The Prokofiev-Svistunov-Ising process is rapidly mixing

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Collaborators

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Ising model - Motivation

- Introduced in 1920 as a model for ferromagnetism
- Hope was to explain the Curie transition:
  - Place iron in a magnetic field
  - Increase field to high value
  - Slowly reduce field to zero
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  - Image processing (archetypal Markov random field)
- Continues to play fundamental role in theoretical/mathematical studies of phase transitions and critical phenomena
The Ising model

- Finite graph $G = (V, E)$
- Configuration space $\Sigma_G = \{-1, +1\}^V$
- Measure
  \[
  \mathbb{P}(\sigma) = \frac{1}{Z} \exp \left( \beta \sum_{ij \in E} \sigma_i \sigma_j + h \sum_{i \in V} \sigma_i \right)
  \]
- Inverse temperature $\beta$
- External field $h$
- $Z$ is the partition function
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- Main physical interest is in certain expectations such as:
  - Two-point correlation $\text{cov}(\sigma_u, \sigma_v) = \mathbb{E}(\sigma_u \sigma_v) - \mathbb{E}(\sigma_u)\mathbb{E}(\sigma_v)$
  - Susceptibility $\chi = \frac{1}{|V|} \sum_{u, v \in V} \text{cov}(\sigma_u, \sigma_v)$
Phase transition

Let $\Lambda_n = \{-n, \ldots, n\}^d \subset \mathbb{Z}^d$, and consider

$$\Sigma^+_{\Lambda_n} = \{\sigma \in \{-1, 1\}^{\mathbb{Z}^d} : \sigma_i = +1 \text{ for all } i \not\in \Lambda_n\}$$

Sequence of Gibbs measures on $\Lambda_n$ converges: $\mathbb{P}^+_{\Lambda_n, \beta, h} \Rightarrow \mathbb{P}_\beta^+$
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Analogous construction for “minus” boundary conditions
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Analogous construction for “minus” boundary conditions

Theorem (Aizenman, Duminil-Copin, Sidoravicius (2014))

1. If $d = 1$, then for any $(\beta, h) \in [0, \infty) \times \mathbb{R}$ there is a unique infinite-volume Gibbs measure
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1. If \( d = 1 \), then for any \( (\beta, h) \in [0, \infty) \times \mathbb{R} \) there is a unique infinite-volume Gibbs measure

2. If \( d \geq 2 \) and \( h \neq 0 \), then for any \( \beta \in [0, \infty) \) there is a unique infinite-volume Gibbs measure
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1. If $d = 1$, then for any $(\beta, h) \in [0, \infty) \times \mathbb{R}$ there is a unique infinite-volume Gibbs measure
2. If $d \geq 2$ and $h \neq 0$, then for any $\beta \in [0, \infty)$ there is a unique infinite-volume Gibbs measure
3. If $d \geq 2$ and $h = 0$, there exists $\beta_c(d) \in (0, \infty)$ such that:
   a. If $\beta \leq \beta_c$, there is a unique infinite-volume Gibbs measure
   b. If $\beta > \beta_c$, then $\mathbb{P}^+_{\beta, 0} \neq \mathbb{P}^-_{\beta, 0}$
Exact solutions

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K-F related Ising partition function to perfect matchings
  - Elegant solution on planar graphs in terms of Pfaffians
  - Only tractable on graphs of bounded (small) genus
  - Method not tractable for $G_n = \mathbb{Z}_n^d$ with $d \geq 3$
Computational Complexity

- **PARTITION**: 
  - Input: Finite graph $G = (V, E)$, and parameters $\beta, h$ 
  - Output: Ising partition function

- **CORRELATION**: 
  - Input: Finite graph $G = (V, E)$, a pair $u, v \in V$, and parameters $\beta, h$ 
  - Output: Ising two-point correlation function

- **SUSCEPTIBILITY**: 
  - Input: Finite graph $G = (V, E)$, and parameters $\beta, h$ 
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**Proposition** (Jerrum-Sinclair 1993; Sinclair-Srivastava 2014)

