The Extremal Function for Petersen Minors

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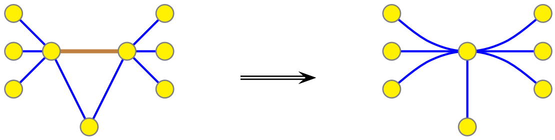
Graph Minors

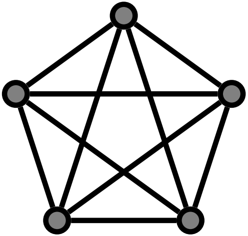
Operations:

1. vertex deletions
2. edge deletions
3. edge contractions

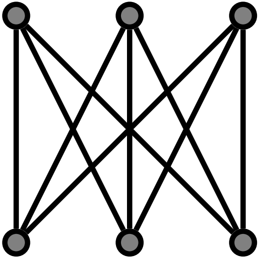


Kuratowski's/Wagner's Theorem

A graph is planar iff it no K_5-minor and no $K_{3,3}$-minor.



K_5



$K_{3,3}$

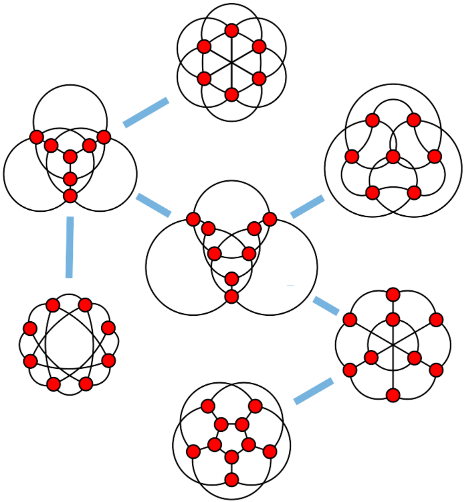
Graph Minor Theorem [Robertson-Seymour]

Every minor closed class can be characterised by a finite set of excluded minors.

Linkless Graphs

Graphs that can be embedded in $\mathbb{R}^3$ such that no two cycles are linked.

Characterisation of Linkless graphs [Robertson, Seymour, Thomas]



Extremal Function

Excluded Minor	Maximum # edges	
K_3	$n-1$	forests
K_4	$2n-3$	[Dirac 1964]
K_5	$3n-6$	[Dirac 1964]
K_6	$4n-10$	[Mader 1968]
K_7	$5n-15$	[Mader 1968]
K_8	$6n-20$	[Jørgensen 1994]
K_9	$7n-27$	[Song, Thomas 2006]
K_t	$\Theta(t\sqrt{\log t})n$	[de la Vega 1983] [Kostochka 1982, 1984] [Thomason 1984, 2001]

Extremal Function

Excluded Minor(s)	Maximum # edges	
K_5 and $K_{3,3}$	$3n-6$	planar
$K_{3,3}$	$3n-5$	[Hall 1943]
Petersen Family	$4n-10$	[Mader68]
$K_{2,2,2}$	(7n-15)/2	[Ding 2013]
$K_{2,t}$	$(t+1)(n-1)/2$	[Chudnovsky,Reed,Seymour 2011]
K_8^-	$(11n-35)/2$	[Song 2005]

Our Main Result

Every graph with $n \geq 2$ vertices and at least $5n - 8$ edges contains a Petersen minor.

Why this is best possible

$(K_9, 2)$-cockades have $5n - 9$ edges, are Petersen minor free.

Petersen Minors

- Tutte's conjecture: Every bridgeless Petersen minor free graph admits a nowhere 0 4-flow.
- Every cubic bridgeless Petersen minor free graph is edge 3-colourable [ERSST].
- A graph has the circuit cover property iff it is Petersen minor free [Alspach, Goddyn, Zhang 1994].

Let G be a minor minimal counterexample

i) G has no Petersen minor

ii) $|E(G)| = 5n - 8$

iii) No minor of G satisfies (ii)

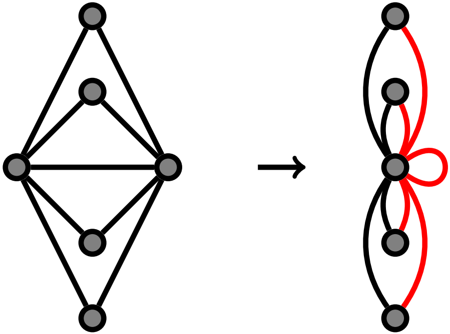
Minimum degree

- minimum degree vertices can be deleted if $\delta(G)$ is small.
- edges can be deleted if $\delta(G)$ is big.

$$6 \leq \delta(G) \leq 9$$

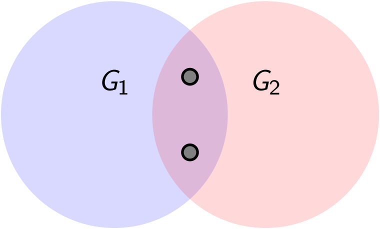
triangles

Every edge is in at least 5 triangles.



