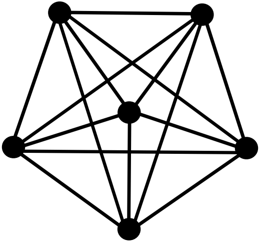
UNIFORM 1-FACTORISATIONS OF CIRCULANT GRAPHS

Sarada Herke

Dr. Barbara Maenhaut
The University of Queensland

March 2014

Consider K_6:



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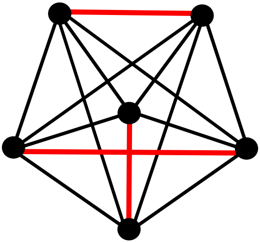

- A 1-factor of a graph G is a spanning 1-regular subgraph of G.

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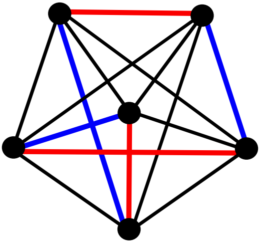

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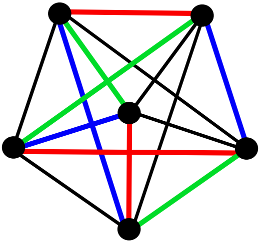

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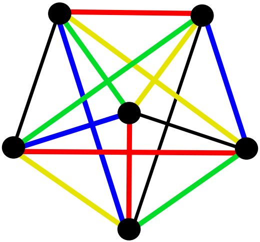

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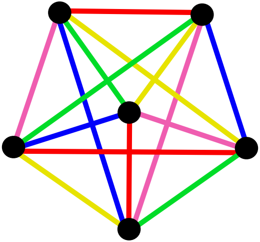

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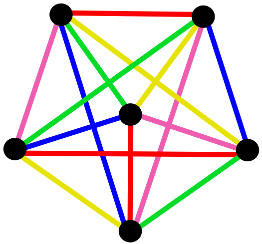

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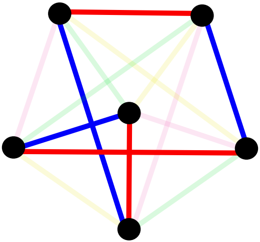

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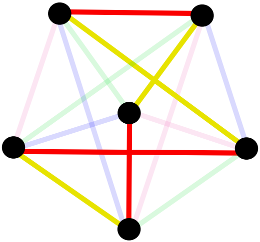

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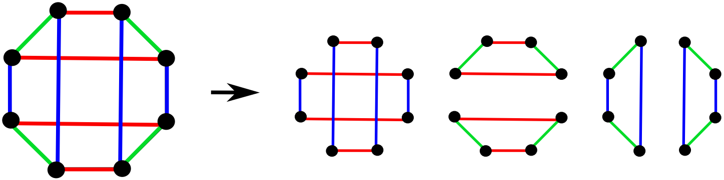

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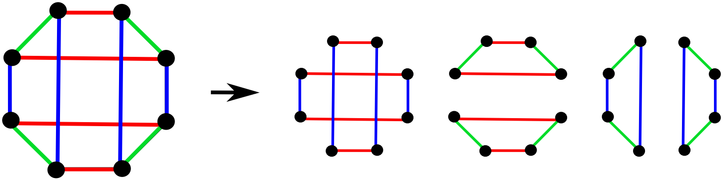

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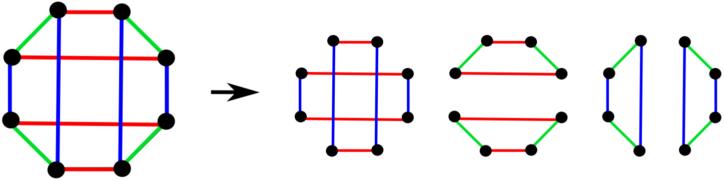

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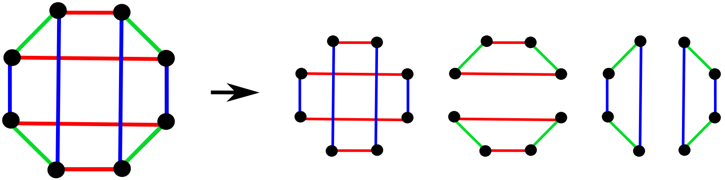

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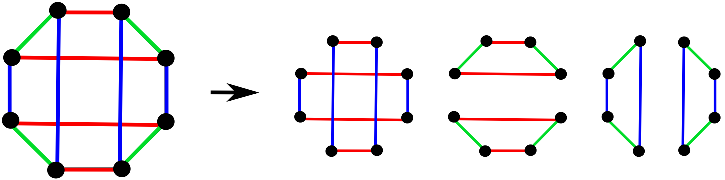

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- A P1F is a U1F of type (n).

