0, \pm 1) \} \subset \mathbb{R}^n$$

- E.g.
  If $n = 3$, $\Sigma = \{ (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) \}$
- Configuration $\omega = \{ W_1 = \omega_1, W_2 = \omega_2, \cdots, W_N = \omega_N \} \in \Sigma^N$, where $N = |V|$. 
Choose configurations in $\Sigma^N$ randomly via Gibbs measure

$$\pi(\omega) = \frac{e^{-\beta H(\omega)}}{Z_N(\beta)}, \ \omega \in \Sigma^N$$

$\beta = 1/T$

$H(\omega)$

$$H = -\sum_{ij} \langle \omega_i, \omega_j \rangle$$

Partition sum $Z_N(\beta)$

$$Z_N(\beta) = \sum_{\omega \in \Sigma^N} e^{-\beta H(\omega)}$$

High temperature, $W_i$ uniformly distributed in $\Sigma$.

Low temperature, $W_i$ prefer to like their neighbors.

$\beta_c$-Critical point
Known results

Square lattice (Nienhuis et al 1982), face-cubic model \( \sim \)
- \( O(n) \) model (\( n \)-vector model) for \( 0 \leq n < 2 \)
- Ashkin-Teller model for \( n = 2 \)
- First-order transition for \( n > 2 \)

Mean-field (or complete graph) (Kim et al, 1975)
- \( n = 1, 2 \), continuous (Ising)
- \( n > 3 \), first-order
- \( n = 3 \), continuous(tricritical)
- \( n = 3 \), first-order (Kim and Levy, 1975)
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Probability distribution of $S_N$ under Gibbs measure

- On the complete graph, Hamiltonian

$$H(\omega) = -\frac{1}{2N} \sum_{i,j=1}^{N} \langle \omega_i, \omega_j \rangle$$

$$= -\frac{1}{2} N S_N^2(\omega)$$

- Probability distribution of $S_N$ in $n$-dimensional cube $\Omega = [-1, 1]^n$?

- Assume $P_N^\beta(S_N = x) \sim e^{-N I_\beta(x)}$, what is the rate function $I_\beta(x)$?
Derive rate function

\[ P_N^\beta(S_N = x) = \frac{1}{Z_N(\beta)} \sum_{\{\omega \in \Sigma^N : S_N(\omega) = x\}} \exp[-\beta H(\omega)] \]

\[ = \frac{1}{Z_N(\beta)} \exp[\beta Nx^2/2] P(S_N = x) \]

Rate function

\[ I_\beta(x) = -\lim_{N \to +\infty} \frac{1}{N} \log P_N^\beta(S_N = x) \]

\[ = I(x) - \frac{\beta}{2} x^2 - \min_{x \in \Omega} [I(x) - \beta x^2/2] \]

Only need to find the global minimum points of

\[ I(x) - \frac{\beta}{2} x^2 \]

in the \( n \)-dimensional cube \([-1, 1]^n\).
- A useful convex duality

\[
\min_{x \in \Omega} [I(x) - \frac{\beta}{2} \langle x, x \rangle] = \min_{u \in \mathbb{R}^n} \left[ \frac{1}{2\beta} \langle u, u \rangle - \lambda(u) \right]
\]

- For face-cubic model

\[
\lambda(u) = \ln \sum_{i=1}^{n} \cosh(u_i)
\]

- Find the global minimum points of

\[
G_{\beta}(u) = \frac{1}{2\beta} \langle u, u \rangle - \ln \sum_{i=1}^{n} \cosh(u_i), \quad \text{with } u \in \mathbb{R}^n
\]
Lemma

Let $\nu$ be a global minimum point of $G_\beta(u)$, then $\nu$ is one of the following $(2n + 1)$ vectors.

\[
\begin{align*}
\nu_0 &= (0, 0, 0, \cdots, 0) \\
\nu_1 &= (a, 0, 0, \cdots, 0) \\
\nu_2 &= (0, a, 0, \cdots, 0) \\
&\vdots \\
\nu_n &= (0, 0, \cdots, 0, a) \\
\nu_{n+i} &= -\nu_i, \ i = 1, 2, \cdots, n
\end{align*}
\]

$0 < a < 1$

\[
G_\beta(u = \nu) = \frac{1}{2\beta} a^2 - \ln[\cosh(a) + n - 1]
\]
Theorem

1. Let $A \subseteq \mathbb{R}^n$. For $1 \leq n \leq 3$,

$$P_N^\beta (S_N \in A) \sim \begin{cases} 
\delta_{\nu_0} (A) & \text{for } 0 < \beta \leq n \\
\frac{1}{2n} \sum_{i=1}^{2n} \delta_{\nu_i} (A) & \text{for } \beta > n 
\end{cases}$$

as $N \to +\infty$.

2. For $n \geq 4$,

$$P_N^\beta (S_N \in A) \sim \begin{cases} 
\delta_{\nu_0} (A) & \text{for } 0 < \beta < \beta' \\
\lambda_0 \delta_{\nu_0} (A) + \lambda_1 \sum_{i=1}^{2n} \delta_{\nu_i} (A) & \text{for } \beta = \beta' \\
\frac{1}{2n} \sum_{i=1}^{2n} \delta_{\nu_i} (A) & \text{for } \beta > \beta'
\end{cases}$$

as $N \to +\infty$, with

$$\lambda_0 = \frac{\kappa_0}{\kappa_0 + 2n\kappa_1}, \quad \lambda_1 = \frac{\kappa_1}{\kappa_0 + 2n\kappa_1},$$

$$\kappa_0 = (\det D^2 G_{\beta_c} (\nu_0))^{-1/2}, \quad \kappa_1 = (\det D^2 G_{\beta_c} (\nu_1))^{-1/2}.$$
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- Rigorously study $n$-component face-cubic model on the complete graph.
- By large deviations analysis, we derive $P_N^\beta(S_N = x) \sim e^{-NI_\beta(x)}$ and explicit form of $I_\beta(x)$.
- For $1 \leq n \leq 3$, continuous phase transition at $\beta_c = n$.
- For $n \geq 4$, first-order phase transition at $\beta_c = \beta'$. 
References

Many thanks for your attention!