Hamilton Decompositions of Infinite Circulant Graphs

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joint work with
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Group $\mathcal{G}$ with identity e and $S \subseteq \mathcal{G} - \{e\}$, inverse-closed

The Cayley graph on the group $\mathcal{G}$ with connection set S, denoted $\mathrm{Cay}(\mathcal{G}, S)$, is the undirected simple graph where

- the vertices are the elements of $\mathcal{G}$ and
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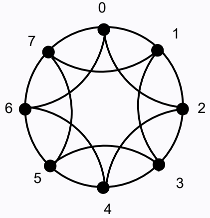
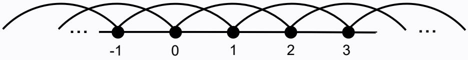


$\mathrm{Cay}(\mathbb{Z}_8, \{1, 2\})$



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Theorem (Bryant, Dean 2015)

There exist $2k$-regular connected Cayley graphs on finite NON-abelian groups that are NOT Hamilton-decomposable.

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Theorem (Bryant, S.H., Maenhaut, Webb)

Every connected Cayley graph on finitely-generated infinite abelian group of infinite valency is Hamilton-decomposable.

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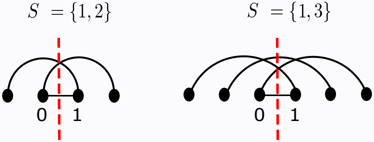

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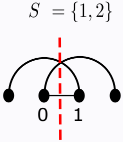

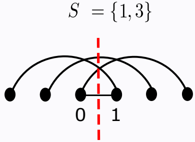

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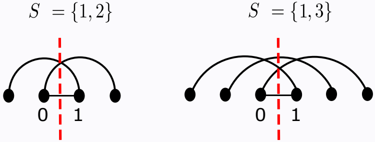

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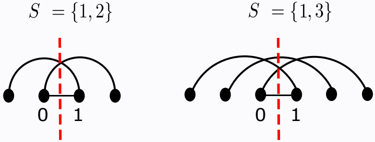

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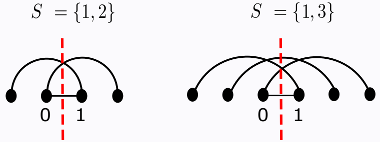

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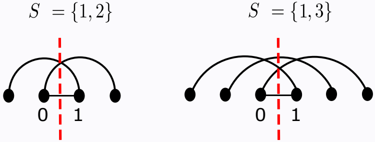

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Is every admissible infinite circulant graph Hamilton-decomposable?

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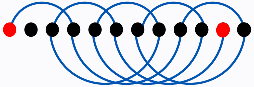

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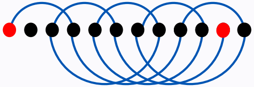

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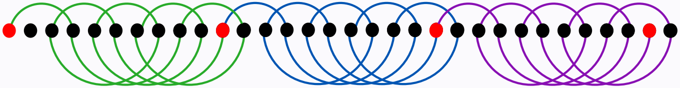

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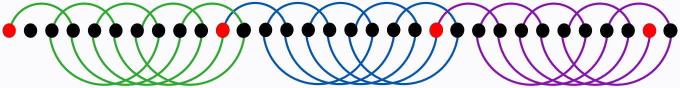

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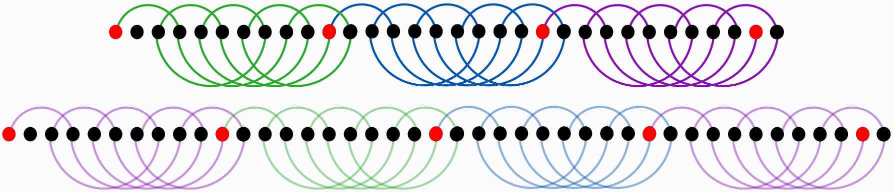

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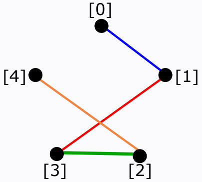

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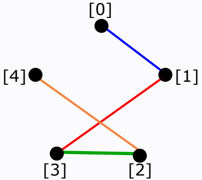