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The complete graph K_{2n} admits a P1F for all $n \geq 2$.

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THEOREM (MESZKA AND ROSA, 2003)

A U1F of the complete graph K_{2n} *where* $2n \leq 16$ *is one of the following:*
a) a P1F
b) a U1F of K_8 *of type* $(4, 4)$
c) a U1F of K_{10} *of type* $(4, 6)$
d) a U1F of K_{12} *of type* $(6, 6)$
e) a U1F of K_{16} *of type* $(4, 4, 4, 4)$

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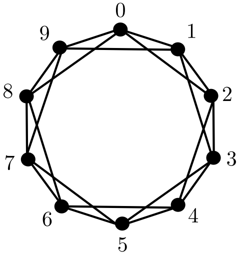
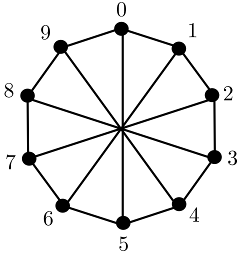
Example: $\mathrm{Circ}(10, \{1, 2\})$ and $\mathrm{Circ}(10, \{1, 5\})$





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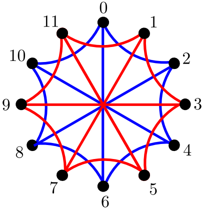

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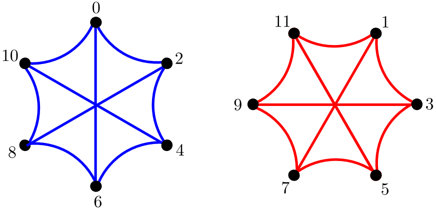

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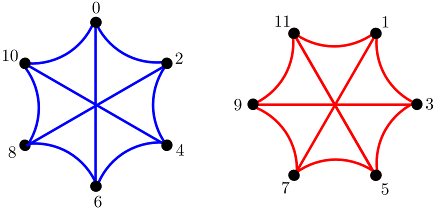

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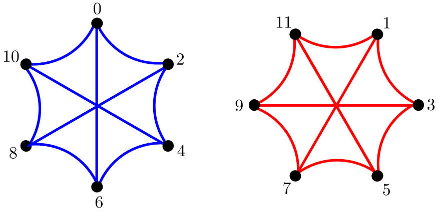

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In fact, $\mathrm{Circ}(n, \{a, b\})$ is connected $\iff$ $\gcd(a, b, n) = 1$.

P1F THEOREM (S.H. AND MAENHAUT)

If $n > 6$, then a connected 3-regular circulant graph G of order n admits a P1F $\iff$ $n \equiv 2 \pmod{4}$ and $G \cong \mathrm{Circ}(n, \{1, \frac{n}{2}\})$.

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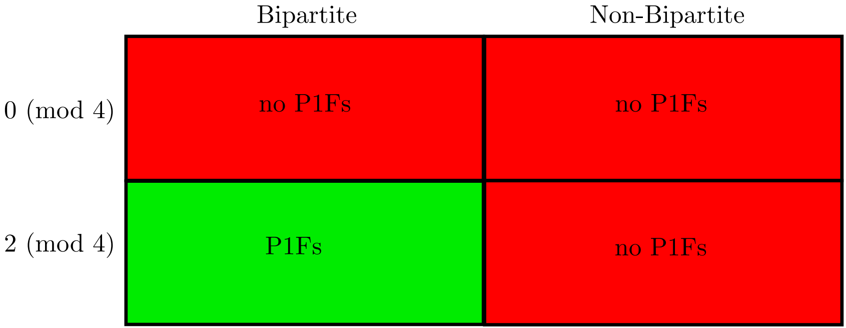

