Uniform generation of random regular graphs

Jane Gao

Joint work with Nick Wormald

~~15th January, 2016~~
27th June, 2016

Why?

- A classical TCS problem;
- Intimate connection with enumeration;
- Testing algorithms with random input;
- Coping with "big data".

Commonly used methods

- Rejection algorithm
- Boltzmann sampler
- MCMC
- Coupling from the past
- Switching algorithm

Generating random d-regular graphs

- Tinhofer '79 – non-uniform random generation.
- Rejection algorithm – uniform sampler for small d ($d = O(\sqrt{\log n})$).
  - A. Békéssy, P. Békéssy and Komlós '72;
  - Bender and Canfield '78;
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- MCMC – approximate sampler.
  - Jerrum and Sinclair '90 – for any d, FPTAS, but no explicit bound on the time complexity;
  - Cooper, Dyer and Greenhill '07 – mixing time bounded by $d^{24}n^9 \log n$;
  - Greenhill '15 – non-regular case, mixing time bounded by $\Delta^{14}M^{10} \log M$.
- Switching algorithm – fast, uniform sampler.
  - McKay and Wormald '90 – for $d = O(n^{1/3})$.
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- Other methods – asymptotic approximate sampler.
  - Steger and Wormald '99 – for $d = n^{1/28}$;
  - Kim and Vu '06 – for $d \leq n^{1/3-\epsilon}$;
  - Bayati, Kim and Saberi '10 – for $d \leq n^{1/2-\epsilon}$;
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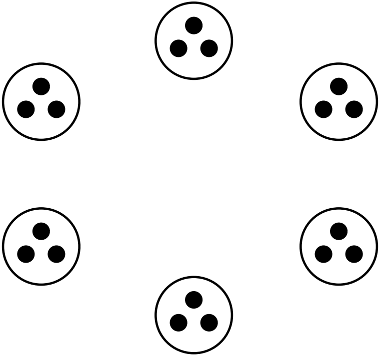

$$n = 6, d = 3$$

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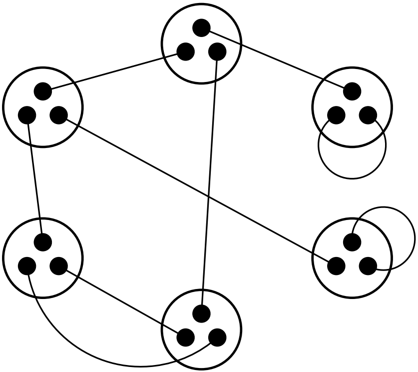

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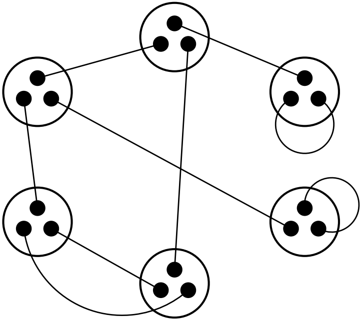

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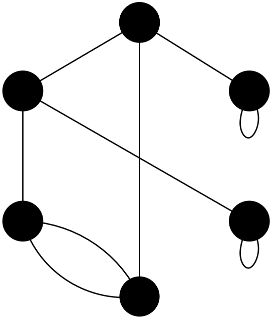

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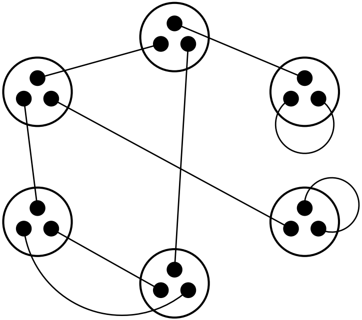

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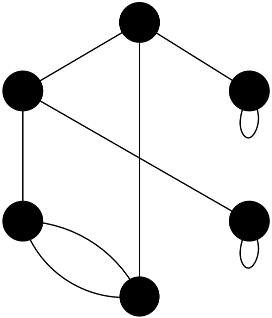
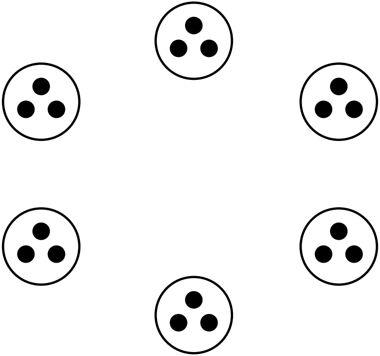


reject!

