Distribution of Symbols in Weighted Random Staircase Tableaux

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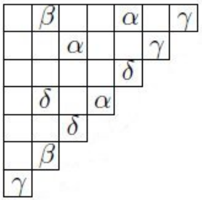
based on a joint work, one with S. Janson (Uppsala U., Sweden), another with A. Parshall (Drexel)

May 12, 2014

Staircase tableaux (Corteel-Williams (2009))

Figure: A staircase tableau of size 7



Filling rules for Greek letters:

- no empty boxes on the diagonal
- empty above α or γ in the same column;
- empty to the left of δ or β in the same row.

Staircase Tableaux, cont.

- Introduced in connection with Asymmetric Exclusion Process (ASEP): a particle model (introduced in 80's) studied by physicists, e.g. Derrida and his co-authors (early 90's -2008); a Markov chain on configurations of $\circ$'s (empty sites) and $\bullet$'s (ocupied sites) of length n.

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- There are also connections of staircase tableaux to Askey-Wilson polynomials, Corteel, Stanley, Stanton, Williams (2012).
- Have life on their own, particularly in connection with other combinatorial structures, especially other types of tableaux (various works by various combinations of Aval, Boussicaut, Corteel, Dasse–Hartaut, Janson, Nadeau, Steingrímsson, Williams, and H. (2009–2013).)

Asymmetric Exclusion Process:

- A Markov chain on configurations of $\circ$'s and $\bullet$'s of length n

$$\circ\ \circ\ \bullet\ \circ\ \bullet\ \bullet\ \circ\ \circ\ \bullet$$

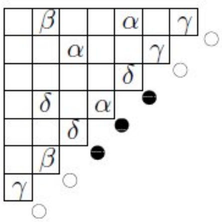
- Transition probabilities:

$A \bullet \circ B$ to $A \circ \bullet B$ (right hopping):	$\frac{u}{n+1}$
$A \circ \bullet B$ to $A \bullet \circ B$ (left hopping):	$\frac{q}{n+1}$
$\circ A$ to $\bullet A$ (entering from the left):	$\frac{\alpha}{n+1}$
$\bullet A$ to $\circ A$ (exiting to the left):	$\frac{\gamma}{n+1}$
$A\bullet$ to $A\circ$ (exiting to the right):	$\frac{\beta}{n+1}$
$A\circ$ to $A\bullet$ (entering from the right):	$\frac{\delta}{n+1}$

For the remaining states it is 0, for not moving at all is the rest.

Connection

Figure: A staircase tableau and its type $(\circ \circ \bullet \bullet \bullet \circ \circ)$.

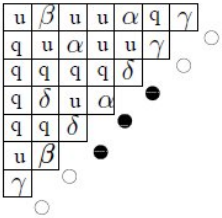


Type of a tableau: Move along the diagonal (NE to SW) and write:

- $\bullet$ for each α or δ;
- $\circ$ for each β or γ.

Connection, cont'd.

Figure: A staircase tableau with u and q



Filling rules for u's and q's:

- first: u's in all boxes to the left of a β and q's in all the boxes to the left of a δ;
- then: u's in all boxes above an α or a δ and q's in all boxes above a β or a γ.

Steady state probabilities for the ASEP:

Corteel and Williams (2009) have shown that for any state σ the steady state probability that the ASEP is in state σ is

$$\frac{Z_\sigma(\alpha, \beta, \gamma, \delta, q, u)}{Z_n(\alpha, \beta, \gamma, \delta, q, u)},$$

where

- $Z_\sigma(\alpha, \beta, \gamma, \delta, q, u) = \sum_{S \text{ of type } \sigma} wt(S);$
- $Z_n(\alpha, \beta, \gamma, \delta, q, u) = \sum_{S \text{ of size } n} wt(S);$
- $wt(S)$ is the product of labels of the boxes of S (a monomial of degree $n(n+1)/2$ in α, β, γ, δ, u and q).

Weighted staircase tableaux

- For combinatorial considerations a simplified version

$$\mathrm{wt}(S) = \alpha^{N_\alpha}\beta^{N_\beta}\gamma^{N_\gamma}\delta^{N_\delta},$$

where $N_{\{\cdot\}}$ is the number of symbols $\cdot$ in the tableau suffices.