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0 (mod 4)	no P1Fs	no P1Fs
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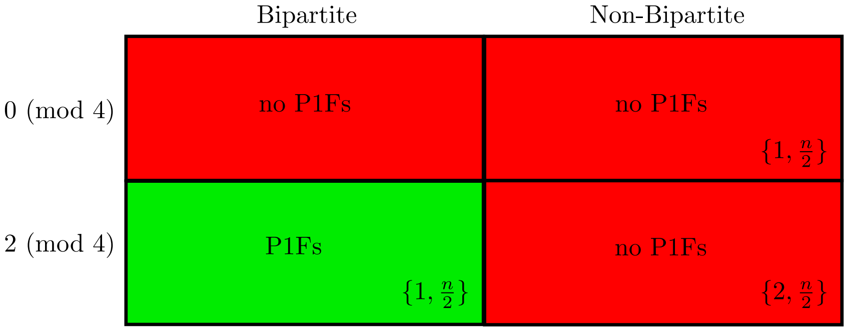

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$\mathrm{Circ}(n, \{a, b\})$ is bipartite $\iff$ a and b are both odd.

n	non-isomorphic connection sets	types of U1Fs
8	$\{1,4\}$	none
10	$\{1,5\}$	(10)
	$\{2,5\}$	none
12	$\{1,6\}$	$(6,6)$
14	$\{1,7\}$	(14)
	$\{2,7\}$	none
16	$\{1,8\}$	none
18	$\{1,9\}$	$(4,4,10)$, $(6,6,6)$ and (18)
	$\{2,9\}$	$(4,4,10)$ and $(6,6,6)$
20	$\{1,10\}$	none
22	$\{1,11\}$	(22)
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24	$\{1,12\}$	$(4,4,6,10)$ and $(6,6,6,6)$
26	$\{1,13\}$	(26)
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28	$\{1,14\}$	none
30	$\{1,15\}$	$(4,4,4,4,14), (4,4,6,6,10),$ $(4,4,6,8,8), (6,6,6,6,6)$ and (30)
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24	$\{1, 12\}$	$(4, 4, 6, 10)$ and $(6, 6, 6, 6)$
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🟪 P1F

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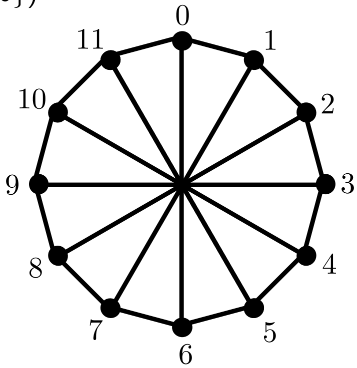

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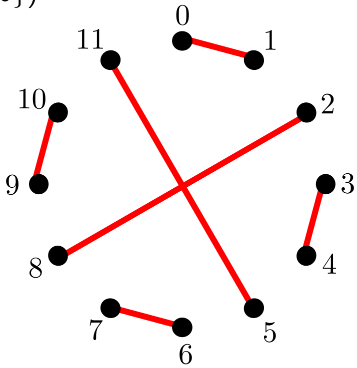

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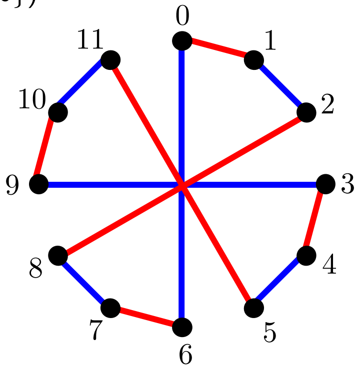

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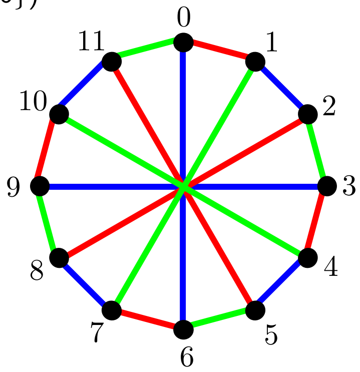

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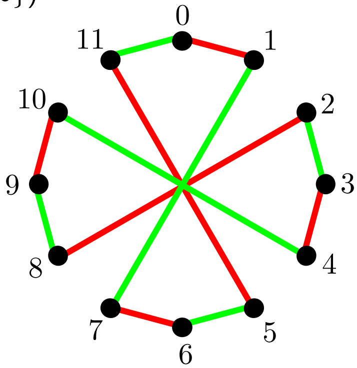