Rejection algorithm



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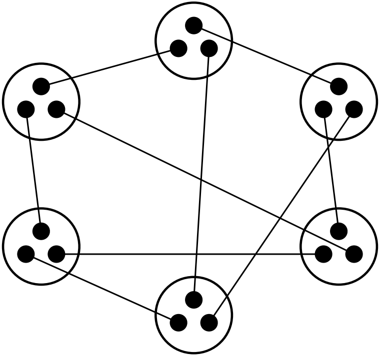

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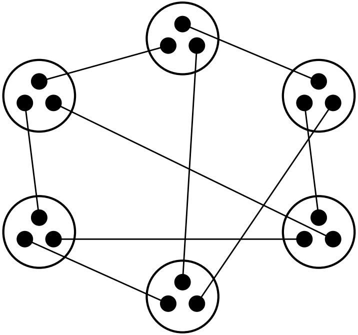

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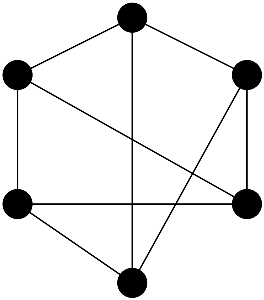

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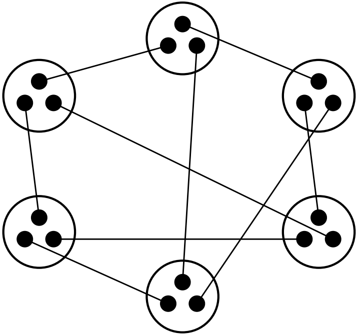

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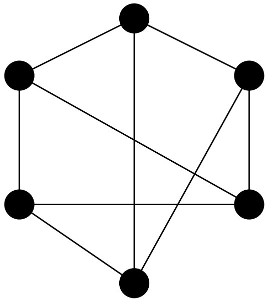
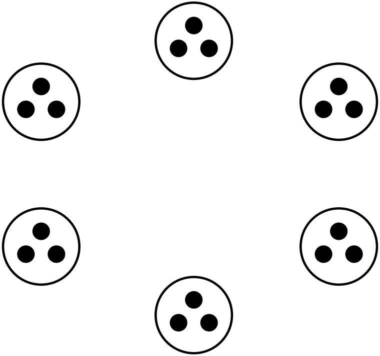


Rejection algorithm

The time complexity is exponential in d^2.

Switching algorithm (McKay and Wormald '90): DEG



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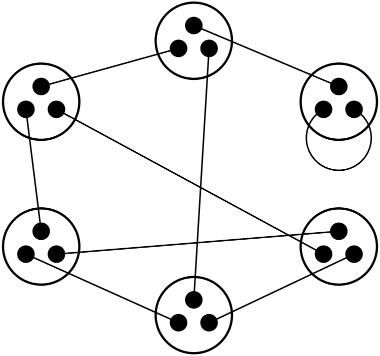

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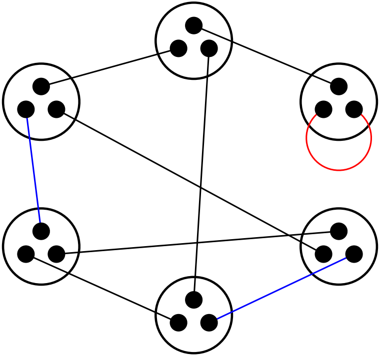
Switching algorithm (McKay and Wormald '90): DEG

For $d = O(n^{1/3})$, with a positive probability (bounded away from 0), there are no double-loops, or multiple edges with multiplicity greater than 2.
So we only need to worry about loops and double-edges.

