Which alternating dimaps are binary functions?

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Work done partly at: Isaac Newton Institute for Mathematical Sciences (Combinatorics and Statistical Mechanics Programme), Cambridge, 2008; University of Melbourne (sabbatical), 2011; and Queen Mary, University of London, 2011.

23 June 2014

Cutset space

Incidence matrix of graph G:

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Often think of this as a *vector*, $\mathbf{f}$, length $2^{|E|}$, entries indexed by subsets of E (or their characteristic vectors).

Binary functions

Indicator functions of cutset spaces are prototypical *binary functions*.

Let E be a finite set (the *ground set*).

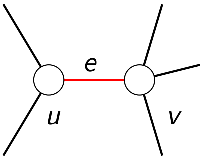
A **binary function** is a function $f : 2^E \to \mathbb{C}$ such that $f(\emptyset) = 1$.

In terms of vectors: it's a $2^{|E|}$-element column vector $\mathbf{f}$, with entries indexed by subsets of E (or their characteristic vectors), such that $f_\emptyset = 1$.

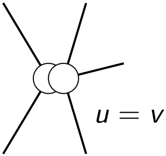
Back to graphs . . .

Contraction and Deletion

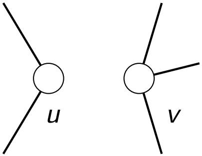
G



G/e



$G \setminus e$



Minors

H is a **minor** of G if it can be obtained from G by some sequence of deletions and/or contractions.

The order doesn't matter. Deletion and contraction **commute**:

$$\begin{aligned} G/e/f &= G/f/e \\ G \setminus e \setminus f &= G \setminus f \setminus e \\ G/e \setminus f &= G \setminus f/e \end{aligned}$$

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Importance of minors:

- excluded minor characterisations
  - planar graphs (Kuratowski, 1930; Wagner, 1937)
  - graphs, among matroids (Tutte, PhD thesis, 1948)
  - Robertson-Seymour Theorem (1985–2004)
- counting
  - Tutte-Whitney polynomial family

Duality and minors

Classical duality for embedded graphs:

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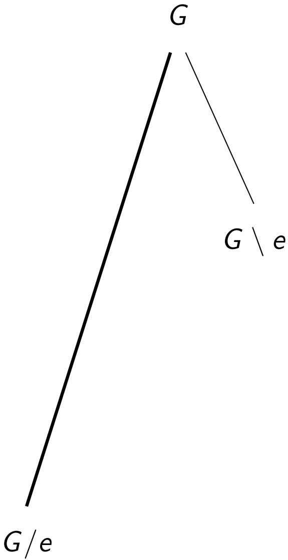
$$\text{contraction} \longleftrightarrow \text{deletion}$$

$$\begin{array}{rcl} (G/e)^* & = & G^* \setminus e \\ (G \setminus e)^* & = & G^*/e \end{array}$$

Duality and minors



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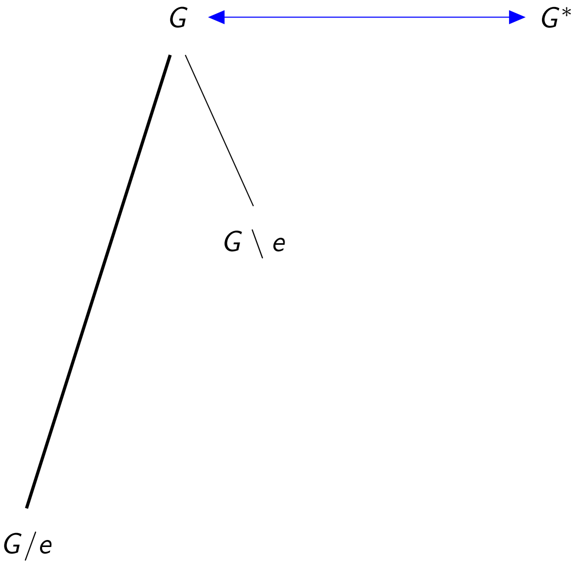

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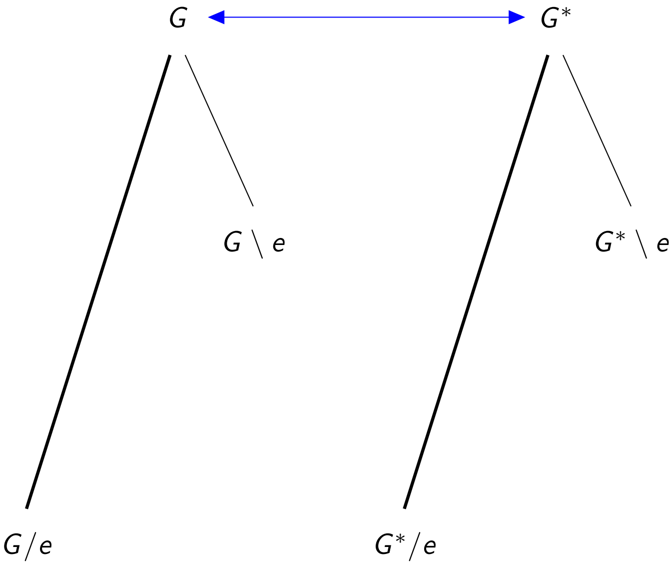

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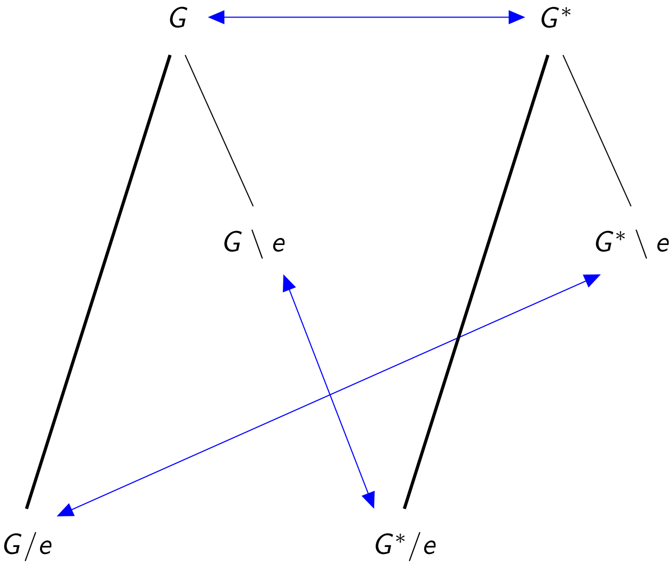
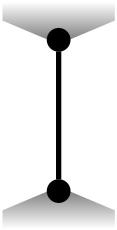


Loops and coloops

loop

coloop = bridge = isthmus



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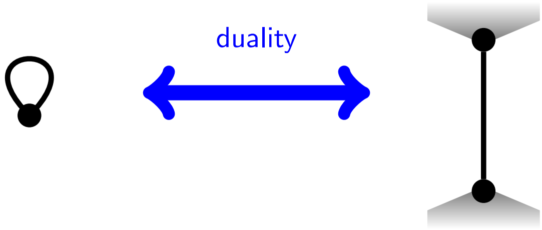

Contraction and deletion in terms of f

Indicator function of cutset space of G:

$$f : 2^E \to \{0,1\}$$

For contraction and deletion of some $e \in E$:
Indicator functions of cutset spaces of ...