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$$\mathrm{wt}(S) = \alpha^{N_\alpha} \beta^{N_\beta} \gamma^{N_\gamma} \delta^{N_\delta},$$

  where $N_{\{\cdot\}}$ is the number of symbols $\cdot$ in the tableau suffices.
- Then the total weight of staircase tableaux of size n is

$$Z_n := \sum_{S \in \mathcal{S}_n} \mathrm{wt}(S) = \prod_{i=1}^{n-1} (\alpha + \beta + \delta + \gamma + i(\alpha + \gamma)(\beta + \delta)).$$

  Corteel, Stanley, Stanton, Williams (2012).

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  Corteel, Stanley, Stanton, Williams (2012).
- $\alpha=\beta=\gamma=\delta=1$ gives

$$|\mathcal{S}_n| := Z_n(1,1,1,1) = \prod_{i=1}^{n-1}(4+4i) = n!4^n,$$

  where $\mathcal{S}_n$ is the set of all staircase tableaux of size n; this has various proofs.

Weighted staircase tableaux, cont.

- For probabilistic considerations define a probability

$$\mathbb{P}(S) = \frac{\mathrm{wt}(S)}{Z_n}, \quad S \in \mathcal{S}_n,$$

(if $\alpha = \beta = \gamma = \delta = 1$ this is the uniform discrete probability measure).

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- We want the general weights; the definition is symmetric w.r.t. α and γ and also β and δ so we consider only two letters (and weights): α and β.
- The resulting probability measure picks a particular staircase tableau S with probability proportional to

$$\alpha^{N_\alpha(S)}\beta^{N_\beta(S)}.$$

Weighted staircase tableaux, cont.

- Under the uniform probability measure Dasse–Hartaut and H. (2013) considered the behavior of various parameters of randomly selected staircase tableau; most interestingly, the number of various symbols on the diagonal.

Weighted staircase tableaux, cont.

- Under the uniform probability measure Dasse–Hartaut and H. (2013) considered the behavior of various parameters of randomly selected staircase tableau; most interestingly, the number of various symbols on the diagonal.
- If we pick a (2–letter) tableau S with probability proportional to

$$2^{N_\alpha(S)+N_\beta(S)}, \quad (\alpha = \beta = 2)$$

  and replace each α by γ and β by δ with probability $1/2$ independently for each occurence and independently of everything else, then the resulting 4–letter staircase tableau has weight

$$2^{N_\alpha(S)+N_\beta(S)}\frac{1}{2^{N_\alpha(S)}2^{N_\beta(S)}} = 1.$$

Main Result for the Diagonal

Let $\alpha, \beta \in (0, \infty]$ and let $a := 1/\alpha$, $b := 1/\beta$. If $(\alpha, \beta) \neq (\infty, \infty)$ Let $A = A_{n,\alpha,\beta}$ be the number of the α's on the diagonal of a staircase tableau.

- The generating function satisfies:

$$g_A(x) = \frac{\Gamma(a+b)}{\Gamma(n+a+b)} p_{n,a,b}(x),$$

where

$$p_{n,a,b}(x) = \sum_k v_{a,b}(n,k) x^k$$

and $(v_{a,b}(n,k))$ satisfies $v_{a,b}(0,0) = 1$, $v_{a,b}(0,k) = 0$ for $k \neq 0$ and for $n \geq 1$

$$v_{a,b}(n,k) = (k+a) v_{a,b}(n-1,k) + (n-k+b) v_{a,b}(n-1,k-1).$$

Comments:

- When $(a, b) = (1, 1)$, $(1, 0)$, or $(0, 1)$, the $(v_{a,b}(n, k))$ (resp. $(p_{n,a,b}(x))$ are the classical Eulerian numbers (resp. polynomials), in different enumerating conventions, e.g.

$$v_{1,1}(n, k) = \left\langle {n+1 \atop k} \right\rangle;$$

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- $\alpha = \infty$ is interpreted as the limit as $\alpha \to \infty$ (same for β); $(\alpha, \beta) = (\infty, \infty)$ is interpreted as $\alpha = \beta \to \infty$.
- results for B, the number of β's on the diagonal, follow from $A + B = n$ or from an involution on $\mathcal{S}_n$ consisting on a reflection of S w.r.t. the NW–SE diagonal and interchanging the roles of α and β.