Switching algorithm (McKay and Wormald '90): DEG



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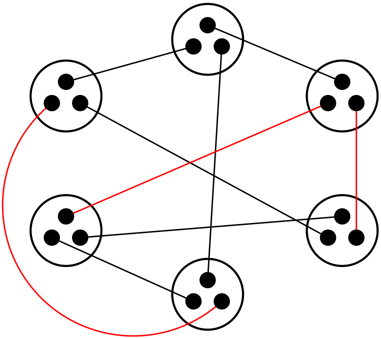

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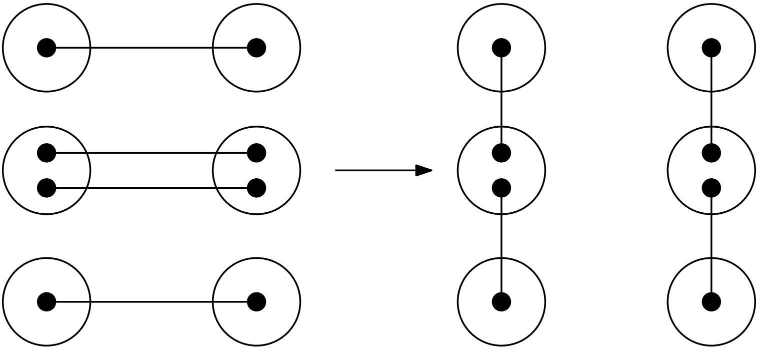

switch away double-edges

Switching algorithm (McKay and Wormald '90): DEG

Let $\mathcal{S}_i$ denote the set of pairings containing exactly i loops (correspondingly, i double-edges, in the phase for double-edge reduction).

A switching coverts a pairing $P \in \mathcal{S}_i$ to $P' \in \mathcal{S}_{i-1}$.

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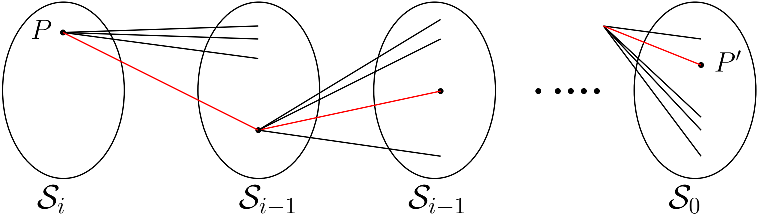

Switching algorithm (McKay and Wormald '90): DEG

- If P is uniformly distributed in $\mathcal{S}_i$, is P' uniformly distributed in $\mathcal{S}_{i-1}$?
- No.
- Because
  - The number $(N(P))$ of ways to perform a switching to $P \in \mathcal{S}_i$ is not uniformly the same for all $P \in \mathcal{S}_i$;
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- Use rejection to enforce uniformity of $P' \in \mathcal{S}_{i-1}$.
- Once a random switching (P, P') is chosen.
  - perform an f-rejection to equalise the probability of a switching for all $P \in \mathcal{S}_i$;
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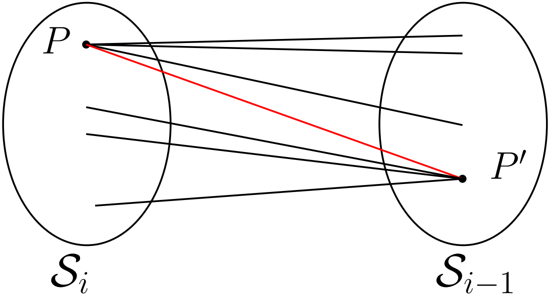

f-rejected with probability $1 - \frac{N(P)}{\max_{P'' \in \mathcal{S}_i}\{N(P'')\}}$;

b-rejected with probability $1 - \frac{\min_{P'' \in \mathcal{S}_{i-1}}\{N'(P'')\}}{N'(P')}$;

Switching algorithm (McKay and Wormald '90): DEG

- Inductively, the final pairing P' is uniformly distributed in $\mathcal{S}_0$, if no rejection has occurred.
- P' represents a uniformly random d-regular graph.
- The probability of a rejection is away from 1 if $d = O(n^{1/3})$.

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For $d \gg n^{1/3}$

For $d \gg n^{1/3}$, DEG is inefficient as the probability of a rejection during the algorithm is very close to 1.