G/e	$G \setminus e$
$f /\!\!/ e : 2^{E\setminus\{e\}} \to \{0,1\}$	$f \setminus\!\!\setminus e : 2^{E\setminus\{e\}} \to \{0,1\}$
$f /\!\!/ e\,(X) = \dfrac{f(X)}{f(\emptyset)}$	$f \setminus\!\!\setminus e\,(X) = \dfrac{f(X) + f(X \cup \{e\})}{f(\emptyset) + f(\{e\})}$

Interpolating between contraction and deletion

(GF, 2004)
For $e \in E$, $X \subseteq E \setminus \{e\}$:

Contraction	Deletion
$(f /\!\!/ e)(X)$	$(f \setminus\!\!\setminus e)(X)$
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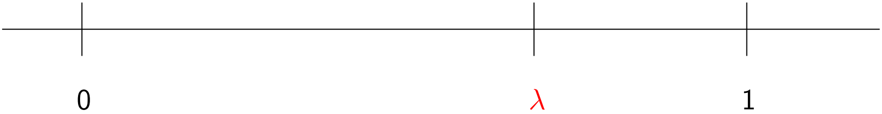

Duality, contraction and deletion

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$$\widehat{f \,\|_{\lambda}\, e} = \hat{f} \,\|_{\lambda^*}\, e$$

(For binary functions, duality = Hadamard transform (GF, 1993).)

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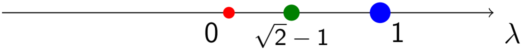
Fixed points:

$$\lambda = \pm\sqrt{2} - 1$$

From λ to μ



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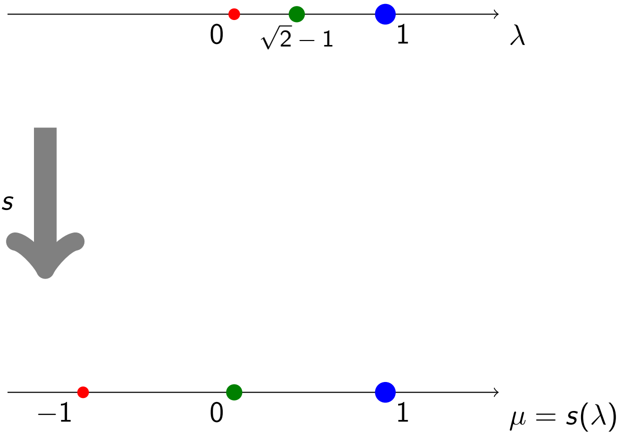

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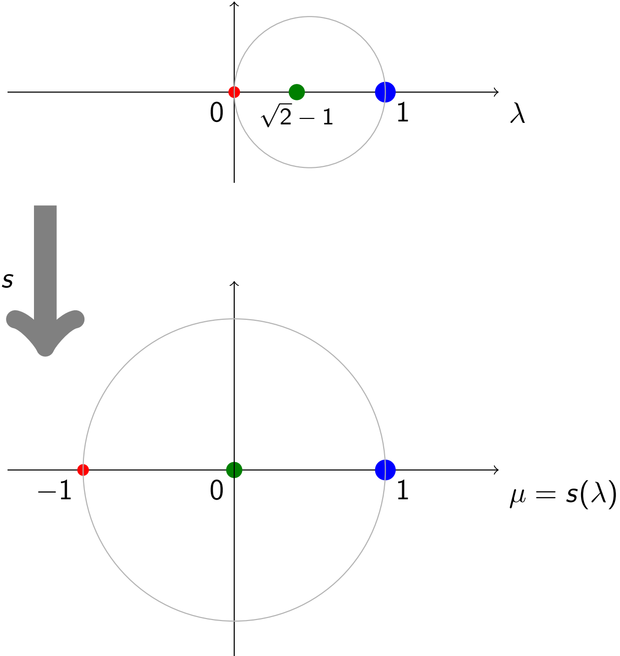

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From λ to μ

$$\mu = s(\lambda) \quad := \quad -(3+2\sqrt{2})\,\frac{\sqrt{2}-1-\lambda}{\sqrt{2}+1+\lambda}$$

$$\lambda = s^{-1}(\mu) \quad := \quad \frac{1+\mu}{\sqrt{2}+1-(\sqrt{2}-1)\mu}$$

Notation:

$$f \,\|_{[\mu]} e \quad := \quad f \,\|_{s^{-1}(\mu)} e$$

The transform $L^{[\mu]}$

$$\begin{aligned}(L^{[\mu]}f)(V) &= (2\sqrt{2})^{-|E|}\times \\ &\sum_{X\subseteq E} (\sqrt{2}-1+(\sqrt{2}+1)\mu)^{|X\cap V|} \\ &\qquad \cdot(1-\mu)^{|X\setminus V|+|V\setminus X|} \\ &\qquad \cdot(\sqrt{2}+1+(\sqrt{2}-1)\mu)^{|E\setminus(X\cup V)|}\ f(X)\end{aligned}$$

Matrix representation:

$$M(\mu) = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}+1+(\sqrt{2}-1)\mu & 1-\mu \\ 1-\mu & \sqrt{2}-1+(\sqrt{2}+1)\mu \end{pmatrix},$$

$$L^{[\mu]}\,\mathbf{f} = M(\mu)^{\otimes m}\,\mathbf{f}$$ (uses m-th Kronecker power)

Special cases:

$\mu = 1$: identity transform

$\mu = -1$: $\sqrt{2}^{|E|}\times$ Hadamard transform (duality)

$\mu = \omega := e^{i\,2\pi/3}$: some kind of "triality"

Properties of the transforms

Composition of transforms $\longleftrightarrow$ multiplication of their parameters:

$$L^{[\mu_1]} L^{[\mu_2]} = L^{[\mu_1 \mu_2]}$$

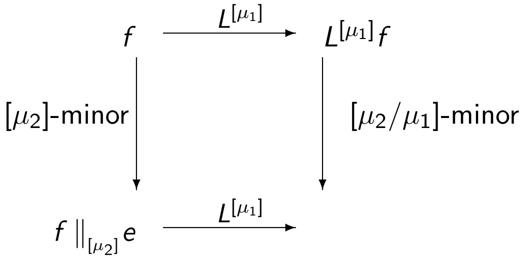
Also have generalisations of Plancherel's and Parseval's theorems.