Consequences (more or less direct):

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- $g_A(x)$ has simple roots on the negative half–line; hence the probabilities

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \frac{v_{a,b}(n,k)}{p_{n,a,b}(1)} \quad 0 \leq k \leq n$$

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- Moreover

$$A_{n,\alpha,\beta} \stackrel{d}{=} \sum_{i=1}^{n} \mathrm{Bern}(p_{i,n}),$$

  where $\mathrm{Bern}(p_{i,n})$ are independent Bernoulli random variables and "$\stackrel{d}{=}$" denotes the equality in distribution.

Consequences, cont.

- Central Limit Theorem holds:

$$\frac{A_{n,\alpha,\beta} - \mathbb{E}A_{n,\alpha,\beta}}{(\mathrm{Var}(A_{n,\alpha,\beta}))^{1/2}} \xrightarrow{d} N(0,1),$$

or, more explicitly,

$$\frac{A_{n,\alpha\beta} - n/2}{\sqrt{n}} \xrightarrow{d} N(0,1/12).$$

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- Moreover, a corresponding local limit theorem holds:

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = (2\pi \mathrm{Var}(A_{n,\alpha,\beta}))^{-1/2} \left(e^{-\frac{(k-EA_{n,\alpha,\beta})^2}{2\mathrm{Var}(A_{n,\alpha,\beta})}} + o(1) \right),$$

  or, more explicitly,

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \sqrt{\frac{6}{\pi n}} \left(e^{-6(k-n/2)^2/n} + o(1) \right),$$

  as $n \to \infty$, uniformly in $k \in \mathbb{Z}$.

A comment about the proof

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- For $\alpha = \beta = 2$ this was done in Dasse–Hartaut, H. (2013) in probabilistic language. We then identified the numbers ($A_{n,2,2}$) in Sloane's EOIS as 'Eulerian numbers of type B'(A060187). They go back to MacMahon (1920) (possibly to Euler (1768). Closed form of their g. f. is in Franssens (2006) and we checked that it satisfies Bender (1973) sufficient condition for the CLT.

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- A new twist here was a *much* heavier use of the properties of polynomials satisfying Eulerian type recurrence:

$$p_{n+1,a,b}(x) = ((n+b)x + a)p_{n,a,b}(x) + x(1-x)p'_{n,a,b}(x).$$

Some special cases

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$$\mathbb{P}(A_{n,1,1} = k) = \frac{v_{1,1}(n,k)}{(n+1)!} = \frac{\left\langle {n+1 \atop k} \right\rangle}{(n+1)!}.$$

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  (known bijectively from the above papers).
- $\alpha = 2$, $\beta = 1$: staircase tableaux without δ's briefly studied in Corteel, Stanley, Stanton, Williams (2012). There are

$$Z_n(2,1) = \prod_{i=0}^{n-1}(3+2i) = (2n+1)!!$$

  such tableaux. Our theorems yield further results on random δ-free staircase tableaux.

Special case: the maximal number of symbols

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- Consider $\alpha = \beta = \infty$ (i.e. $\alpha = \beta \to \infty$). Only tableaux with the maximum number of symbols ($2n - 1$ as we will see) have non–zero probability. The g.f. is obtained by extracting the highest order term from

$$\begin{aligned} Z_n(\alpha, \beta) &= \prod_{i=0}^{n-1} (\alpha + \beta + i\alpha\beta) \sim (\alpha + \beta) \prod_{i=1}^{n-1} (i\alpha\beta) \\ &= (n-1)!(\alpha^n \beta^{n-1} + \alpha^{n-1}\beta^n). \end{aligned}$$

Hence, there are $(n-1)!$ tableaux with n α's and $n-1$ β's and $(n-1)!$ with $n-1$ α's and n β's (and $\left\langle {n-1 \atop k-1} \right\rangle$ of them have k α's on the diagonal). By flipping, there are $2^{2n}(n-1)!$ (4–letter) staircase tableaux with the maximal number of letters (Corteel, Dasse–Hartaut (2011)).