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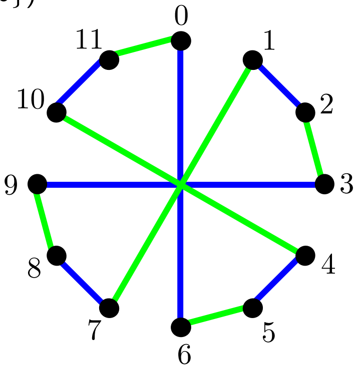

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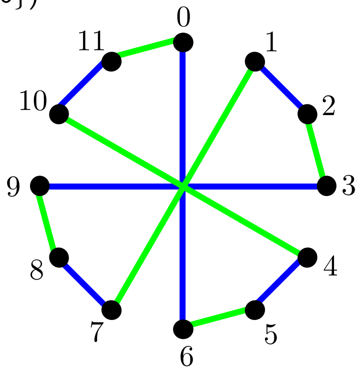

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$n \equiv 6 \pmod{12} \implies \mathrm{Circ}(n, \{2, \frac{n}{2}\})$ *has a type* $(6, 6, ..., 6)$ *U1F.*

n	non-isomorphic connection sets	types of U1Fs
8	$\{1,4\}$	none
10	$\{1,5\}$	(10)
	$\{2,5\}$	none
12	$\{1,6\}$	$(6,6)$
14	$\{1,7\}$	(14)
	$\{2,7\}$	none
16	$\{1,8\}$	none
18	$\{1,9\}$	$(4,4,10)$, $(6,6,6)$ and (18)
	$\{2,9\}$	$(4,4,10)$ and $(6,6,6)$
20	$\{1,10\}$	none
22	$\{1,11\}$	(22)
	$\{2,11\}$	none
24	$\{1,12\}$	$(4,4,6,10)$ and $(6,6,6,6)$
26	$\{1,13\}$	(26)
	$\{2,13\}$	none
28	$\{1,14\}$	none
30	$\{1,15\}$	$(4,4,4,4,14), (4,4,6,6,10),$ $(4,4,6,8,8), (6,6,6,6,6)$ and (30)
	$\{2,15\}$	$(4,4,4,4,14), (4,4,6,6,10),$ $(4,4,6,8,8)$ and $(6,6,6,6,6)$

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	$\{2, 9\}$	$(4, 4, 10)$ and $(6, 6, 6)$
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24	$\{1, 12\}$	$(4, 4, 6, 10)$ and $(6, 6, 6, 6)$
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28	$\{1, 14\}$	none
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THEOREM (S.H. AND MAENHAUT, 2013)

Suppose $n > 6$ is even. A connected circulant graph $\mathrm{Circ}(n, \{2, \frac{n}{2}\})$ admits a U1F if and only if $n \equiv 6 \pmod{12}$.

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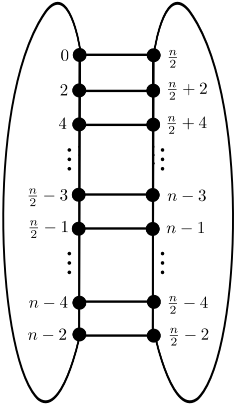
By the P1F Theorem, the U1F is not a P1F.

$$\begin{aligned} G \text{ connected } &\implies n \equiv 2 \pmod{4} \\ &\implies \text{the length 2 edges form 2 disjoint } \tfrac{n}{2}\text{−cycles} \end{aligned}$$

Call the length $\frac{n}{2}$ edges cross edges.



a = number of cross edges in belonging to A
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$$a+b+c=\frac{n}{2}$$

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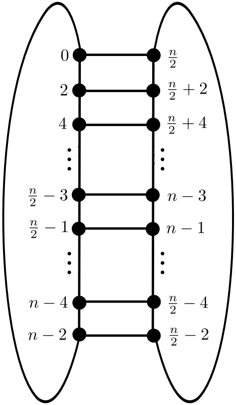

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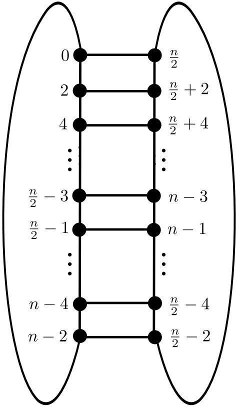
Aim to show $a=b=c$

If edges $\{0, 2\}$ and $\{\frac{n}{2}, \frac{n}{2} + 2\}$ belong to different 1-factors...