$[\mu]$-minors

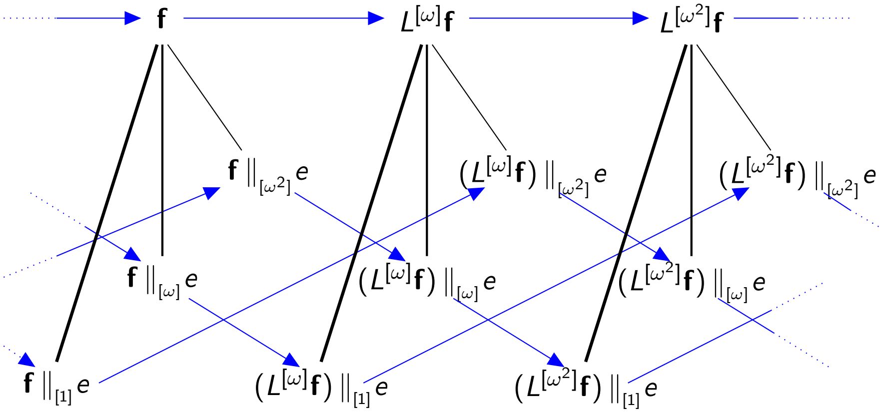
Theorem

$$(L^{[\mu_1]} f)\,\|_{[\mu_2/\mu_1]}\, e = \mathit{ScalingFactor}(f, \mu_1, \mu_2) \cdot L^{[\mu_1]}(f\,\|_{[\mu_2]}\, e)$$

Up to constant factors:



$[\omega]$-minors



Alternating dimaps

Alternating dimap (Tutte, 1948):

- directed graph without isolated vertices,
- 2-cell embedded in a disjoint union of orientable 2-manifolds,
- each vertex has even degree,
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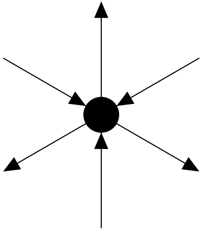

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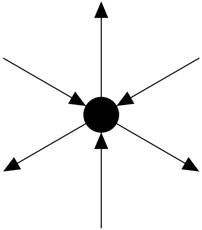

Genus $\gamma(G)$ of an alternating dimap G:

$$V - E + F = 2(k(G) - \gamma(G))$$

Alternating dimaps

Three special partitions of $E(G)$:

- *clockwise faces*
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- *in-stars*

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Each defines a permutation of $E(G)$. These permutations satisfy

$$\sigma_i \sigma_c \sigma_a = 1$$

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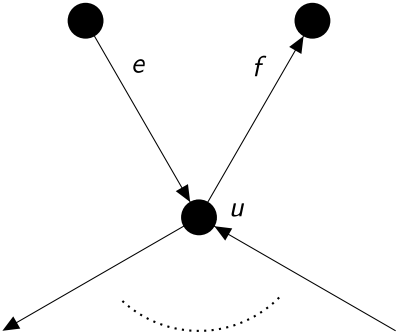

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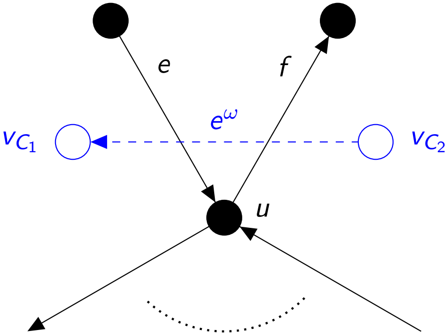

Minor operations

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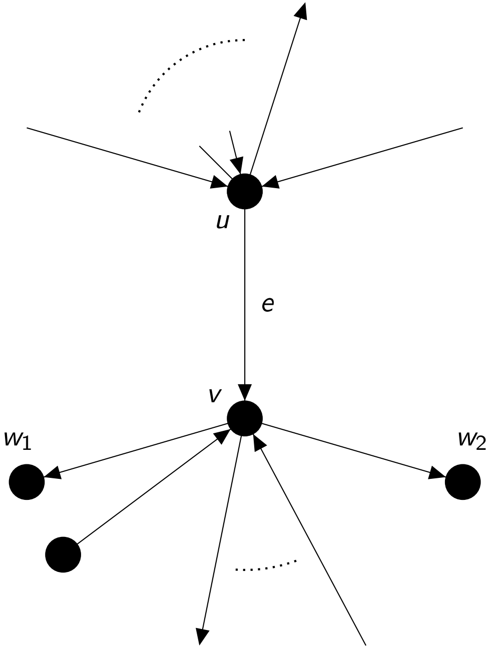
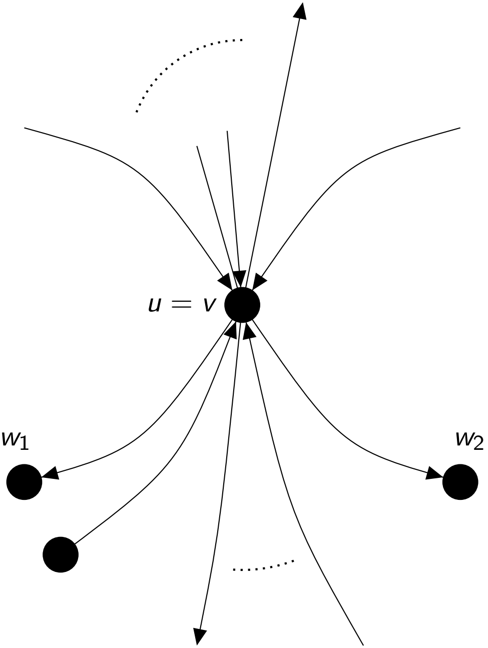


