The Extremal Function for Petersen Minors

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Graph Minors

Operations:

1. vertex deletions
2. edge deletions
3. edge contractions
Kuratowski’s/Wagner’s Theorem

A graph is planar iff it no $K_5$-minor and no $K_{3,3}$-minor.
Every minor closed class can be characterised by a finite set of excluded minors.
Linkless Graphs

Graphs that can be embedded in $\mathbb{R}^3$ such that no two cycles are linked.
Characterisation of Linkless graphs [Robertson, Seymour, Thomas]
# Extremal Function

<table>
<thead>
<tr>
<th>Excluded Minor</th>
<th>Maximum # edges</th>
<th>forests</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_3$</td>
<td>$n - 1$</td>
<td>forests</td>
</tr>
<tr>
<td>$K_4$</td>
<td>$2n - 3$</td>
<td>[Dirac 1964]</td>
</tr>
<tr>
<td>$K_5$</td>
<td>$3n - 6$</td>
<td>[Dirac 1964]</td>
</tr>
<tr>
<td>$K_6$</td>
<td>$4n - 10$</td>
<td>[Mader 1968]</td>
</tr>
<tr>
<td>$K_7$</td>
<td>$5n - 15$</td>
<td>[Mader 1968]</td>
</tr>
<tr>
<td>$K_8$</td>
<td>$6n - 20$</td>
<td>[Jørgensen 1994]</td>
</tr>
<tr>
<td>$K_9$</td>
<td>$7n - 27$</td>
<td>[Song, Thomas 2006]</td>
</tr>
<tr>
<td>$K_t$</td>
<td>$\Theta(t \sqrt{\log t})n$</td>
<td>[de la Vega 1983, Kostochka 1982, 1984, Thomason 1984, 2001]</td>
</tr>
</tbody>
</table>
## Extremal Function

<table>
<thead>
<tr>
<th>Excluded Minor(s)</th>
<th>Maximum # edges</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_5$ and $K_{3,3}$</td>
<td>$3n - 6$</td>
<td>planar</td>
</tr>
<tr>
<td>$K_{3,3}$</td>
<td>$3n - 5$</td>
<td>[Hall 1943]</td>
</tr>
<tr>
<td>Petersen Family</td>
<td>$4n - 10$</td>
<td>[Mader 68]</td>
</tr>
<tr>
<td>$K_{2,2,2}$</td>
<td>$(7n-15)/2$</td>
<td>[Ding 2013]</td>
</tr>
<tr>
<td>$K_{2,t}$</td>
<td>$(t + 1)(n - 1)/2$</td>
<td>[Chudnovsky, Reed, Seymour 2011]</td>
</tr>
<tr>
<td>$K_8$</td>
<td>$(11n - 35)/2$</td>
<td>[Song 2005]</td>
</tr>
</tbody>
</table>
Our Main Result

Every graph with $n \geq 2$ vertices and at least $5n - 8$ edges contains a Petersen minor.
Why this is best possible

\((K_9, 2)\)-cockades have \(5n - 9\) edges, are Petersen minor free.
Petersen Minors

- Tutte’s conjecture: Every bridgeless Petersen minor free graph admits a nowhere 0 4-flow.
- Every cubic bridgeless Petersen minor free graph is edge 3-colourable [ERSST].
- A graph has the circuit cover property iff it is Petersen minor free [Alspach, Goddyn, Zhang 1994].
Let $G$ be a minor minimal counterexample

i) $G$ has no Petersen minor

ii) $|E(G)| = 5n - 8$

iii) No minor of $G$ satisfies (ii)
Minimum degree

- minimum degree vertices can be deleted if \( \delta(G) \) is small.
- edges can be deleted if \( \delta(G) \) is big.

\[ 6 \leq \delta(G) \leq 9 \]
Every edge is in at least 5 triangles.
Connectivity

- $G$ is 3-connected.
Connectivity

- $G$ is 3-connected.
- There is some small degree vertex on either side of any 3-cut.
MASSIVE ASSUMPTION!

All small degree vertices have degree 7.
Each edge is in 5 triangles
Small degree vertices have dense neighbourhoods

$K_8$ minus a matching of size at most 3.
Pick \( v \) and \( C \) to minimize \( |V(C)| \)
Where are we going with this?

- Pick $v$ and $C$ to minimize $|V(C)|$
Where are we going with this?

- Pick $v$ and $C$ to minimize $|V(C)|$
- find a small degree vertex $u$ in $C$
Where are we going with this?

- Pick \( v \) and \( C \) to minimize \( |V(C)| \)
- find a small degree vertex \( u \) in \( C \)
- find a component \( C' \) of \( G \) minus the neighbours of \( u \) inside \( C \)
Look for a Petersen minor
Look for a Petersen minor

Occurs when $|V(C)| \geq 2$ and $|N(C)| \geq 4$
What if $|V(C)| = 1$?
$G$ has more than 9 vertices.
$G$ is 3-connected
\(\delta(G) \geq 6\)
We can find a Petersen minor.
Each component has exactly 3 neighbours
There is a small degree vertex on either side of each 3-cut.
$u$ has degree 7 by our assumption
Where is $v$?
Finding $C'$

$C'$

$u$

$D$ $v$
$E$ is connected.
$E$ contains no neighbour of $v$.
$E$ contains $u$. 
Where is $E$?

$E$ contains no neighbour of $v$.
$E$ contains $u$.

$E$ is connected.
Where is $E$?

$E$ is connected.
$E$ contains no neighbour of $v$.
$E$ contains $u$. 
Where is $C'$?
Where is $C'$?

$C'$ is in $C$.
Every Petersen minor free graph is 9-colourable. This is best possible.
Further Questions

What if we increase connectivity?

- 3-connected Petersen minor free graphs can have $5n - 12$ edges.
- 5-connected Petersen Minor free graphs can have $5n - 15$ edges.
- 6-connected Petersen Minor free graphs can have $4n - 10$ edges (apex graphs).
- We know of no $\geq 10$-vertex 7-connected Petersen minor free graph.