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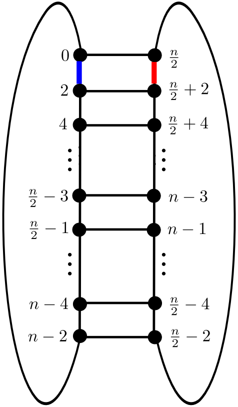

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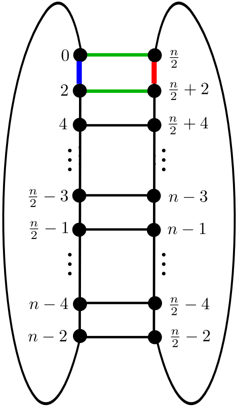

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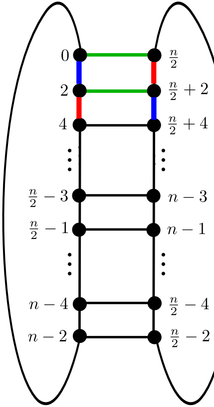

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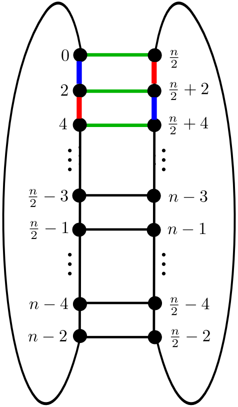

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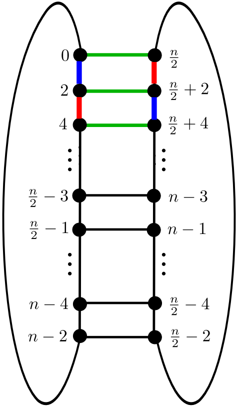

$(\Rightarrow\Leftarrow)$

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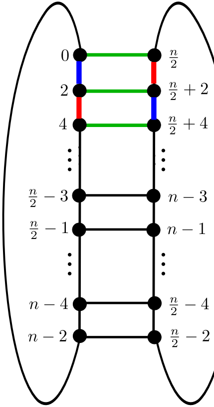
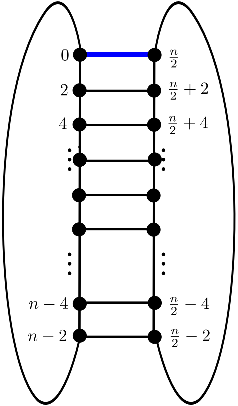


$(\Rightarrow\Leftarrow)$

Similarly, $\{i, i + 2\}$ and $\{i + \frac{n}{2}, i + \frac{n}{2} + 2\}$ belong to the same 1-factor.

Consider $A \cup B$.



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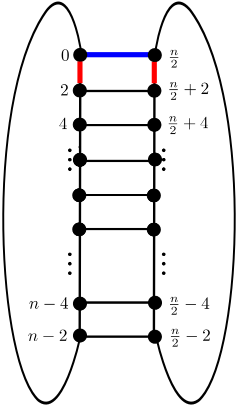

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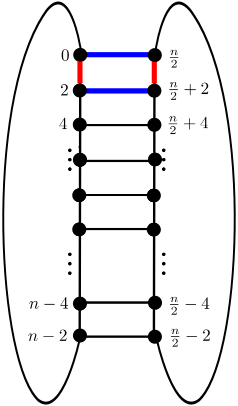

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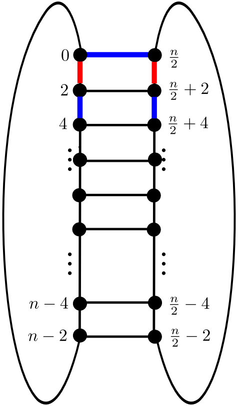

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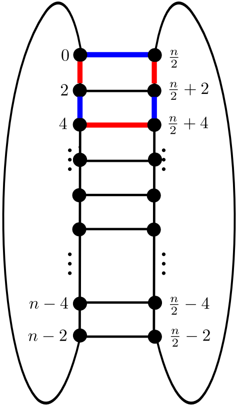