Minor operations

$G[1]e$



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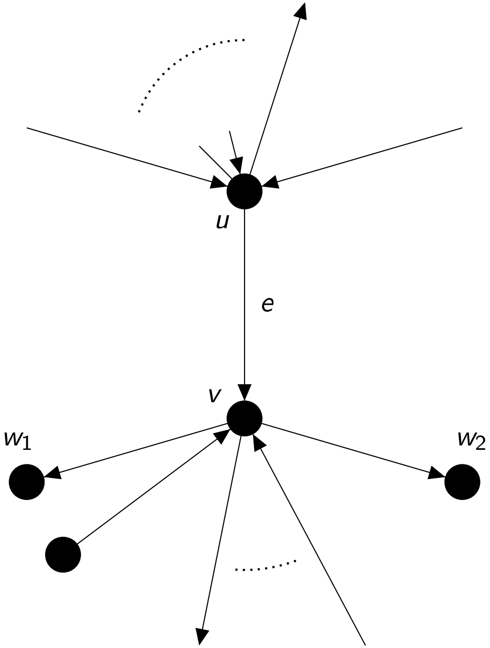
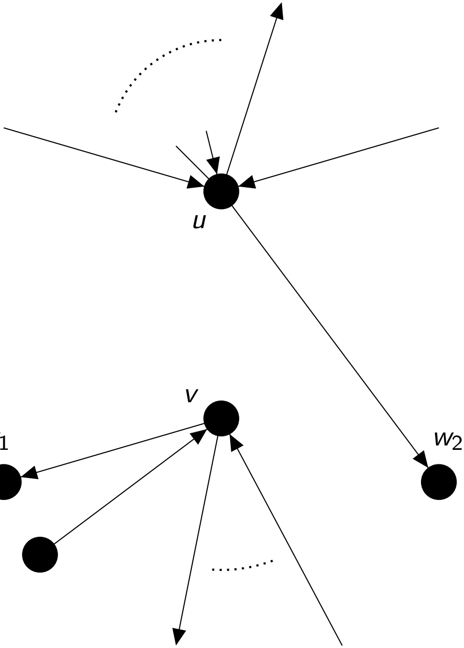


Minor operations

$G[\omega]e$



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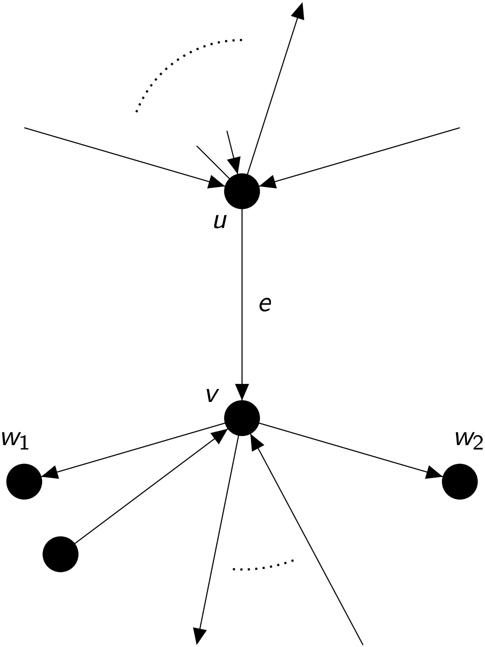
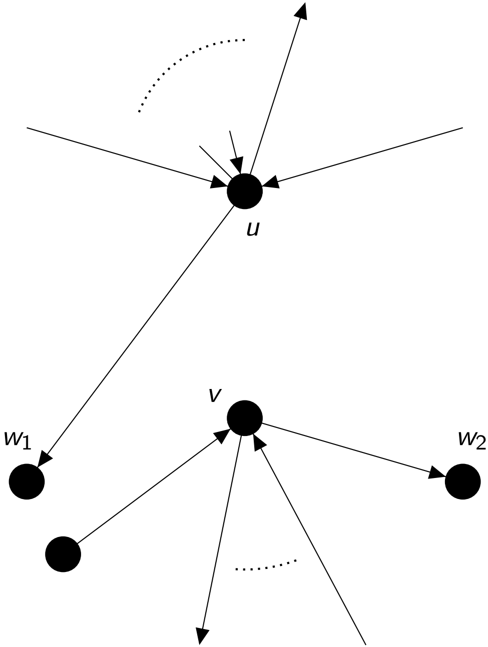


Minor operations

$G[\omega^2]e$



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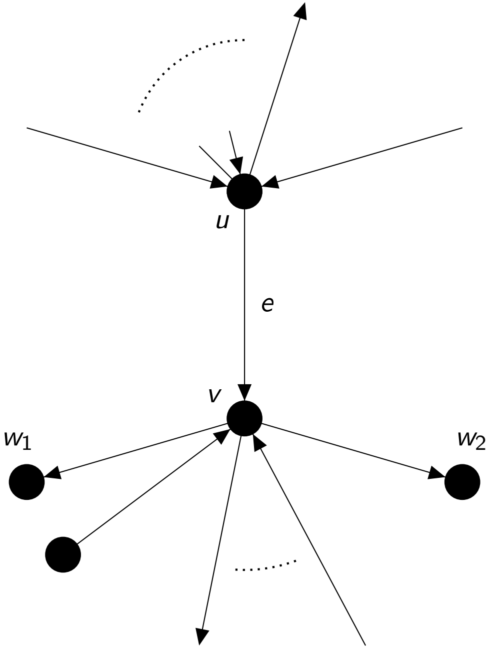

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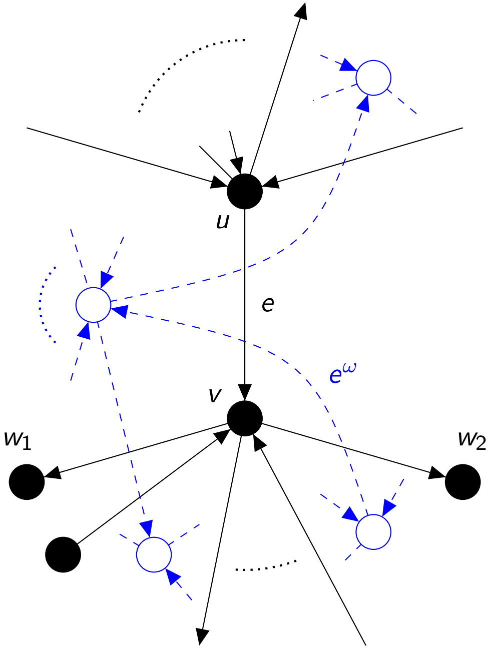