This is because

- variation of $N(P)$ is too big for $P \in \mathcal{S}_i$, causing big f-rejection probability;
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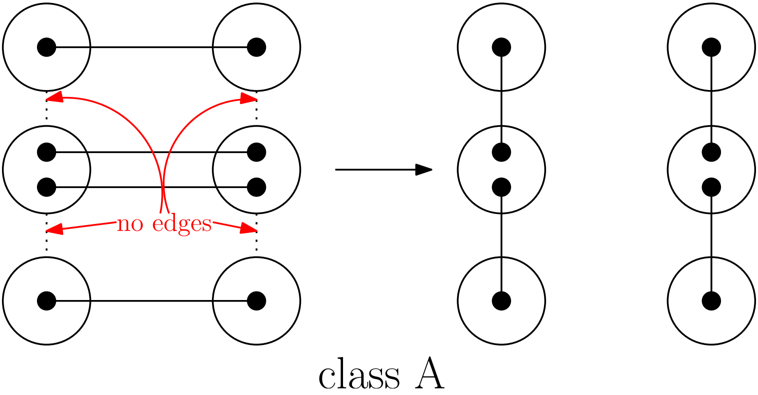
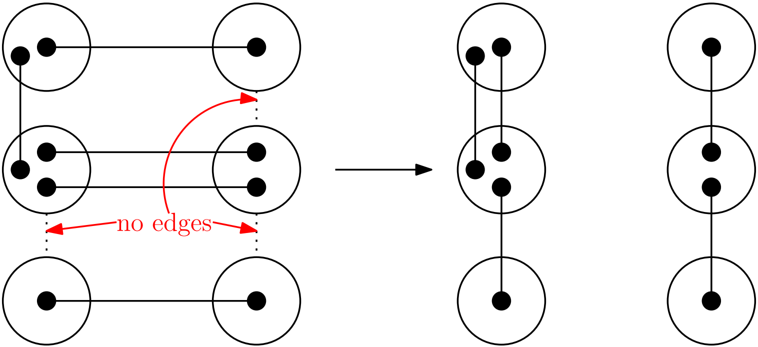
REG: reduce f-rejection

To reduce f-rejection, we allow some switchings that were forbidden before, these switchings are categorised into different classes (e.g. class B).

Class A: the typical one



Class B: the one forbidden by DEG

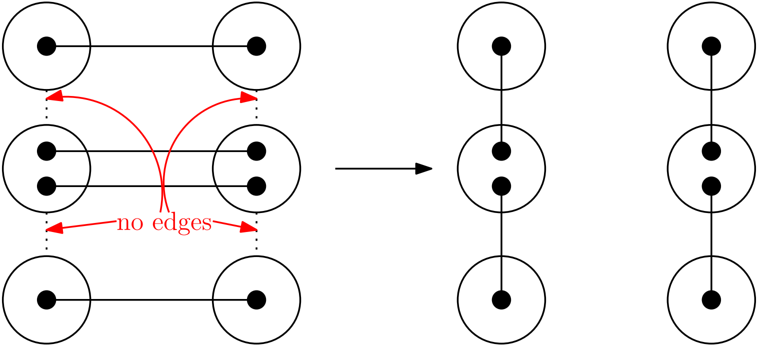


class B

Reduce b-rejection

To reduce b-rejection, we perform occasionally some other types of switchings that will boost the pairings that were underrepresented, i.e. with small value of $N'(P')$.

Type I



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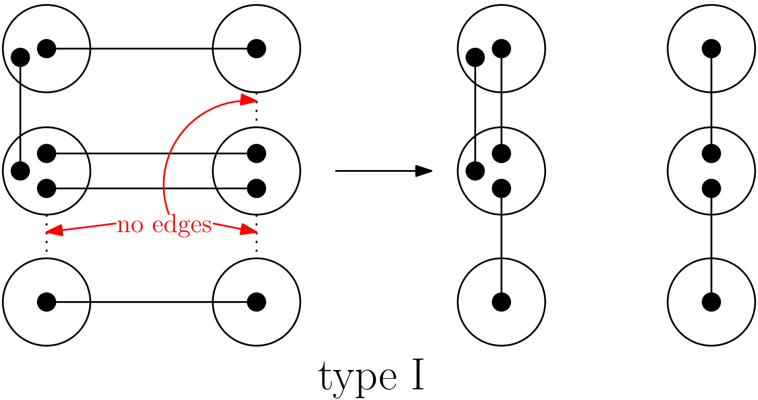
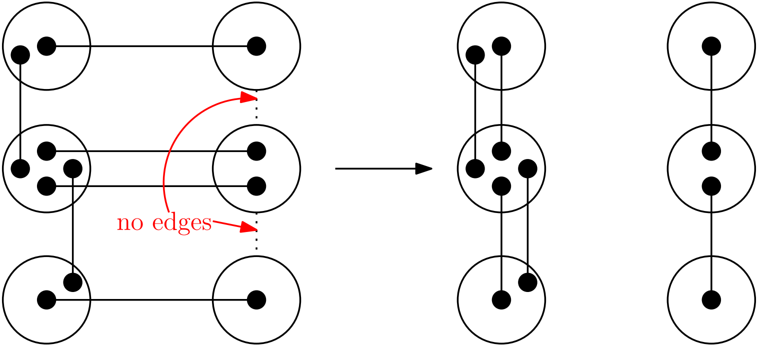