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- **Where are the symbols located?**

Position of symbols

Most of the off-diagonal boxes are empty. Let $S_{n,\alpha,\beta}(i,j)$ be a content of the (i,j)th box (enumerated as in a matrix). We have

- **Off–diagonal boxes:**

$$\begin{aligned}
\mathbb{P}(S_{n,\alpha,\beta}(i,j)=\alpha) &= \frac{j-1+b}{(i+j+a+b-1)(i+j+a+b-2)}, \\
\mathbb{P}(S_{n,\alpha,\beta}(i,j)=\beta) &= \frac{i-1+a}{(i+j+a+b-1)(i+j+a+b-2)}, \\
\mathbb{P}(S_{n,\alpha,\beta}(i,j)\neq\emptyset) &= \frac{1}{i+j+a+b-1}.
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- **Diagonal boxes:** Let $S_n(j) := S_{n,\alpha,\beta}(n+1-j,j)$ be the symbol on the diagonal in the jth column and let $1 \le j_1 < \cdots < j_\ell \le n$. Then

$$\mathbb{P}\big(S_n(j_1)=\cdots=S_n(j_\ell)=\alpha\big)=\prod_{k=1}^{\ell}\frac{j_k-k+b}{n-k+a+b}.$$

Position of symbols: subtableaux

These results follow from a key observation: For $i + j \leq n + 1$ let $S_{n,\alpha,\beta}[i, j]$ be a subtableau obtained from $S_{n,\alpha,\beta}$ by removing the top $i - 1$ rows and the left $j - 1$ columns. Then

Theorem

Let $\alpha, \beta \in (0, \infty]$ and $i + j \leq n + 1$. The subtableau $S_{n,\alpha,\beta}[i, j]$ of $S_{n,\alpha,\beta}$ has the same distribution as $S_{n-i-j+2,\hat{\alpha},\hat{\beta}}$, where $\hat{\alpha}^{-1} = \alpha^{-1} + i - 1$ and $\hat{\beta}^{-1} = \beta^{-1} + j - 1$.

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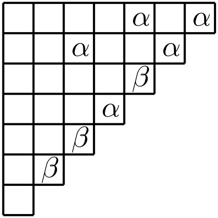
- the proof is not dificult, but *crucially* needs the notion of weights.
- Once this is known make a box you are interested in a NW corner box; the distibution of its entry is computable (and *trivial* for diagonal boxes).

Main step in a proof: $S[1,2]$



Fill the new column.

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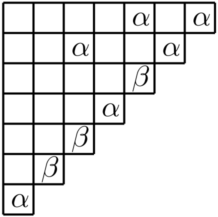

Put α at the bottom and leave all other boxes empty.

Main step in a proof: $S[1,2]$

$*$				α		α
$*$		α			α	
				β		
$*$			α			
		β				
	β					
β						

Put β at the bottom and fill (or not) the $*$-ed boxes.

Main step in a proof: $S[1,2]$

Take a fixed tableau S of size $n-1$. The probability that when the first column is removed from a random tableau of size n we obtain precisely our S, is proportional to the sum of the weights of all extensions of S to a tableau of size n. When the calculations are carried out this sum turns out to be

$$(\beta+\alpha)(1+\beta)^n \alpha^{N_\alpha} \left(\frac{\beta}{1+\beta}\right)^{N_\beta},$$

so that the probability is proportional to

$$\alpha^{N_\alpha} \hat{\beta}^{N_\beta} := \alpha^{N_\alpha} \left(\frac{\beta}{1+\beta}\right)^{N_\beta}$$

That is $\hat{\beta} = \beta/(1+\beta)$ i.e. $\hat{\beta}^{-1} = \beta^{-1} + 1$.

