The Erdős-Ko-Rado Theorem and the Treewidth of the Kneser Graph

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Erdős-Ko-Rado Theorem

- Let $[n]$ denote the set of elements $\{1, \ldots, n\}$.
- A *k-set* of $[n]$ is a subset of $[n]$ of k elements.
- There are $\binom{n}{k}$ such k-sets.
- Denote the set of all k-sets by $\binom{[n]}{k}$.

Erdős-Ko-Rado Theorem

Question

Given a collection of k-sets of $[n]$ denoted $\mathcal{A}$, such that any two k-sets of $\mathcal{A}$ intersect, how large can $|\mathcal{A}|$ be?

Also

Question

What might $\mathcal{A}$ look like when it maximises $|\mathcal{A}|$?

Easy Case

- If $n < 2k$, then $\mathcal{A} = \binom{[n]}{k}$.
- Hence we assume $n \geq 2k$.

A Naïve Answer

- Let $\mathcal{A}$ contain all k-sets that contain the element 1.
- Clearly the sets of $\mathcal{A}$ pairwise intersect.
- $|\mathcal{A}| = \binom{n-1}{k-1}$.
- This is in fact best possible.

Erdős-Ko-Rado Theorem

This proof due to Gyula O.H. Katona.

- Let $\mathcal{A}$ be a collection of pairwise intersecting k-sets
- Let C be a cyclic order of $[n]$.
- Consider the pairs (a, C), where $a \in \mathcal{A}$ and a forms a contiguous block in C.
- We shall double-count the number of pairs (a, C).

Erdős-Ko-Rado Theorem

- For a fixed a, there are $k!(n-k)!$ pairs (a, C).
- Hence $\#(a, C) = |\mathcal{A}|k!(n-k)!$

Erdős-Ko-Rado Theorem

- For a fixed C, how many pairs (a, C) are there?
- If (a, C) is a pair, then a forms a contiguous block in C.
- If (b, C) is also a pair, then the block for b must intersect the block for a.
- Naïvely, there are at most $2(k-1)$ possible b.
- However, as $n \geq 2k$, it is only possible to get at most half of these.
- Hence for a fixed C there are at most k pairs (a, C).

Erdős-Ko-Rado Theorem

- There are $(n-1)!$ choices of C.
- Hence $\#(a, C) \leq k(n-1)!$

Erdős-Ko-Rado Theorem

$$|\mathcal{A}|k!(n-k)! = \#(a, C) \leq k(n-1)!$$

$$|\mathcal{A}| \leq \frac{k(n-1)!}{k!(n-k)!}$$

$$|\mathcal{A}| \leq \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Erdős-Ko-Rado Theorem

Answer

Thus, the naïve choice of $\mathcal{A}$ is best possible.

Answer

If $n > 2k$, then the naïve choice is the unique maximal $\mathcal{A}$. (When $n = 2k$, can also consider all k-sets not containing element 1.)

Extensions of the Erdős-Ko-Rado Theorem

There has been some work on generalising the Erdős-Ko-Rado Theorem in different directions.

- Allow sets in $\mathcal{A}$ to have less than k elements, with the added proviso that none is a subset of another.
- This turns out to be exactly equivalent to Erdős-Ko-Rado, due to a result by Sperner.
- Alternatively, allow a certain bounded amount of non-intersection in $\mathcal{A}$; that is, each set in $\mathcal{A}$ is allowed to be non-intersecting with at most d others.

Cross-Intersecting Families

Question

Given two collections of k-sets of $[n]$ denoted $\mathcal{A}$ and $\mathcal{B}$, such that every k-set of $\mathcal{A}$ intersects every k-set of $\mathcal{B}$, how large can $|\mathcal{A}||\mathcal{B}|$ be?

Also

Question

What might $\mathcal{A}, \mathcal{B}$ look like when maximising $|\mathcal{A}||\mathcal{B}|$?

Cross-Intersecting Families

Answered by Pyber, then Matsumoto and Tokushige

Answer

$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$

Answer

If $|\mathcal{A}||\mathcal{B}| = \binom{n-1}{k-1}^2$*, then* $\mathcal{A} = \mathcal{B} = \{$*all sets containing element* i *for fixed* $i\}$.

Note no requirement that $\mathcal{A}, \mathcal{B}$ be disjoint.