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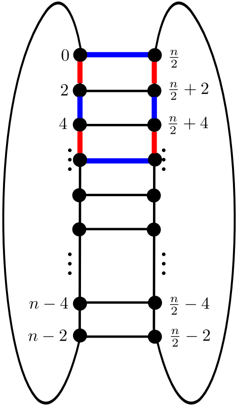

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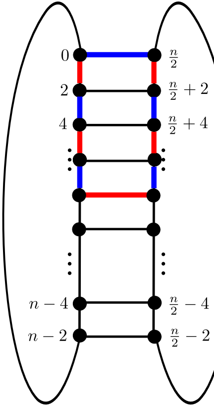

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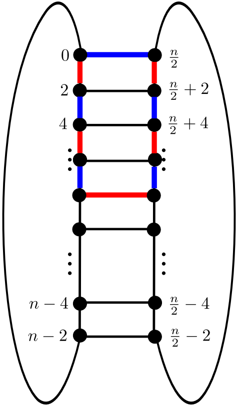

Each possible cycle uses exactly 2 cross edges

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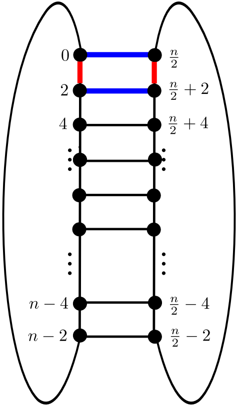

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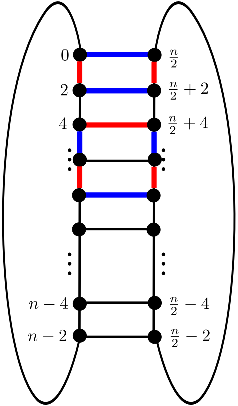

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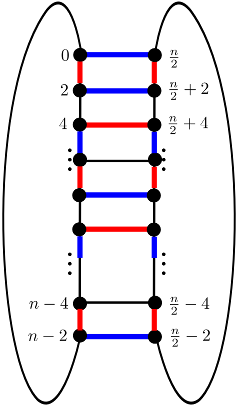

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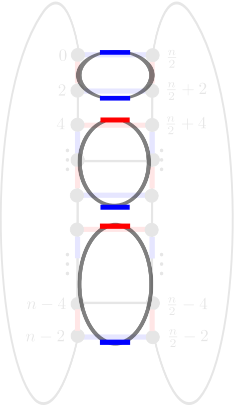

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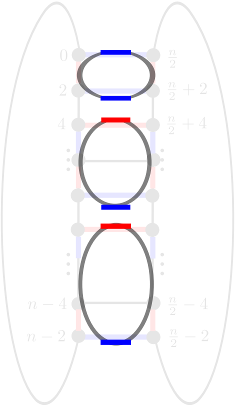

Each possible cycle uses exactly 2 cross edges

$$a + b = 2 \text{ (number of cycles in } A \cup B)$$

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Similarly, by considering the other two pairs of 1-factors,

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n divisible by 6 and $n \equiv 2 \pmod{4}$ and $\implies n \equiv 6 \pmod{12}$

□

OPEN PROBLEM

Characterise the connected 3-regular circulant graphs of the form $\mathrm{Circ}(n, \{1, \frac{n}{2}\})$ *that admit a U1F of any type.*

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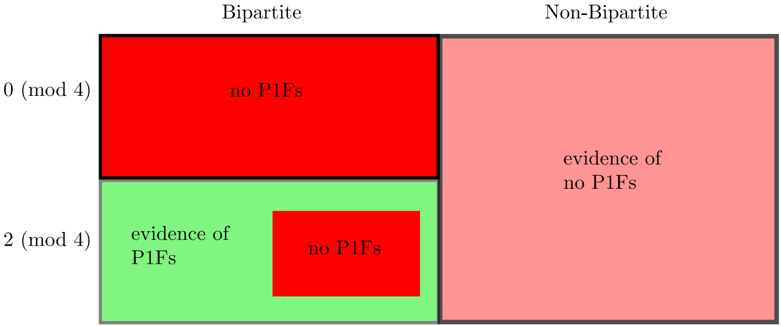
Characterise the connected 4-regular circulant graphs that admit P1Fs.



Bipartite

Non-Bipartite

0 (mod 4)

no P1Fs

evidence of
no P1Fs

2 (mod 4)

evidence of
P1Fs

no P1Fs

Questions?