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$$(\beta+\alpha)(1+\beta)^n \alpha^{N_\alpha}\left(\frac{\beta}{1+\beta}\right)^{N_\beta},$$

so that the probability is proportional to

$$\alpha^{N_\alpha}\hat{\beta}^{N_\beta} := \alpha^{N_\alpha}\left(\frac{\beta}{1+\beta}\right)^{N_\beta}$$

That is $\hat{\beta} = \beta/(1+\beta)$ i.e. $\hat{\beta}^{-1} = \beta^{-1}+1$.
Removing the first row follows from symmetry, the general case by iterating this procedure.

Joint (p.) g.f. for the total number of symbols

Let $\alpha, \beta \in (0, \infty]$, and let $a := \alpha^{-1}$, $b := \beta^{-1}$. The joint probability generating function of N_α and N_β for the random staircase tableau is

$$\mathbb{E}x^{N_\alpha}y^{N_\beta} = \prod_{i=0}^{n-1} \frac{\alpha x + \beta y + i\alpha\beta xy}{\alpha + \beta + i\alpha\beta} = \prod_{i=0}^{n-1} \frac{bx + ay + ixy}{a + b + i}.$$

That is because

$$\begin{aligned}\mathbb{E}x^{N_\alpha}y^{N_\beta} &= \sum_{k,m} x^k y^m \mathbb{P}(N_\alpha = k, N_\beta = m) = \sum_{k,m} x^k y^m \frac{\alpha^k \beta^m}{Z(\alpha, \beta)} \\ &= \frac{Z(\alpha x, \beta y)}{Z(\alpha, \beta)} = \prod_{i=0}^{n-1} \frac{\alpha x + \beta y + i\alpha\beta xy}{\alpha + \beta + i\alpha\beta}\end{aligned}$$

$$\mathbb{E}x^{N_\alpha}y^{N_\beta} = \prod_{i=0}^{n-1} \frac{bx + ay + ixy}{a + b + i}$$

means that

$$(N_\alpha, N_\beta) \overset{d}{=} \Big(\sum_i I_i, \sum_i J_i \Big),$$

where (I_i, J_i) are independent pairs of random variables such that

$$\mathbb{P}(I_i = \iota, J_i = \iota') = \begin{cases} 0, & (\iota, \iota') = (0, 0), \\ \frac{b}{a+b+i}, & (\iota, \iota') = (1, 0), \\ \frac{a}{a+b+i}, & (\iota, \iota') = (0, 1), \\ \frac{i}{a+b+i}, & (\iota, \iota') = (1, 1). \end{cases}$$

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In particular, the marginal distributions are

$$I_i \stackrel{d}{=} \mathrm{Bern}\Big(1 - \frac{a}{a+b+i}\Big), \quad J_i \stackrel{d}{=} \mathrm{Bern}\Big(1 - \frac{b}{a+b+i}\Big).$$

Hence, for example

$$\mathbb{E}N_\alpha = \sum_i \Big(1 - \frac{a}{a+b+i}\Big) = n - \sum_i \frac{a}{a+b+i}.$$

NW Symbol

We can compute the distribution of a symbol in the NW corner as follows: the expected total number of symbols α in $S = S_{n,\alpha,\beta}$ is

$$\mathbb{E}N_\alpha = \sum_{i=0}^{n-1} \Big(1 - \frac{a}{a+b+i}\Big).$$

If we delete the first column, the remaining part $S[1,2]$ is an S_{n-1,α_1,β_1} with $a_1 := \alpha_1^{-1} = a$ and $b_1 := \beta_1^{-1} = b+1$; hence the expected number of α's in $S[1,2]$ is

$$\sum_{i=0}^{n-2} \Big(1 - \frac{a}{a+b+1+i}\Big) = \sum_{i=1}^{n-1} \Big(1 - \frac{a}{a+b+i}\Big).$$

Taking the difference we see that

$$\mathbb{E}\Big(\#\alpha \text{ in the first column}\Big) = 1 - \frac{a}{a+b} = \frac{b}{a+b}.$$

NW Corner, cont'd.