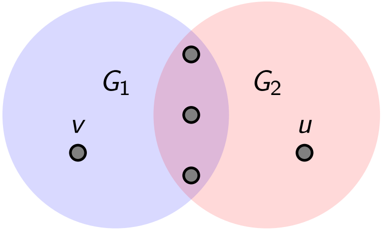
Connectivity

- G is 3-connected.



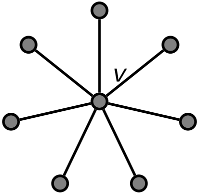
Connectivity

- G is 3-connected.
- There is some small degree vertex on either side of any 3-cut.

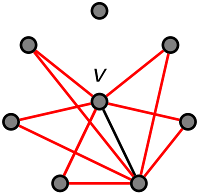


MASSIVE ASSUMPTION!

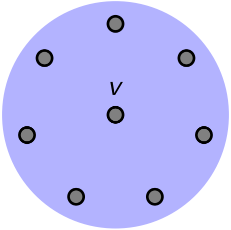
All small degree vertices have degree 7.



Each edge is in 5 triangles

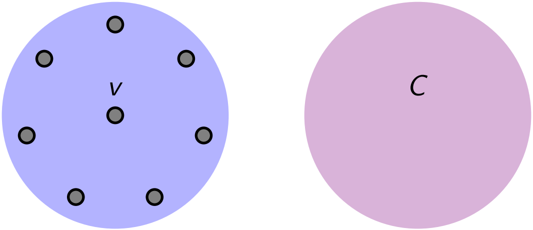


Small degree vertices have dense neighbourhoods

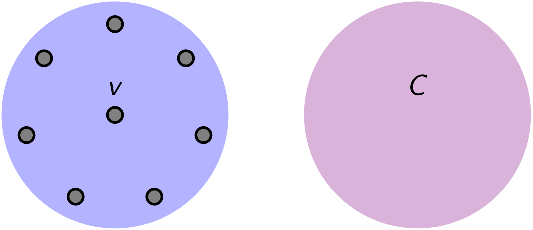


K_8 minus a matching of size at most 3.

Pick v and C to minimize $|V(C)|$



Where are we going with this?



- Pick v and C to minimize $|V(C)|$

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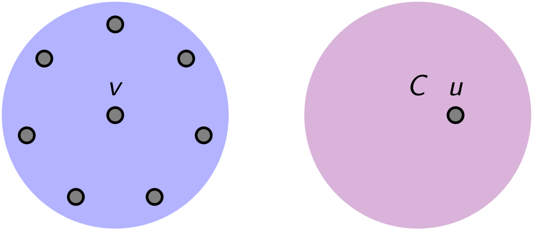
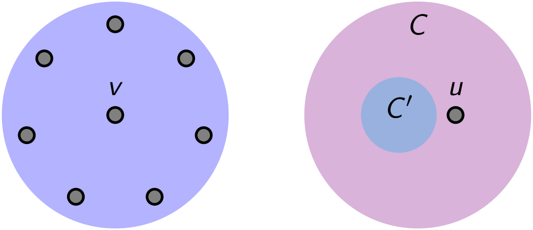


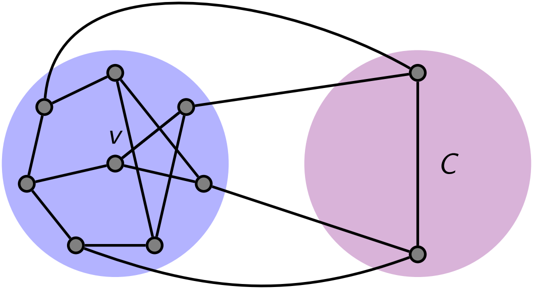
- Pick v and C to minimize $|V(C)|$
- find a small degree vertex u in C

Where are we going with this?