Minor operations

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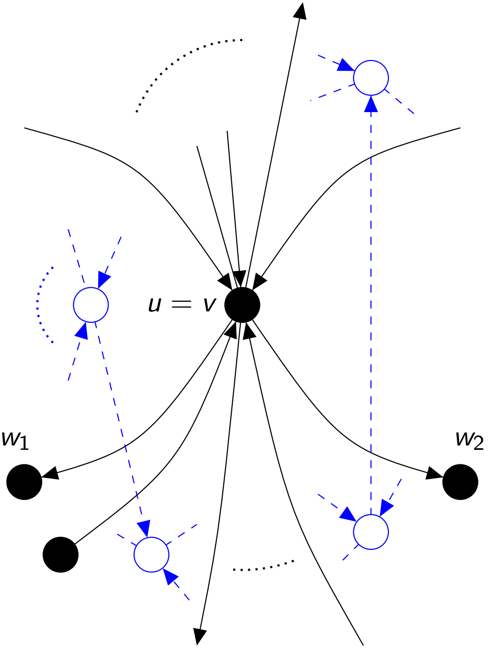
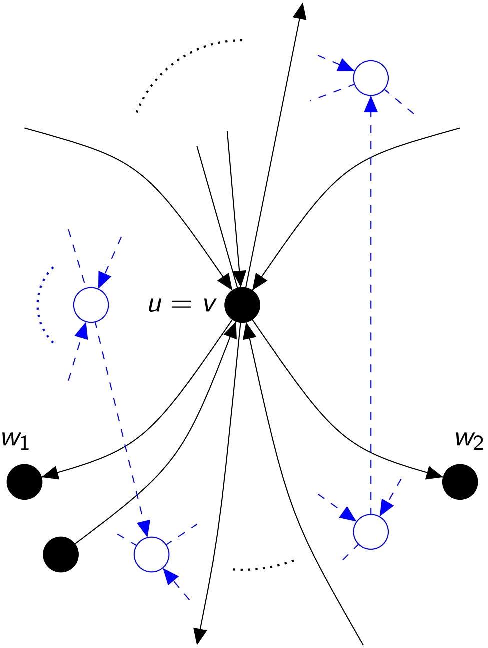


Minor operations

$(G[1]e)^\omega = G^\omega[\omega^2]e^\omega$



Minor operations

$$\begin{aligned}
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\end{aligned}$$

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\end{aligned}$$

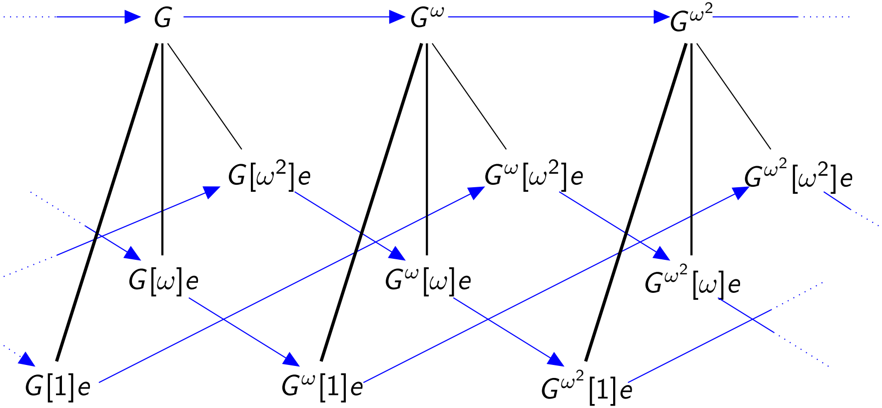
Theorem

If $e \in E(G)$ *and* $\mu, \nu \in \{1, \omega, \omega^2\}$ *then*

$$G^{\mu}[\nu]e^{\omega} = (G[\mu\nu]e)^{\mu}.$$

Same pattern as established for generalised minor operations on binary functions (GF, 2008/2013. . .).

Minor operations



Relationships

triangulated triangle

$\updownarrow$

alternating dimaps

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bicubic map (reduction: Tutte 1975)

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Eulerian triangulation

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$\updownarrow$ (Cavenagh & Lisoněck, 2008)

spherical latin bitrade

Ultraloops, triloops, semiloops

ultraloop

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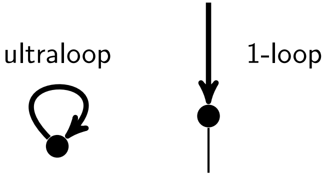

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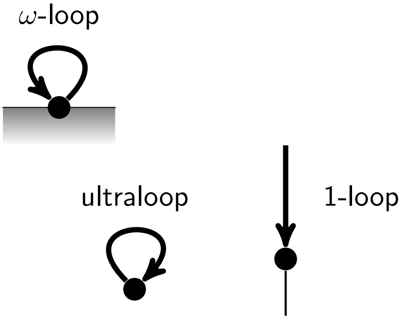

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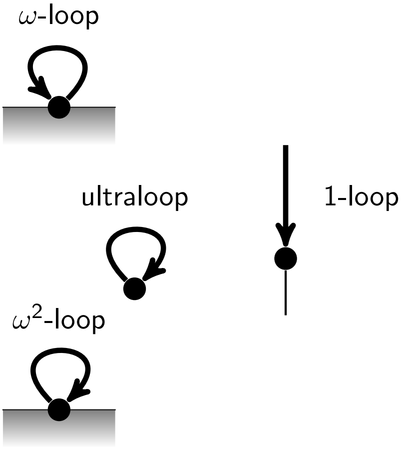
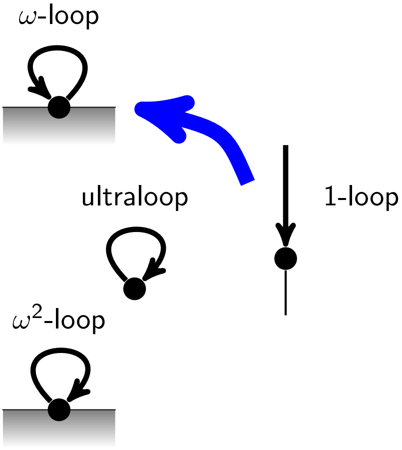