PARTITION, SUSCEPTIBILITY and CORRELATION are #P-hard.
Markov-chain Monte Carlo

- Construct a transition matrix $P$ on $\Omega$ which:
  - Is ergodic
  - Has stationary distribution $\pi(\cdot)$
- Generate random samples with (approximate) distribution $\pi$
- Estimate $\pi$ expectations using sample means
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- **Mixing time** quantifies the rate of convergence
  $$t_{\text{mix}}(\delta) := \min \{ t : d(t) \leq \delta \}$$
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  - If $t_{\text{mix}} = O(\text{poly}(n))$ we have **rapid mixing**
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  - $|\Omega| = 2^n$ so rapid mixing implies only logarithmically-many states need be visited to reach approximate stationarity
Markov chains for the Ising model

- Glauber process (arbitrary field) 1963
  - Lubetzky & Sly (2012): Rapidly mixing on boxes in $\mathbb{Z}^2$ at $h = 0$ iff $\beta \leq \beta_c$
  - Levin, Luczak & Peres (2010): Precise asymptotics on $K_n$ for $h = 0$
  - Not used by computational physicists
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  - Simulates high-temperature graphs for $h > 0$
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  - No empirical results - not used by computational physicists
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- **Prokofiev-Svistunov worm process (zero field)** 2001
  - No rigorous results currently known
  - Empirically, best method known for susceptibility (Deng, G., Sokal)
  - Widely used by computational physicists
Theorem (Collevecchio, G., Hyndman, Tokarev 2014+)

For any temperature, the mixing time of the PS process on graph $G = (V, E)$ satisfies

$$t_{\text{mix}}(\delta) = O(\Delta(G)m^2n^5)$$

with $n = |V|$, $m = |E|$ and $\Delta(G)$ the maximum degree.
Mixing time bound for PS process

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Only Markov chain for the Ising model currently known to be rapidly mixing at the critical point for boxes in $\mathbb{Z}^d$
High-temperature expansions and the PS measure

Let $C_k = \{A \subseteq E : (V, A) \text{ has } k \text{ odd vertices} \}$
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- PS measure defined on the configuration space $C_0 \cup C_2$

$$\pi(A) \propto x^{|A|} \begin{cases} n, & A \in C_0, \\ 2, & A \in C_2. \end{cases}$$
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  - Ising susceptibility $\chi = \frac{1}{\pi(C_0)}$
  - Ising two-point correlation function $\mathbb{E}(\sigma_u \sigma_v) = \frac{n}{2} \frac{\pi(C_{uv})}{\pi(C_0)}$
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- PS measure is stationary distribution of PS process
Fully-polynomial randomized approximation schemes

**Definition**

An **fpras** for an Ising property $f$ is a randomized algorithm such that for given $G$, $T$, and $\xi, \eta \in (0, 1)$ the output $Y$ satisfies

$$\mathbb{P}[(1 - \xi)f \leq Y \leq (1 + \xi)f] \geq 1 - \eta$$

and the running time is bounded by a polynomial in $n, \xi^{-1}, \eta^{-1}$. 
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  - Let $R(G)$ be our upper bound for $t_{mix}(\delta)$ with $\delta = \xi/[8S(n)]$
  - Let $Y = 1_{\mathcal{A}}$
  - Run the PS process $T = R(G)$ time steps and record $Y_T$
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Combine rapid mixing of PS process with general fpras construction of Jerrum-Sinclair (1993):

- Let $\mathcal{A} \subseteq \mathcal{C}_0 \cup \mathcal{C}_2$ with $\pi(\mathcal{A}) \geq 1/S(n)$ for some polynomial $S(n)$
- The following defines an fpras for $\pi(\mathcal{A})$:
  - Let $R(G)$ be our upper bound for $t_{\text{mix}}(\delta)$ with $\delta = \xi/[8S(n)]$
  - Let $Y = 1_{\mathcal{A}}$
  - Run the PS process $T = R(G)$ time steps and record $Y_T$
  - Independently generate $72\xi^{-2}S(n)$ such samples and take the sample mean
Fully-polynomial randomized approximation schemes