Question

What if $\mathcal{A} \cap \mathcal{B} = \emptyset$*?*

A Graph Theoretic Interpretation

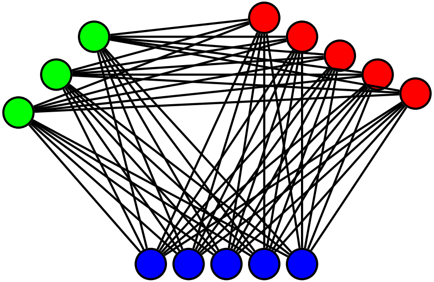
- Let $G(n, k)$ denote the intersection graph with vertex set $\binom{[n]}{k}$.
  - Two vertices are adjacent iff their k-sets intersect.
- A collection of pairwise intersecting k-sets corresponds to a clique in $G(n, k)$.
- Hence Erdős-Ko-Rado states that $\omega(G(n, k)) = \binom{n-1}{k-1}$.

A Graph Theoretic Interpretation

- If $\mathcal{A}, \mathcal{B}$ are (disjoint) cross-intersecting families, then they form a complete bipartite subgraph.
  - Note this is not necessarily an induced subgraph.
- Hence, we can think of finding a large pair of cross-intersecting families as trying to determine an upper bound on the order of a complete bipartite subgraph.
- Essentially, this is now a problem in extremal graph theory.

A Few Technicalities

- We might as well ask the more general question, and try to determine the upper bound on the order of a complete multipartite subgraph.



A Few Technicalities

- In this case, it makes sense to try and maximise the number of vertices in the subgraph, i.e. $|\mathcal{A}| + |\mathcal{B}|$ instead of $|\mathcal{A}||\mathcal{B}|$.
- However, this leads to an obvious problem: Set $\mathcal{A} = V(G(n,k)), \mathcal{B} = \emptyset$; this maximises $|\mathcal{A} \cup \mathcal{B}|$.
- To avoid this, say no part of the complete multipartite subgraph contains too many vertices.

The Largest Multipartite Subgraph of $G(n, k)$

Question

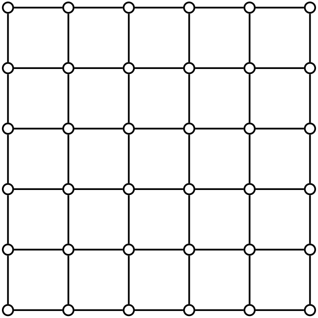
Say $p \in [\frac{2}{3}, 1)$. If H is a complete multipartite graph, a subgraph of $G(n, k)$, and no colour class of H contains more than $p|H|$ vertices, how large can $|H|$ be?

The benefit to this interpretation is that we can now use results of graph structure theory to determine $|H|$.

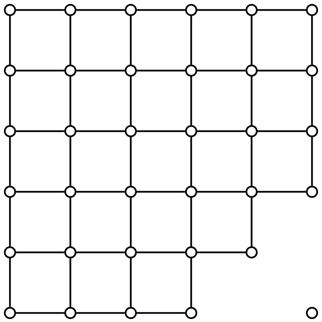
Separators

- It is a standard question in graph theory to determine minimal vertex cuts.
  - Connectivity etc.
- Sometimes, however, this is not really sufficient.
- For example $\kappa(G) \leq \delta(G)$.

Connectivity of the Grid



Connectivity of the Grid



Sometimes, it is desirable to find a set $X \subseteq V(G)$ such that deleting X doesn't just separate a small number of vertices from the rest of the graph.

Separators

For a fixed $p \in [\frac{2}{3}, 1)$, a *p-separator* X is a set of vertices such that no component C of $G - X$ contains more than $p|G - X|$ vertices.

- Since $p \geq \frac{2}{3}$, this is equivalent to saying that $G - X$ can be partitioned into two parts A, B with no edge between them and $|A|, |B| \leq p|G - X|$.

Separators

- Graph separators are of independent interest.
- Applications to dynamic programming.

The Largest Multipartite Subgraph of $G(n, k)$

Recall our question:

Question

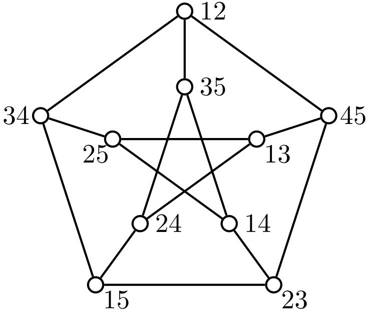
Say $p \in [\frac{2}{3}, 1)$. If H is a complete multipartite graph, a subgraph of $G(n, k)$, and no colour class of H contains more than $p|H|$ vertices, how large can $|H|$ be?

This is equivalent to finding a (small) p-separator in the complement of $G(n, k)$.