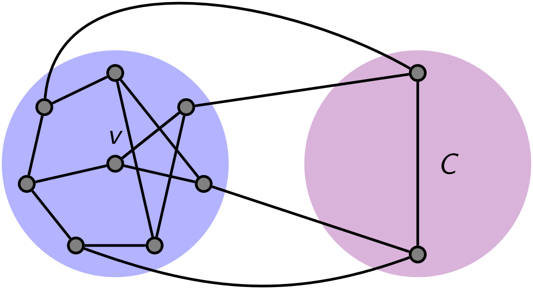


- Pick v and C to minimize $|V(C)|$
- find a small degree vertex u in C
- find a component C' of G minus the neighbours of u inside C

Look for a Petersen minor

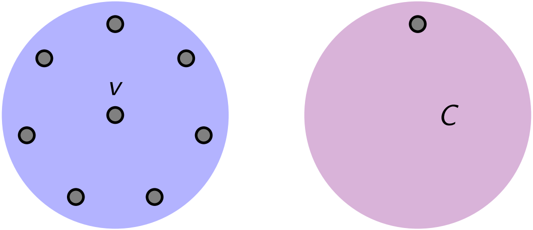


Look for a Petersen minor

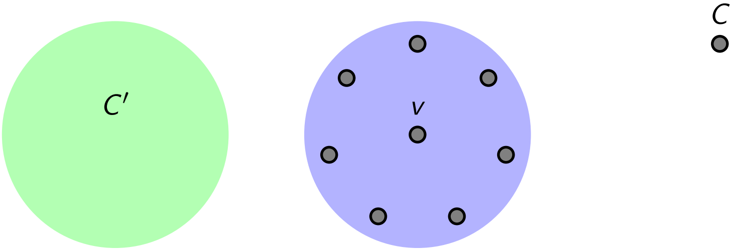


Occurs when $|V(C)| \geq 2$ and $|N(C)| \geq 4$

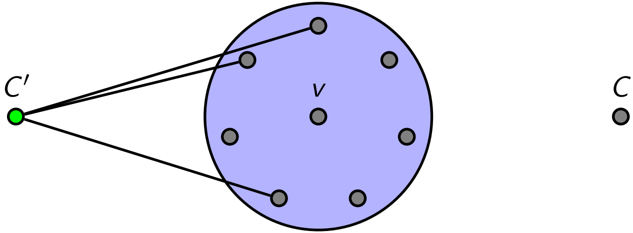
What if $|V(C)| = 1$?



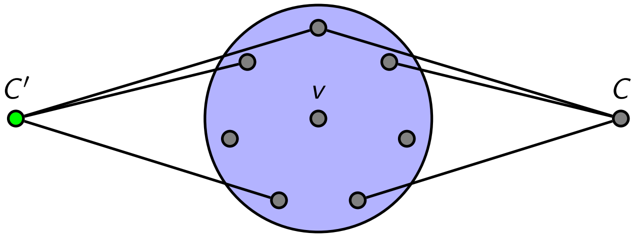
G has more than 9 vertices.



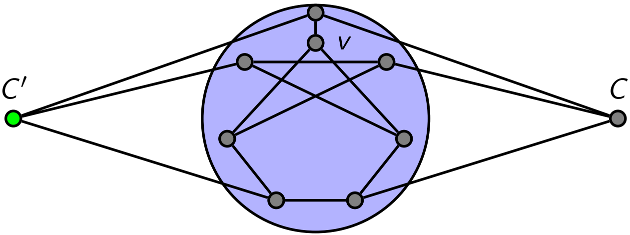
G is 3-connected



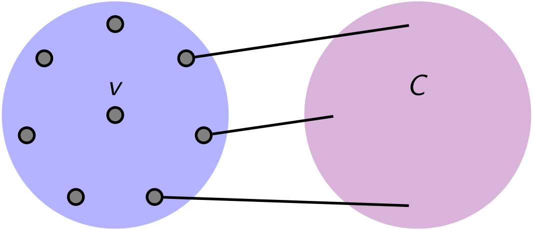
$\delta(G) \geq 6$



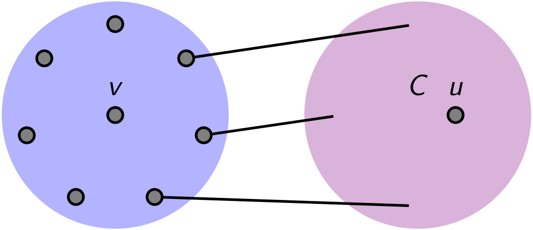
We can find a Petersen minor



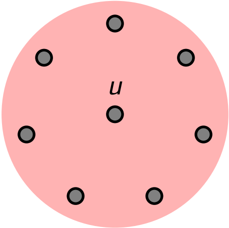
Each component has exactly 3 neighbours



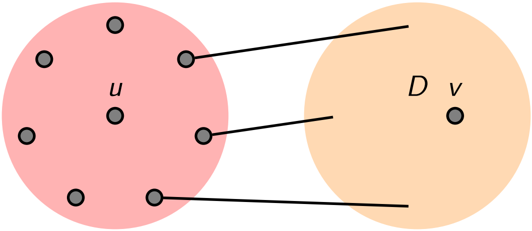
There is a small degree vertex on either side of each 3-cut