Definition

An **fpras** for an Ising property \( f \) is a randomized algorithm such that for given \( G, T \), and \( \xi, \eta \in (0, 1) \) the output \( Y \) satisfies

\[
\mathbb{P}[(1 - \xi)f \leq Y \leq (1 + \xi)f] \geq 1 - \eta
\]

and the running time is bounded by a polynomial in \( n, \xi^{-1}, \eta^{-1} \).

Combine rapid mixing of PS process with general fpras construction of Jerrum-Sinclair (1993):

- Let \( A \subseteq C_0 \cup C_2 \) with \( \pi(A) \geq 1/S(n) \) for some polynomial \( S(n) \)
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  - Independently generate \( 72\xi^{-2}S(n) \) such samples and take the sample mean
  - Repeat \( 6 \log \lceil 1/\eta \rceil + 1 \) such experiments and take the median
Prokofiev-Svistunov process

PS proposals:
Prokofiev-Svistunov process

PS proposals:
- If $A \in C_0$: [Diagram of a grid with selected paths]
Prokofiev-Svistunov process

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Introduction

Main Theorem

Proof

Discussion

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Metropolize proposals with respect to PS measure $\pi(\cdot)$
Proof of rapid mixing

- We use the **path method**
- Consider the **transition graph** $G = (\mathcal{V}, \mathcal{E})$ of the PS process
  - $\mathcal{V} = \mathcal{C}_0 \cup \mathcal{C}_2$
  - $\mathcal{E} = \{AA' : P(A, A') > 0\}$
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- Specify paths $\gamma_{I,F}$ in $G$ between pairs of states $I, F$
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Lemma (Jerrum-Sinclair-Vigoda (2004))

Consider MC with state space $\Omega$ and stationary distribution $\pi$. Let $S \subset \Omega$, and specify paths $\Gamma = \{\gamma_{I,F} : I \in S, F \in S^c\}$. Then

$$t_{\text{mix}}(\delta) \leq \log \left( \frac{1}{\delta \min_{A \in \Omega} \pi(A)} \right) \left[ 2 + 4 \left( \frac{\pi(S)}{\pi(S^c)} + \frac{\pi(S^c)}{\pi(S)} \right) \right] \varphi(\Gamma)$$

where

$$\varphi(\Gamma) := \left( \max_{(I,F) \in S \times S^c} |\gamma_{I,F}| \right) \max_{AA' \in \mathcal{E}} \left\{ \sum_{(I,F) \in S \times S^c \atop \gamma_{I,F} \ni AA'} \frac{\pi(I)\pi(F')}{\pi(A)P(A,A')} \right\}$$
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- We choose $S = C_2$. 

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—we choose $S = C_2$. Elementary to show:

$$2 \frac{mx}{n \ mx + 1} \leq \frac{\pi(C_2)}{\pi(C_0)} \leq n - 1,$$

and

$$\pi(A) \geq \left( \frac{x}{8} \right)^m$$

for all $A$.
Lemma (Jerrum-Sinclair-Vigoda (2004))

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  $$\frac{2}{n \text{mx}} + 1 \leq \frac{\pi(C_2)}{\pi(C_0)} \leq n-1, \quad \text{and} \quad \pi(A) \geq \left( \frac{x}{8} \right)^m \quad \text{for all } A$$

- Therefore:

  $$t_{\text{mix}}(\delta) \leq \left( \log \left( \frac{8}{x} \right) - \frac{\log \delta}{m} \right) \left( 3 + \frac{1}{\text{mx}} \right) 2mn \varphi(\Gamma)$$
Choice of Canonical Paths

- To transition from $I$ to $F$
  - Flip each $e \in I \Delta F$
Choice of Canonical Paths

