Cycles of given size in a dense graph

Daniel J. Harvey

Monash University

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Corrádi-Hajnal Theorem

Theorem (Corrádi, Hajnal (1963))

Every graph with minimum degree $\geq 2k$ and at least $3k$ vertices contains k (vertex) disjoint cycles.

Corrádi-Hajnal Theorem

There have been many extensions/generalisations/variants of this result:

Theorem (Egawa et. al. (2003))

Every graph with at least $3k$ vertices such that $\deg(x) + \deg(y) \geq 4k - 1$ *for all non-adjacent vertices* x, y *contains* k *disjoint cycles.*

Theorem (Justesen (1989))

Every graph with $n \geq 3k$ *vertices and* $\max\{(2k-1)(n-k), \binom{3k-1}{2} + (n - 3k + 1)\}$ *edges contains* k *disjoint cycles.*

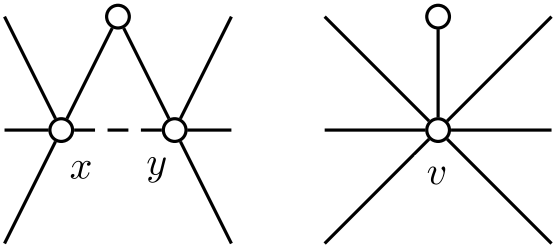
Corrádi-Hajnal Theorem

Theorem (Wang (2012))

Let $k \geq 2$. Every graph with minimum degree $\geq 2k$ and at least $4k$ vertices contains k (vertex) disjoint cycles of length at least 4, except in a few restricted cases.

Minors

Graph H is a *minor* of G if (a graph isomorphic to) H can be constructed from G by repeated *vertex deletion*, *edge deletion* and *edge contraction*.



Minors

Theorem (Mader (1967))

Every graph with average degree at least 2^{t-2} contains K_t as a minor.

Theorem (Kostochka (1982,1984) Thomason (1984,2001))

Every graph with average degree $(\alpha + o(1))t\sqrt{\ln t}$ contains K_t as a minor.

Minors

- One possible extension is to consider the average degree required to force an H-minor, for some other fixed H.
- Myers and Thomason (2005) answered this question when H is sufficiently dense.
- $K_{s,t}$ has been well studied (for $s \ll t$). (Kühn & Osthus (2005), Kostochka & Prince (2008))

Cycle Minors

If we let H be the graph consisting of k disjoint triangles, we obtain a similar result to Corrádi-Hajnal.

$$G \text{ contains an } r\text{-length cycle as a minor.}$$
$$\updownarrow$$
$$G \text{ contains a } (\geq r)\text{-length cycle as a subgraph.}$$

Cycle Minors

What average degree is required to force the existence of an H-minor, when H is the graph consisting of k disjoint copies of K_3?

Theorem

Every graph with average degree $\geq 4k - 2$ contains k copies of K_3 as a minor.

Minor-Minimal Graphs

- We say G is *minor-minimal* (with respect to average degree) if every proper minor of G has average degree lower than G.
- $\delta(G) > \frac{1}{2}d(G)$,
- $\tau(G) > \frac{1}{2}d(G) - 1$, where $\tau(G)$ is the minimum number of common neighbours for the endpoints of an edge.

Proof

- Say $d(G) \geq 4k - 2$,
- Let G' be a minor-minimal minor of G,
- $\delta(G') > 2k - 1$ and $|V(G')| \geq 4k - 1$,
- So G' contains k disjoint cycles, by Corrádi-Hajnal.

Main Theorem

Theorem (Harvey, Wood (2015))

Let H be the graph consisting of $k(\geq 6)$ disjoint r-cycles. Every graph G with average degree at least $\frac{4}{3}kr$ contains H as a minor.

$r = 3$ case — direct proof

- $d(G) \geq 4k - 2 \rightarrow k$ disjoint copies of K_3 as a minor.
- This proof is "shorter" than the other version, because Corrádi-Hajnal is longer.
- The techniques we use here generalise to the general r case.

$r = 3$ case — direct proof

- Let G be minor-minimal.
  - $\delta(G) \geq 2k$
  - $\tau(G) \geq 2k - 1$
- Partition $V(G)$ into disjoint cycles of length ≤ 3
  - Pretend that K_2, K_1 are cycles.
- Choose this partition so that $\#K_3$ is maximised, then $\#K_2$ is maximised.

$r = 3$ case — direct proof

- Suppose that there are at least two K_2's, call them vw and xy.