The Kneser Graph

- The complement of $G(n, k)$ is the *Kneser Graph* $\mathrm{Kn}(n, k)$.
- Each vertex is a k-set; two k-sets are adjacent if they do *not* intersect.
- Kneser graphs are of independent interest.
- $\chi(\mathrm{Kn}(n, k)) = n - 2k + 2$.
- Famously, the Petersen graph is $\mathrm{Kn}(5, 2)$.

The Petersen Graph as a Kneser Graph



Key Result

Result

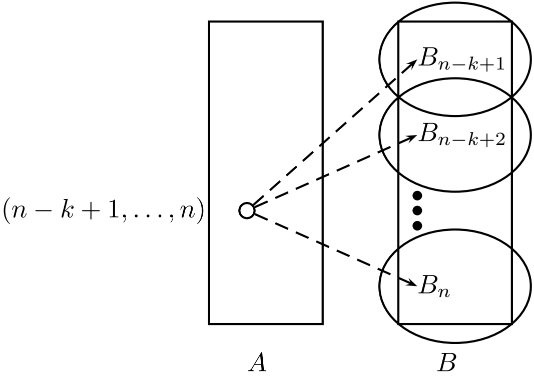
Say n is sufficiently large with respect to p and k.
If X is a p-separator of $\mathrm{Kn}(n, k)$ then $|X| \geq \binom{n-1}{k}$.

This means that $|H| \leq \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$.

Basic Sketch of Proof

- Assume for the sake of a contradiction that $|X| < \binom{n-1}{k}$, and say $G - X$ is partitioned into A, B.
- Then $|A \cup B| > \binom{n-1}{k-1}$.
- Let A_i denote the subset of A using element i, and A_{-i} denote the subset of A not using element i.
- The proof follows mainly by "iteration".

Basic Sketch of Proof



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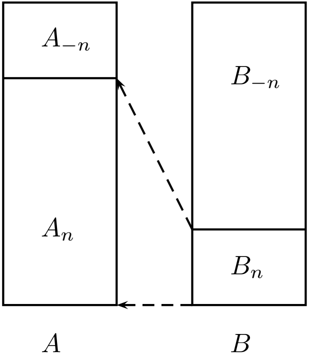

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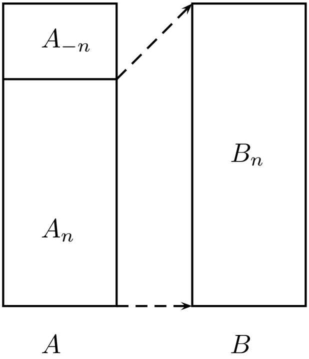

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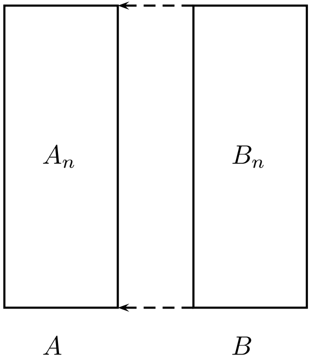


The Largest Multipartite Subgraph of $G(n, k)$

Question

Say $p \in [\frac{2}{3}, 1)$. If H is a complete multipartite graph, a subgraph of $G(n, k)$, and no colour class of H contains more than $p|H|$ vertices, how large can $|H|$ be?

Answer

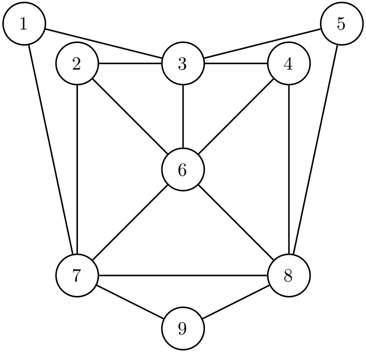
Assuming n is large, $|H| \leq \binom{n-1}{k-1}$.

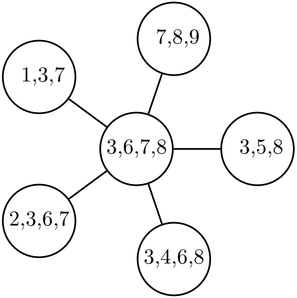
The separator result has more applications, however.

Tree Decompositions

A *tree decomposition* of a graph G is:

- a tree T with
- a *bag* of vertices of G for each node of T...