- To transition from $I$ to $F$
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- If $(I, F) \in \mathcal{C}_2 \times \mathcal{C}_0$ then $I \Delta F \in \mathcal{C}_2$
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- $I \triangle F = A_0 \cup \left( \bigcup_{i \geq 1} A_i \right)$
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One can show that $\varphi(\Gamma) \leq \Delta(G)n^4m/2$
Bounding the congestion

- **Define measure** $\Lambda$
  
  $\Lambda(A) = x^{|A|} \begin{cases} 
  n, & A \in C_0 \\
  2, & A \in C_2 \\
  1, & A \in C_4 
  \end{cases}$

- $\pi(A) = \frac{\Lambda(A)}{\Lambda(C_0 \cup C_2)}$

If $T = AA'$ is a maximally congested transition

$$\varphi(\Gamma) = \sum_{(I,F) \in \mathcal{P}(T)} \frac{\pi(I)\pi(F)}{\pi(A)P(A, A')} \max_{(I,F) \in S \times S^c} |\gamma_{I,F}|$$
Bounding the congestion

- **Define measure** $\Lambda$

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\varphi(\Gamma) \leq \sum_{(I,F) \in \mathcal{P}(T)} \frac{\pi(I)\pi(F)}{\pi(A)P(A, A')} m
\]
Bounding the congestion

- Define measure $\Lambda$

\[ \Lambda(A) = x^{|A|} \begin{cases} 
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\end{cases} \]

- $\pi(A) = \frac{\Lambda(A)}{\Lambda(C_0 \cup C_2)}$

If $T = AA'$ is a maximally congested transition

\[ \varphi(\Gamma) \leq \sum_{(I,F) \in \mathcal{P}(T)} \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)\Lambda(C_0 \cup C_2)} \frac{m}{\Lambda(A)\Lambda(C_0 \cup C_2)} \]
Bounding the congestion

- **Define measure** \( \Lambda \)

  \[
  \Lambda(A) = x^{|A|} \begin{cases} 
  n, & A \in C_0 \\
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  \pi(A) = \frac{\Lambda(A)}{\Lambda(C_0 \cup C_2)}
  \]

If \( T = AA' \) is a maximally congested transition

\[
\varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I,F) \in \mathcal{P}(T)} \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)\rho(A, A')}
\]
Bounding the congestion

- **Define measure** \( \Lambda \)
  \[
  \Lambda(A) = x^{|A|} \begin{cases} 
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  2, & A \in C_2 \\
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\varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I,F) \in \mathcal{P}(T)} \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)P(A, A')}
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Bounding the congestion

Define measure $\Lambda$

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\Lambda(A) = x^{\lfloor |A| \rfloor} \begin{cases} 
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2, & A \in C_2 \\
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\pi(A) = \frac{\Lambda(A)}{\Lambda(C_0 \cup C_2)}
\]

Define for each $T = AA' \in \mathcal{E}$

\[
\eta_T : C_2 \times C_0 \rightarrow C_0 \cup C_2 \cup C_4
\]

\[
\eta_T(I, F) = I \triangle F \triangle (A \cup A')
\]

\[
\frac{\Lambda(I) \Lambda(F)}{\Lambda(A) P(A, A')} \leq 4\Delta(G)n \Lambda(\eta_T(I, F))
\]

If $T = AA'$ is a maximally congested transition

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\varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I, F) \in \mathcal{P}(T)} \frac{\Lambda(I) \Lambda(F)}{\Lambda(A) P(A, A')}
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Bounding the congestion

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If $T = AA'$ is a maximally congested transition

$$\varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I, F) \in \mathcal{P}(T)} \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)P(A, A')}$$

$$\leq 4 \Delta(G)n m \frac{1}{\Lambda(C_0 \cup C_2)} \sum_{(I, F) \in \mathcal{P}(T)} \Lambda(\eta_T(I, F))$$
Bounding the congestion