Ultraloops, triloops, semiloops



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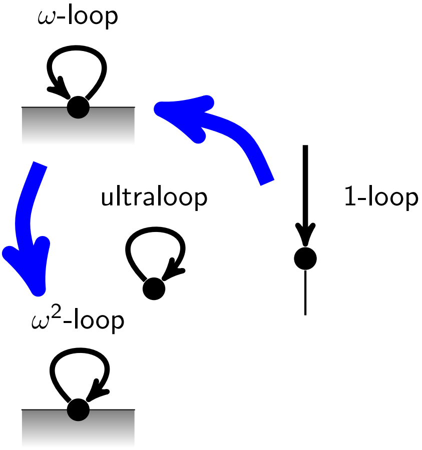

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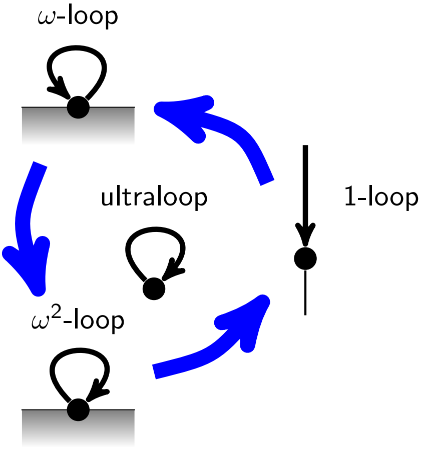

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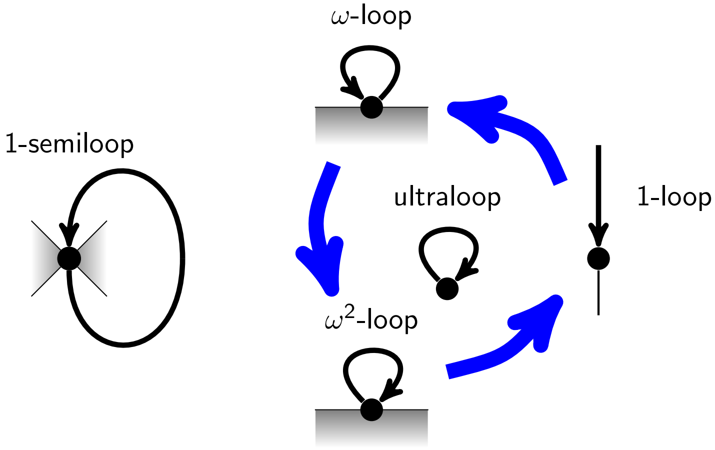
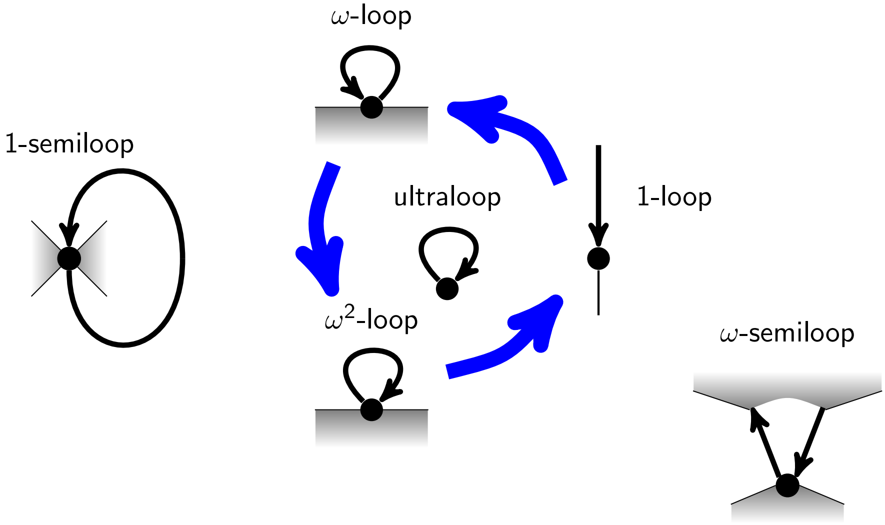


Ultraloops, triloops, semiloops



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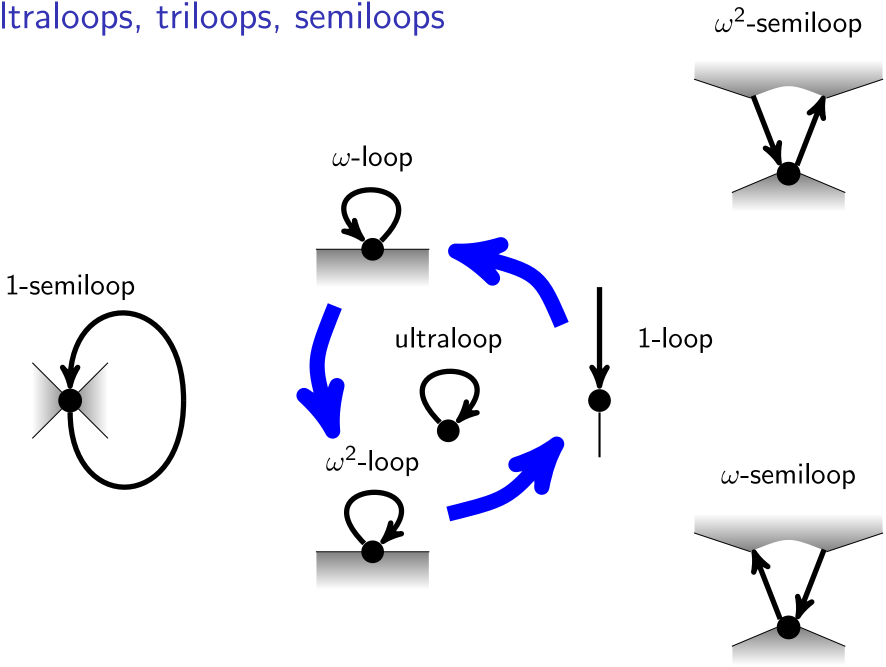