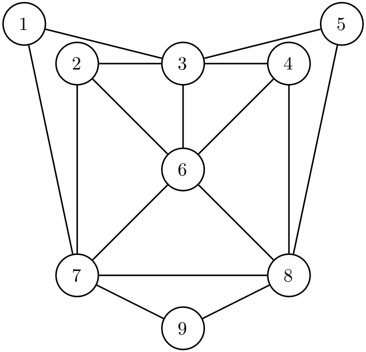


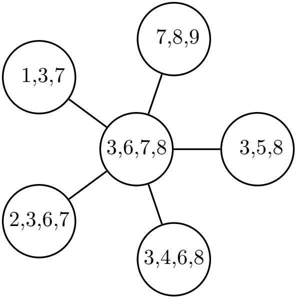


Tree Decompositions

...such that

1. each $v \in V(G)$ is in at least one bag,
2. for each $v \in V(G)$, the bags containing v form a connected subtree of T,
3. for each $uv \in E(G)$, there is a bag containing u and v.





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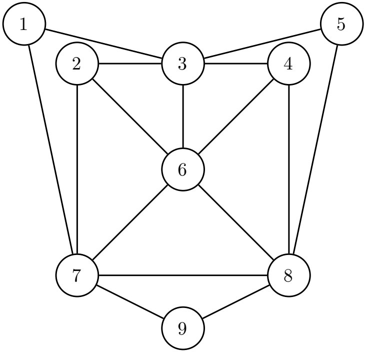

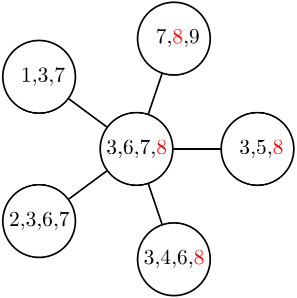

Tree Decompositions

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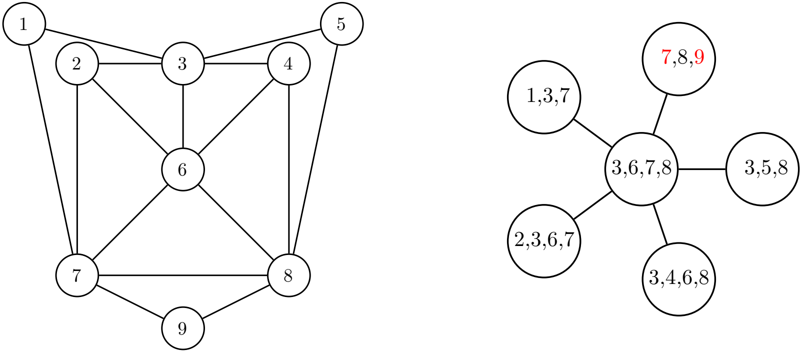
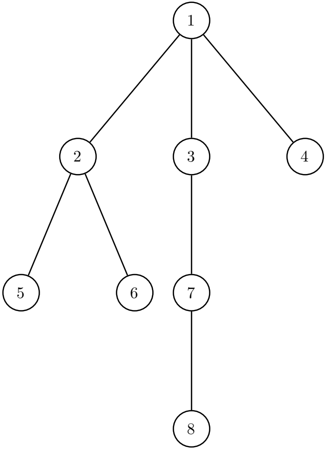


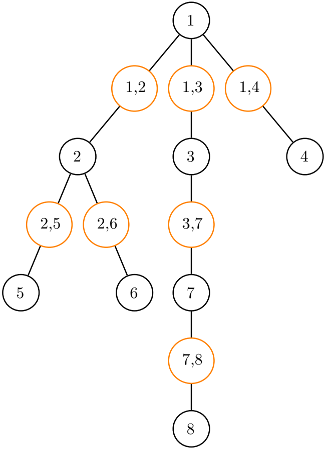
Treewidth

- The *width* of a tree decomposition is the size of its largest bag, minus 1.
- The *treewidth* $\text{tw}(G)$ is the minimum width over all tree decompositions.

Example Tree Decompositions

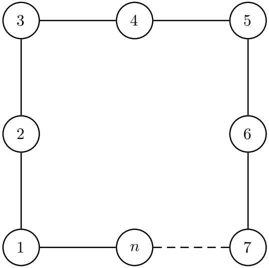
- tw(G) $= 1$ iff G is a forest.

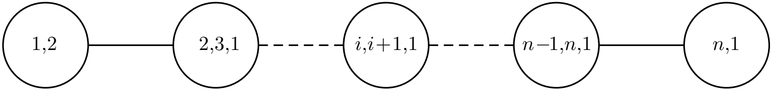




Example Tree Decompositions

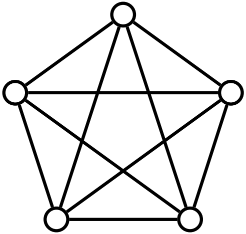
- If G is a cycle, $\text{tw}(G) = 2$.