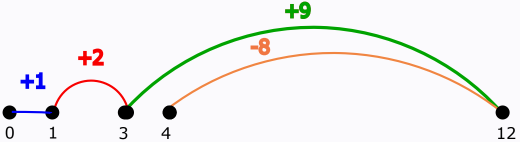

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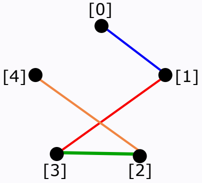

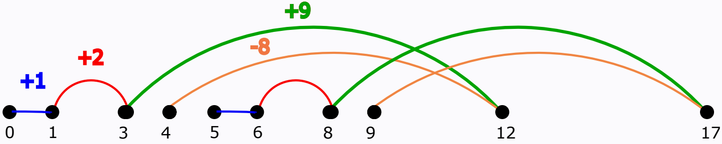

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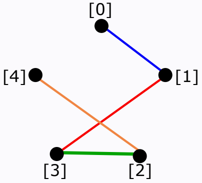

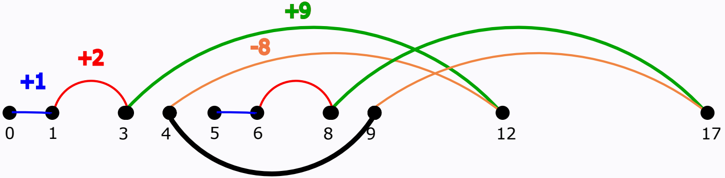

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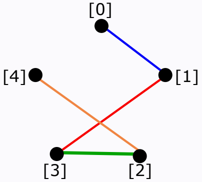

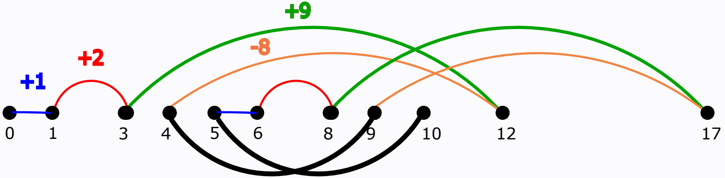

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Buratti's Conjecture (2007)

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No Hamilton path in K_9 with the following multisets for edge lengths:

$$\{1,3,3,3,3,3,3,3\} \qquad \{2,3,3,3,3,3,3,3\}$$

$$\{3,3,3,3,3,3,3,3\} \qquad \{3,3,3,3,3,3,3,4\}$$

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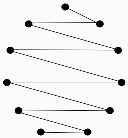

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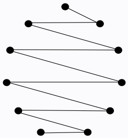

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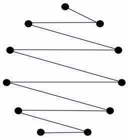

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6-Regular Infinite Circulants

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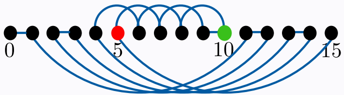

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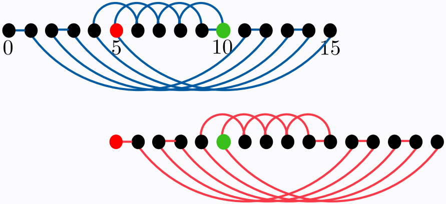

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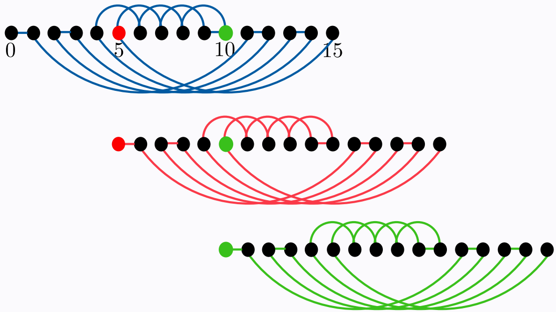

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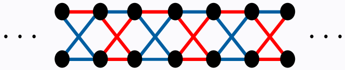

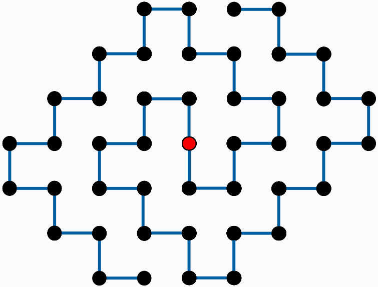

Ben Reiniger

Thanks!