- **Define measure** $\Lambda$
  
  $$\Lambda(A) = x^{|A|} \begin{cases} 1, & A \in C_4 \\ 2, & A \in C_2 \\ n, & A \in C_0 \end{cases}$$

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- $\eta_T$ is an injection

If $T = AA'$ is a maximally congested transition

$$\varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I, F) \in \mathcal{P}(T)} \frac{\Lambda(I) \Lambda(F)}{\Lambda(A) \Lambda(P(A, A'))} \leq 4 \Delta(G) n m \sum_{(I, F) \in \mathcal{P}(T)} \Lambda(\eta_T(I, F))$$
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  \]

- **Define for each** $T = AA' \in \mathcal{E}$
  - $\eta_T : C_2 \times C_0 \to C_0 \cup C_2 \cup C_4$
  - $\eta_T(I, F) = I \triangle F \triangle (A \cup A')$
  - $\frac{\Lambda(I)\Lambda(F)}{\Lambda(A)\mathcal{P}(A, A')} \leq 4\Delta(G)n\Lambda(\eta_T(I, F))$

- $\eta_T$ *is an injection*

If $T = AA'$ is a maximally congested transition

\[
\varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I,F) \in \mathcal{P}(T)} \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)\mathcal{P}(A, A')}
\]

\[
\leq 4\Delta(G)n\frac{1}{m} \sum_{(I,F) \in \mathcal{P}(T)} \Lambda(\eta_T(I, F))
\]

\[
\leq 4\Delta(G)n\frac{\Lambda(C_0 \cup C_2 \cup C_4)}{\Lambda(C_0 \cup C_2)}
\]
Bounding the congestion

Define measure $\Lambda$

\[ \Lambda(A) = \begin{cases} n, & A \in C_0 \\ 2, & A \in C_2 \\ 1, & A \in C_4 \end{cases} \]

\[ \pi(A) = \frac{\Lambda(A)}{\Lambda(C_0 \cup C_2)} \]

Define for each $T = AA' \in \mathcal{E}$

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\[ \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)P(A, A')} \leq 4\Delta(G)n\Lambda(\eta_T(I, F)) \]

$\eta_T$ is an injection

If $T = AA'$ is a maximally congested transition

\[ \varphi(\Gamma) \leq \frac{m}{\Lambda(C_0 \cup C_2)} \sum_{(I, F) \in P(T)} \frac{\Lambda(I)\Lambda(F)}{\Lambda(A)P(A, A')} \]

\[ \leq 4\Delta(G)nm \frac{1}{\Lambda(C_0 \cup C_2)} \sum_{(I, F) \in P(T)} \Lambda(\eta_T(I, F)) \]

\[ \leq 4\Delta(G)nm \frac{\Lambda(C_0 \cup C_2 \cup C_4)}{\Lambda(C_0 \cup C_2)} = 4\Delta(G)nm \left(1 + \frac{\Lambda(C_4)}{\Lambda(C_0) + \Lambda(C_2)}\right) \]
Bounding the congestion cont. . .

The final step is to note that the high-temperature expansion implies

\[
\frac{\Lambda(C_4)}{\Lambda(C_0)} \leq \frac{1}{n} \binom{n}{4}
\]

so that

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\varphi(\Gamma) \leq 4 \Delta(G) nm \left(1 + \frac{\Lambda(C_4)}{\Lambda(C_0)}\right)
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Bounding the congestion cont. . .

The final step is to note that the high-temperature expansion implies

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\]
\[
\leq 4 \Delta(G) nm \frac{n^3}{8}
\]
Bounding the congestion cont. . .

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\[
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Bounding the congestion cont. . .