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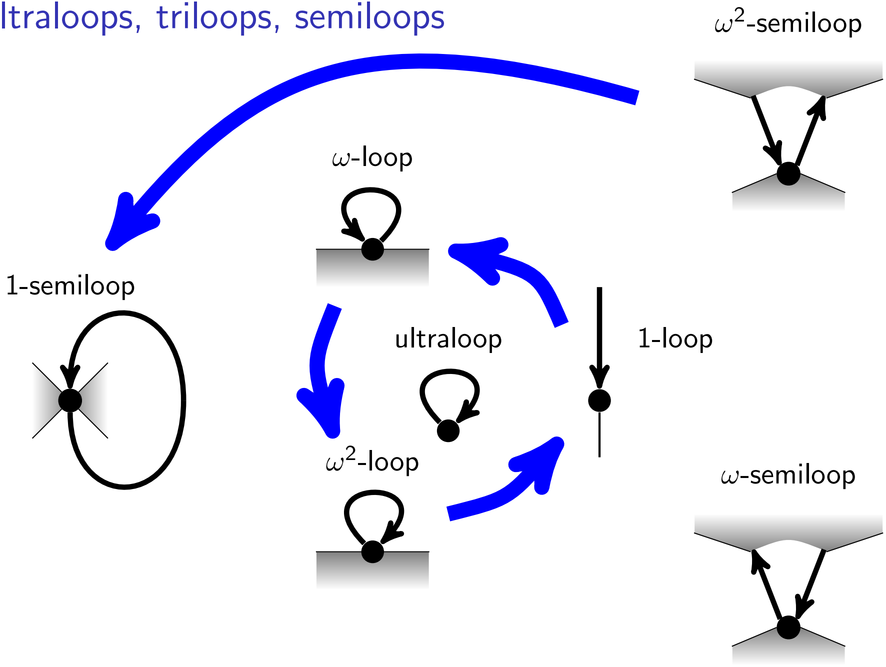
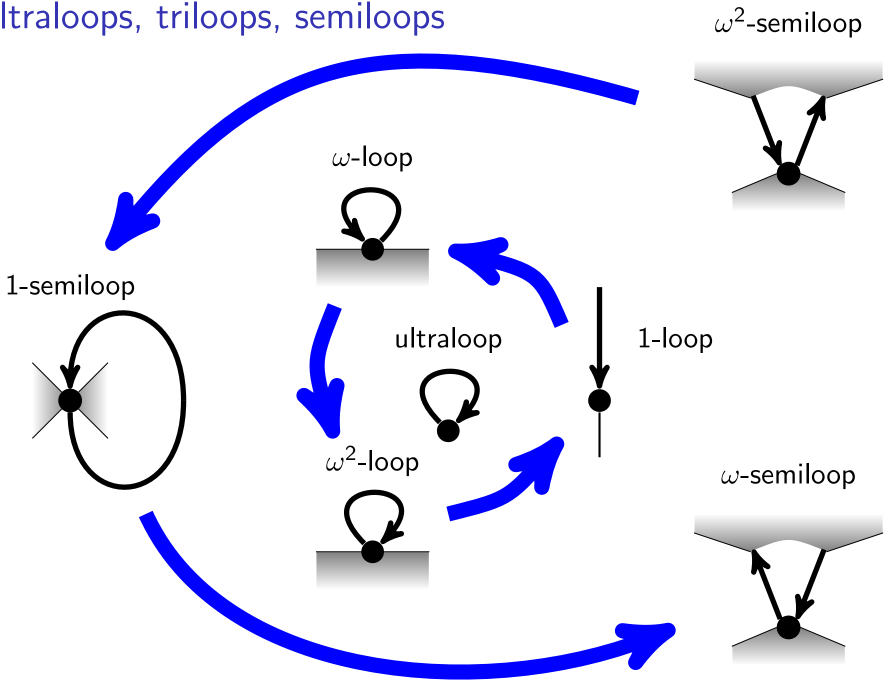
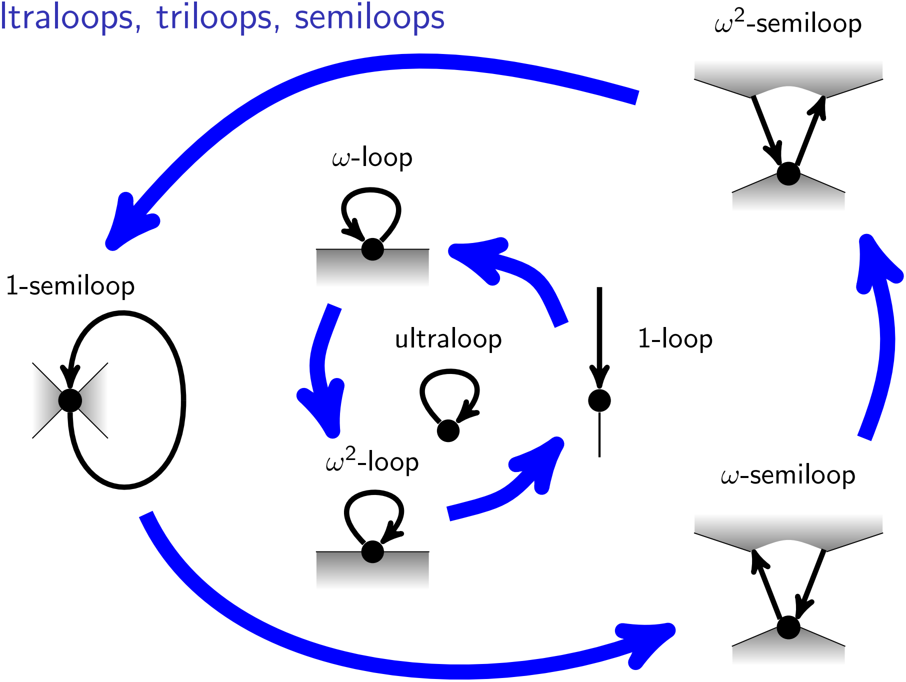


Ultraloops, triloops, semiloops



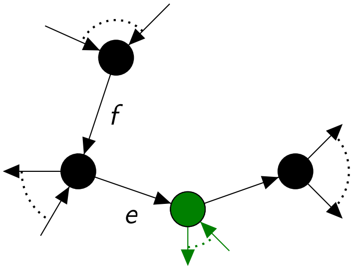
Ultraloops, triloops, semiloops



Non-commutativity

Some bad news: sometimes,

$$G[\mu]e[\nu]f \neq G[\nu]f[\mu]e$$



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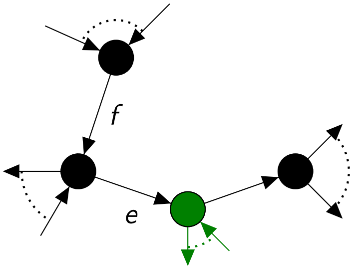