$r = 3$ case — direct proof

- Suppose that there are at least two K_2's, call them vw and xy.
- Let $p_i := \#$ of neighbours of vw in the i^{th} copy of K_3.
- Let $q_i := \#$ of neighbours of xy in the i^{th} copy of K_3.

$r = 3$ case — direct proof

$$\sum_{i=1}^{\#K_3}(p_i + q_i) = \sum_{i=1}^{\#K_3} p_i + \sum_{i=1}^{\#K_3} q_i \geq 2\tau(G) \geq 4k - 2$$

Since $\#K_3 \leq k - 1$, there exists an i such that $p_i + q_i \geq 5$.

$r = 3$ case — direct proof

- Hence, there can be at most one K_2.
- By a similar argument, there is at most one K_1.

$r = 3$ case — direct proof

- Hence $|V(G)| \leq 3(k-1) + 2 + 1 = 3k$.
- However $|V(G)| \geq 4k - 1$, so $k = 1$.
- But $d(G) \geq 2$ forces the existence of a single cycle.

□

General Case

- Now we consider the extension to the case of general r.
- Again, consider a minor-minimal G.
  - $\delta(G) > \frac{2}{3}kr$,
  - $\tau(G) > \frac{2}{3}kr - 1$.

General Case

- Now we consider the extension to the case of general r.
- Again, consider a minor-minimal G.
  - $\delta(G) > \frac{2}{3}kr$,
  - $\tau(G) > \frac{2}{3}kr - 1$.
- Partition $V(G)$ into disjoint cycles of length $\leq r$.
- Choose partition so that $\#$ cycles of length r is maximised, then $\#$ cycles of length $r-1$, ...

General Case

- Consider an edge vw in the largest $(< r)$-cycle and an edge xy in the second largest $(< r)$-cycle.
- We essentially show that if there are $< k$ cycles of length r then it is possible to construct a better partition.

Key Lemma

Lemma

Let $C_1, \ldots, C_t$ be a set of disjoint r-cycles, q an integer such that $1 \leq q \leq r-1$, and $S, T \subseteq V(C_1) \cup \cdots \cup V(C_t)$ such that $|S|, |T| > \frac{2}{3}tr$.
There exists a path P in some C_i such that P contains q vertices, P has both end vertices in S, and there exists a vertex of T in $C_i - P$.

General Case

- If we cannot construct a better partition, then it follows that $|V(G)|$ is too small given $d(G)$.
- (There are also a few specialised cases to consider, which I'll omit.)

Sharpness

Our main theorem is almost sharp when $r = 3$;

Theorem (Harvey, Wood (2015))

Let H be the graph consisting of $k(\geq 6)$ disjoint r-cycles. Every graph G with average degree at least $\frac{4}{3}kr$ contains H as a minor.

Extensions

Can we improve $\frac{4}{3}$? When $r = 4$, $d(G) \geq 4k - 1$ is sufficient and best possible. Recall the following.

Theorem (Wang (2012))

Every graph with minimum degree $\geq 2k$ and at least $4k$ vertices contains k (vertex) disjoint cycles of length at least 4, except in a few restricted cases.

Extensions

Can we improve $\frac{4}{3}$?

Conjecture

For every integer $k \geq 2$ and odd integer $r \geq 3$, every graph with average degree at least $(r+1)k - 2$ contains k disjoint cycles of length at least r.

Conjecture

For every integer $k \geq 3$ and even integer $r \geq 6$, every graph with average degree at least $rk - 2$ and at least rk vertices contains k disjoint cycles of length at least r.

Extensions

These conjectures are similar to previous conjectures by Wang:

Conjecture (Wang (2012))

For every integer $k \geq 2$ and odd integer $r \geq 3$, every graph G with at least rk vertices and minimum degree at least $\frac{r+1}{2}k$ contains k disjoint cycles of length at least r.

Conjecture (Wang (2012))

For every integer $k \geq 3$ and even integer $r \geq 6$, every graph G with at least rk vertices and minimum degree at least $\frac{r}{2}k$ contains k disjoint cycles of length at least r, unless k is odd and $rk + 1 \leq |V(G)| \leq rk + r - 2$.

Extensions

- Alternatively, we could allow H to be any t-vertex 2-regular graph
  - That is, the cycles of H may have different orders.

Conjecture (Reed, Wood (2014))

Every graph with average degree $\frac{4}{3}t - 2$ contains every t-vertex 2-regular graph as a minor.

Extensions

Finally, we could consider the following:

Conjecture

Let H be a t-vertex 2-regular graph with c odd order components, such that H is not a single cycle of even order of $\frac{t}{4}$ cycles of order 4. Every graph with average degree at least $t + c - 2$ and at least t vertices contains H as a minor.