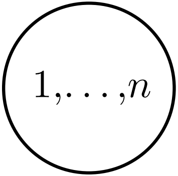




Example Tree Decompositions

- $\text{tw}(G) = |V(G)| - 1$ iff G is a complete graph.





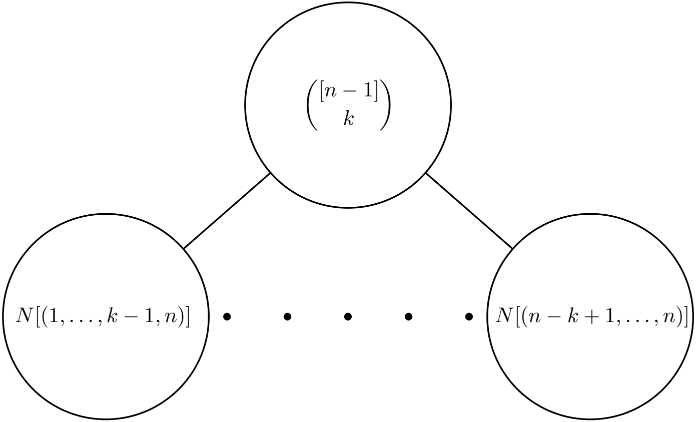
Treewidth

- Treewidth is core to the important Graph Minor Theorem by Robertson and Seymour.
  - Specifically, the Graph Minor Structure Theorem, which essentially defines how to construct any graph with no H-minor, for a fixed graph H.
- Treewidth also has algorithmic applications; certain NP-Hard problems can be solved in polynomial time on graphs with bounded treewidth.

Treewidth and Separators

- G has a $\frac{2}{3}$-separator of order $\text{tw}(G) + 1$.
- Hence $\text{tw}(\text{Kn}(n, k)) \geq \binom{n-1}{k} - 1$, when n is sufficiently large.

Treewidth of the Kneser Graph



- Thus $\mathrm{tw}(\mathrm{Kn}(n,k)) \leq \binom{n-1}{k} - 1$.

Treewidth of the Kneser Graph

- Hence $\mathrm{tw}(\mathrm{Kn}(n,k)) = \binom{n-1}{k} - 1$ when n is sufficiently large.
- Specifically, $n \geq 4k^2 - 4k + 3$.

Open Questions

- Obviously, the goal would be to improve the lower bound on n.
- If $n < 3k - 1$, can prove that $\text{tw}(\text{Kn}(n,k)) < \binom{n-1}{k} - 1$.
- This suggests the following conjecture:

Not Actually a Conjecture

$\text{tw}(\text{Kn}(n,k)) = \binom{n-1}{k} - 1$ *when* $n \geq 3k - 1$ *and* $k \geq 2$.

However, this isn't quite true...

Open Questions

- tw(Kn(n, 2)) is completely determined;
  tw(Kn(n, 2)) $= \binom{n-1}{2} - 1$ when $n \geq 6 = 3k$.
- Perhaps, conjecture the following:

Conjecture

tw(Kn(n, k)) $= \binom{n-1}{k} - 1$ *when* $n \geq 3k$ *and* $k \geq 2$.

Open Questions

- $\mathrm{tw}(\mathrm{Kn}(n,2))$ is completely determined; $\mathrm{tw}(\mathrm{Kn}(n,2)) = \binom{n-1}{2} - 1$ when $n \geq 6 = 3k$.
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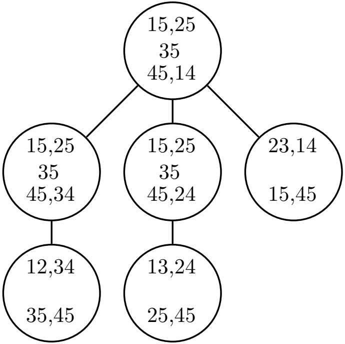
Conjecture

$\mathrm{tw}(\mathrm{Kn}(n,k)) = \binom{n-1}{k} - 1$ *when* $n \geq 3k$ *and* $k \geq 2$*.*

- However, if $k = 2$ and $n = 3k - 1 = 5$, then $\mathrm{Kn}(n,k)$ is the Petersen graph.

Open Questions



Conjecture

$\mathrm{tw}(\mathrm{Kn}(n,k)) = \binom{n-1}{k} - 1$ *when* $n \geq 3k - 1$ *and* $k \geq 2$*; except for the Petersen graph.*