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$$\leq 4 \Delta(G) nm \frac{n^3}{8}$$

$$= \frac{\Delta(G) n^4 m}{2}$$

□
Can we obtain sharper results if we focus on special families of graphs, such as $G = \mathbb{Z}_L^d$?
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The PS process is closely related to a modification of the “lamplighter walk” in which lamps are always switched when visited. Can we use this similarity to say something more precise when $G = \mathbb{Z}_L^d$?
Discussion

- Can we obtain sharper results if we focus on special families of graphs, such as $G = \mathbb{Z}_L^d$?
- The PS process is closely related to a modification of the “lamplighter walk” in which lamps are always switched when visited. Can we use this similarity to say something more precise when $G = \mathbb{Z}_L^d$?
- Study related spin models using similar methods?
Mixing time bound for PS process

Theorem (Collevecchio, G., Hyndman, Tokarev 2014+)

The mixing time of the PS process on graph $G = (V, E)$ with parameter $x \in (0, 1)$ satisfies

$$t_{\text{mix}}(\delta) \leq \left( \log \left( \frac{8}{x} \right) - \frac{\log \delta}{m} \right) \left( 3 + \frac{1}{m x} \right) \Delta(G) m^2 n^5,$$

with $n = |V|$, $m = |E|$ and $\Delta(G)$ the maximum degree.
Let $\partial A = \{ v \in V : v \text{ has odd degree in } (V, A) \}$
High-temperature expansions and the PS measure

- Let $\partial A = \{v \in V : v$ has odd degree in $(V, A)\}$
- Let $C_W := \{A \subseteq E : \partial A = W\}$ for $W \subseteq V$
High-temperature expansions and the PS measure

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- $\lambda(\cdot)$ defined by $\lambda(S) = \sum_{A \in S} x^{\mid A \mid}$ for $S \subseteq \{A \subseteq E\}$
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If $x = \tanh \beta$ then

$$E_{G, \beta}^{(\text{Ising})} \left( \prod_{v \in W} \sigma_v \right) = \frac{\lambda(C_W)}{\lambda(C_0)}$$
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If $x = \tanh \beta$ then

$$
\mathbb{E}_{\mathcal{G},\beta}^{(\text{Ising})} \left( \prod_{v \in W} \sigma_v \right) = \frac{\lambda(C_W)}{\lambda(C_0)}
$$

- PS measure defined on the configuration space $C_0 \cup C_2$

$$
\pi(A) = \frac{\lambda(A)}{n\lambda(C_0) + 2\lambda(C_2)} \begin{cases} 
n, & A \in C_0, \\
2, & A \in C_2.
\end{cases}
$$
High-temperature expansions and the PS measure

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- PS measure defined on the configuration space $C_0 \cup C_2$

$$\pi(A) = \frac{\lambda(A)}{n \lambda(C_0) + 2 \lambda(C_2)} \begin{cases} n, & A \in C_0, \\ 2, & A \in C_2. \end{cases}$$

- Ising susceptibility $\chi = \frac{1}{\pi(C_0)}$

- Ising two-point correlation function $\mathbb{E}(\sigma_u \sigma_v) = \frac{n \pi(C_{uv})}{2 \pi(C_0)}$
High-temperature expansions and the PS measure

- Let $\partial A = \{v \in V : v$ has odd degree in $(V, A)\}$
- Let $C_W := \{A \subseteq E : \partial A = W\}$ for $W \subseteq V$
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- $\lambda(\cdot)$ defined by $\lambda(S) = \sum_{A \in S} x^{|A|}$ for $S \subseteq \{A \subseteq E\}$

If $x = \tanh \beta$ then

$$\mathbb{E}_{G, \beta}^{(\text{Ising})} \left( \prod_{v \in W} \sigma_v \right) = \frac{\lambda(C_W)}{\lambda(C_0)}$$

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$$\pi(A) = \frac{\lambda(A)}{n\lambda(C_0) + 2\lambda(C_2)} \begin{cases} n, & A \in C_0, \\ 2, & A \in C_2. \end{cases}$$

- Ising susceptibility $\chi = \frac{1}{\pi(C_0)}$

- Ising two-point correlation function $\mathbb{E}(\sigma_u \sigma_v) = \frac{n}{2} \frac{\pi(C_{uv})}{\pi(C_0)}$

- PS measure is stationary distribution of PS process