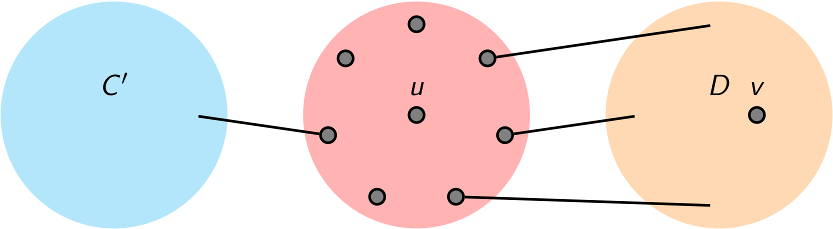
u has degree 7 by our assumption

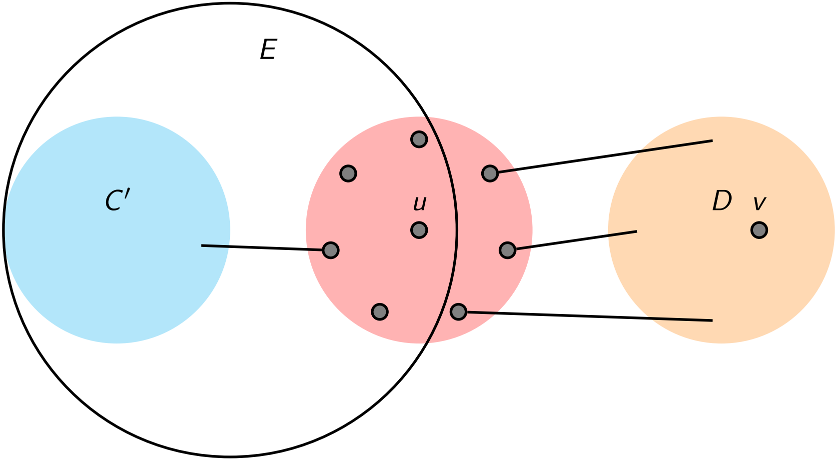


Where is v?



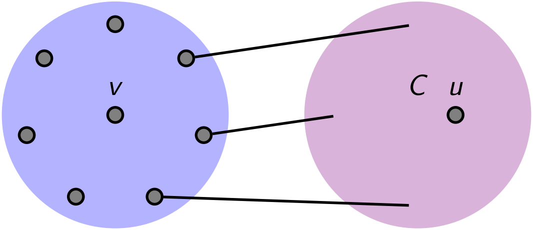
Finding C'





E is connected.
E contains no neighbour of v.
E contains u.

Where is E?



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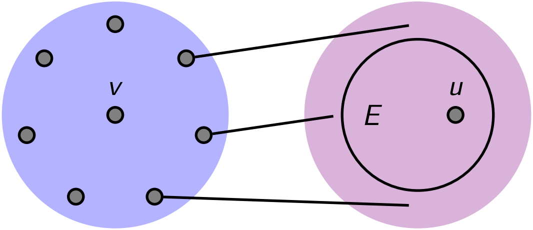

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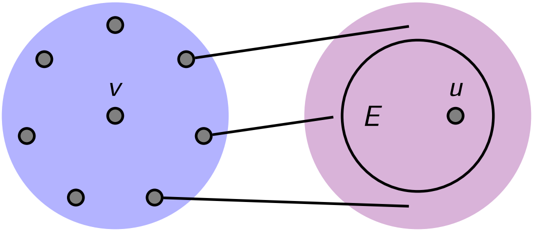
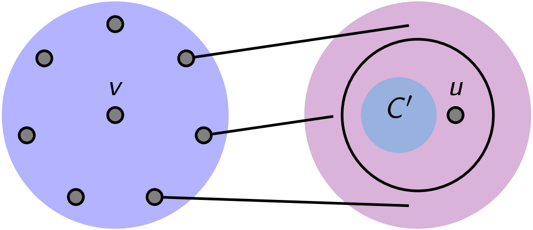
Where is C'?



Where is C'?



C' is in C.

Application to colouring

Every Petersen minor free graph is 9-colourable.
This is best possible.

Further Questions

What if we increase connectivity?

- 3-connected Petersen minor free graphs can have $5n - 12$ edges.
- 5-connected Petersen Minor free graphs can have $5n - 15$ edges.
- 6-connected Petersen Minor free graphs can have $4n - 10$ edges (apex graphs).
- We know of no ≥ 10-vertex 7-connected Petersen minor free graph.