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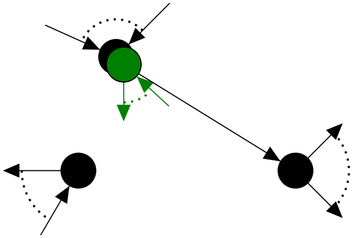
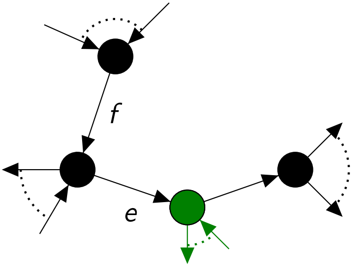


$G[\omega]f[1]e$



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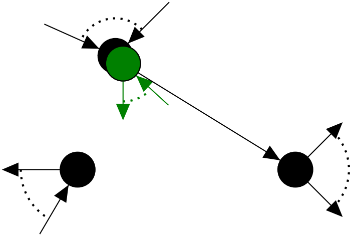
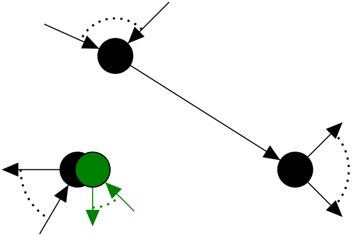
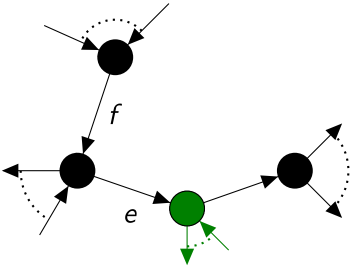


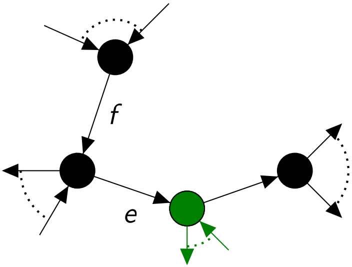
$G[\omega]f[1]e$



$G[1]e[\omega]f$



$$G[\omega]f[1]e \neq G[1]e[\omega]f$$



$$G[\omega]f[1]e \neq G[1]e[\omega]f$$

Theorem

Except for the above situation and its trials, reductions commute.

$$G[\mu]f[\nu]e = G[\nu]e[\mu]f$$

Corollary

If $\mu = \nu$, or one of e, f is a triloop, then reductions commute.

Which alternating dimaps “are” binary functions?

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Not all: for alternating dimaps, reductions do not commute in general, whereas for binary functions, they do.

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Definition

A *strict binary representation* of a minor-closed set $\mathcal{A}$ of alternating dimaps is a triple (F, ε, ν) such that

(a) $F : \mathcal{A} \to \{\text{binary functions}\}$

(b) $\varepsilon = (\varepsilon_G \mid G \in \mathcal{A})$ is a family of bijections $\varepsilon_G : E(G) \to E(F(G))$;

(c) $\nu \in \mathbb{C}$ with $|\nu| = 1$;

(d) $F(G^{(\omega)}) \simeq L^{[\omega]} F(G)$ for all $G \in \mathcal{A}$;

(e) $F(G[\mu]e) \simeq F(G) \parallel_{[\nu\mu]} \varepsilon_G(e)$ for all $G \in \mathcal{A}$, $e \in E(G)$ and $\mu \in \{1, \omega, \omega^2\}$.

Which alternating dimaps are binary functions?

Definitions

$$\begin{aligned} C_1 &:= \text{ultraloop} \\ iC_1 &= \text{disjoint union of } i \text{ ultraloops} \\ 0C_1 &= \text{empty alternating dimap} \end{aligned}$$

$$\begin{aligned} \mathcal{U}_k &= \{iC_1 \mid i = 0, \ldots, k\} \\ \mathcal{U}_\infty &= \{iC_1 \mid i \in \mathbb{N} \cup \{0\}\} \end{aligned}$$

Theorem

If $\mathcal{A}$ is a minor-closed class of alternating dimaps which has a strict binary representation then

- $\mathcal{A} = \emptyset$, *or*
- $\mathcal{A} = \mathcal{U}_k$ *for some* k, *or*
- $\mathcal{A} = \mathcal{U}_\infty$.

Which alternating dimaps are binary functions?

Proof. (Outline) If $\mathcal{A} = \emptyset$: done. So suppose $\mathcal{A} \neq \emptyset$.

Since $\mathcal{A}$ is minor-closed, it must contain the empty alt. dimap $0C_1$. It must be represented by $f : 2^{\emptyset} \to \mathcal{C}$ with $f(\emptyset) = 1$, i.e., $\mathbf{f} = (1)$.

If $|\mathcal{A}| = 1$ then we are done. This F gives a strict binary representation, and $\mathcal{A} = \mathcal{U}_0$.

If $|\mathcal{A}| \geq 2$, then it must contain the ultraloop C_1. Its image $F(C_1)$ is given by

$$F(C_1) = \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}.$$

Proof: C_1 is self-trial, so $F(C_1)$ must be too. So $F(C_1)$ must be an eigenvector for eigenvalue 1 of the matrix $M(\omega)$.

If $|\mathcal{A}| = 2$ then we are done. This F gives a strict binary representation, and $\mathcal{A} = \{\text{empty}, \text{ultraloop}\} = \mathcal{U}_1$.

Which alternating dimaps are binary functions?

Suppose $|\mathcal{A}| \geq 3$. Then $\mathcal{A}$ must have at least one alternating dimap G_2 on two edges.
For any such G_2, all reductions give the ultraloop C_1.
So all reductions of $F(G_2)$ give $F(C_1) = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$.
Then show that $F(G_2) = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}^{\otimes 2}$.
Therefore $F(G_2)$ is self-trial, so G_2 must be too.
So $G_2 = 2C_1$ (the only self-trial alternating dimap on two edges).
So far, we have at most one alternating dimap in $\mathcal{A}$ with each possible number of edges (0, 1, 2).
Show by induction that $\mathcal{A}$ has at most one member with k edges, and that it is kC_1, with

$$F(kC_1) = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}^{\otimes k}.$$

This is (the guts of) the strict binary representation. □

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