Type II



type II, class B

New features

- Each switching is associated with a type $\tau \in \mathcal{T}$ and a class $\alpha \in \mathcal{A}$;
- The Markov chain among $(\mathcal{S}_i)_{i \geq 0}$ cycles;
- The algorithm needs to decide which type of switchings to be performed in each step.

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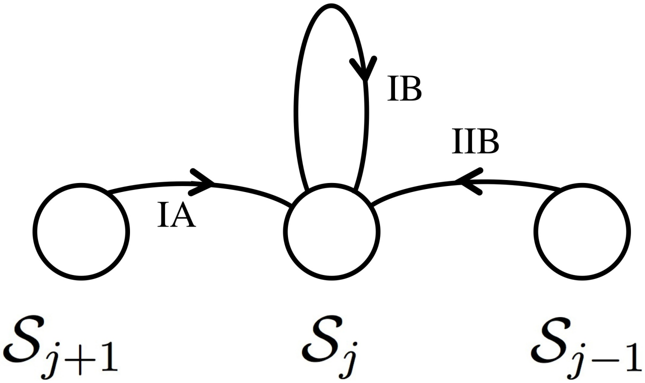
Transitions into $\mathcal{S}_j$



New elements of the analysis

- We no longer equalise the transition probabilities out of or into pairings in a certain $\mathcal{S}_i$, instead we equalise the expected number of times each pairing is visited in a given $\mathcal{S}_i$.
- This equalisation is done within each switching class $\alpha \in \mathcal{A}$.
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Algorithm framework

Given $P \in \mathcal{S}_i$,

(i) If $i = 0$, output P.

(ii) Choose a type: choose τ with probability $\rho_\tau(i)$, and with the remaining probability, $1 - \sum_\tau \rho_\tau(i)$, perform a t-rejection. Then select u.a.r. one of the type τ switchings that can be performed on P.

(iii) Let P' be the element that the selected switching would produce if applied to P, let α be the class of the selected switching and let $i' = \mathcal{S}(P')$. Perform an f-rejection with probability $1 - N_\tau(P)/\max\{N_\tau(P'')\}$ and then perform a b-rejection with probability $1 - \min\{N'_\alpha(P'')\}/N'_\alpha(P')$;

(iv) if no rejection occurred, replace P with P'.

(v) Repeat until $i = 0$.

The rest of the definition

- The algorithm is defined after we fix all parameters $\rho_\tau(i)$, $\tau \in \{I, II\}$.
- These probabilities are designed to equalise the expected number of times each pairing is visited in a certain $\mathcal{S}_i$.
- Thereby we deduce a system of equations that $\rho_\tau(i)$ must satisfy.
- We describe an efficient scheme to find a desirable solution to the system of equations.
- done!

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The rest of the definition

- The algorithm is defined after we fix all parameters $\rho_\tau(i)$, $\tau \in \{I, II\}$.
- These probabilities are designed to equalise the expected number of times each pairing is visited in a certain $\mathcal{S}_i$.
- Thereby we deduce a system of equations that $\rho_\tau(i)$ must satisfy.
- We describe an efficient scheme to find a desirable solution to the system of equations.
- done!

Time complexity

Theorem (Gao and Wormald '15)

REG generates d*-regular graphs uniformly at random. For* $d = o(\sqrt{n})$*, the expected running time of REG for generating one graph is* $O(d^3 n)$*.*

Approximate sampler: REG*

Most of the running time of REG is spent on computing the probability of a b-rejection in each step.

If we ignore b-rejections, we get a linear-running-time approximate sampler REG*.

Theorem (Gao and Wormald '15)

For $d = o(\sqrt{n})$, REG generates a random d-regular graph whose total variation distance from the uniform distribution is $o(1)$. The expected running time of REG* is $O(dn)$.*

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Future directions

- Larger d.
- General degree sequences.
- Heavily-tailed degree sequences such as power-law sequences.
- New switchings.