Now delete the first row of S. The remainder $S[2,1]$ is an S_{n-1,α_2,β_2} with $a_2 := \alpha_2^{-1} = a+1$ and $b_2 := \beta_2^{-1} = b$. Hence

$$\mathbb{E}\Big(\#\alpha \text{ in boxes } (2,1),\ldots,(n,1)\Big) = \frac{b_2}{a_2+b_2} = \frac{b}{a+b+1},$$

and taking the difference again we obtain

$$\mathbb{P}\Big(S_{n,\alpha,\beta}(1,1)=\alpha\Big) = \frac{b}{a+b} - \frac{b}{a+b+1} = \frac{b}{(a+b)(a+b+1)}.$$

Hence also

$$\mathbb{P}\Big(S_{n,\alpha,\beta}(1,1)=\beta\Big) = \frac{a}{(a+b)(a+b+1)}$$

and

$$\mathbb{P}\Big(S_{n,\alpha,\beta}(1,1)=0\Big) = 1 - \frac{1}{a+b+1} = \frac{a+b}{a+b+1}.$$

Joint distribution of diagonal boxes

- Other diagonals: **Conjecture:** the number of α's/β's/symbols on the diagonal $(k_n - j, j)$ is asymptotically Poisson as $k_n \to \infty$ with $n \to \infty$.

Joint distribution of diagonal boxes

- Other diagonals: **Conjecture:** the number of α's/β's/symbols on the diagonal $(k_n - j, j)$ is asymptotically Poisson as $k_n \to \infty$ with $n \to \infty$.
- For $k_n = n$ (just above the main diagonal) the conjecture is correct (A. Parshall, H. (2014+)):

Theorem

As $n \to \infty$ the number of α's on the 2nd diagonal converges in distribution to a Poisson random variable with parameter 1, i.e.

$$\mathbb{P}(\#\{j : \ S(n-j,j) = \alpha\} = k) \to e^{-1/2}\frac{1}{k!2^k}, \quad 1 \le k < n$$

and the number of symbols converges to a Poisson variable with parameter 1:

$$\mathbb{P}(\#\{j : \ S(n-j,j) \neq \emptyset\} = k) \to e^{-1}\frac{1}{k!}, \quad 1 \le k < n.$$

Sketch of the proof

Poisson random variable with parameter $\lambda > 0$ is characterized by

$$\mathbb{E}(X)_r = \mathbb{E}X(X-1)\dots(X-(r-1)) = \lambda^r, \quad r \geq 1$$

and for convergence of (X_n) to such variable it is enough to show

$$\mathbb{E}(X_n)_r \to \lambda^r, \quad \text{as } n \to \infty, \quad r \geq 1.$$

Also, if $X = \sum_j I_j$ is the sum of the indicator random variables

$$\mathbb{E}z^X = \mathbb{E}\prod_j (1 + I_j(z-1)) = 1 + \sum_{r\geq 1}(z-1)^r \sum_{j_1<\dots<j_r} \mathbb{E}\prod_{m=1}^{r} I_{j_m}$$

so that

$$\mathbb{E}(X)_r = \frac{d^r(\mathbb{E}z^X)}{dz^r}_{|z=1} = r! \sum_{j_1<\dots<j_r} \mathbb{P}(\prod_{m=1}^{r} I_{j_m}).$$

Sketch of the proof, cont.

If I_j is the indicator that there is an α in the $(n-j,j)$ box then:

$$\mathbb{P}(\prod_{k=1}^{r} I_{j_k}) = \prod_{k=1}^{r} \frac{b+j_{r-k+1}-2r+2k-1}{(a+b+n-2r+2k-1)(a+b+n-2r+2k-2)}.$$

provided $j_k \leq j_{k+1}-2, \ \forall k=1,2,\ldots,r-1$ and is zero otherwise.

Sketch of the proof, cont.

If I_j is the indicator that there is an α in the $(n-j,j)$ box then:

$$\mathbb{P}(\prod_{k=1}^{r} I_{j_k}) = \prod_{k=1}^{r} \frac{b + j_{r-k+1} - 2r + 2k - 1}{(a+b+n-2r+2k-1)(a+b+n-2r+2k-2)}.$$

provided $j_k \leq j_{k+1} - 2, \ \forall k = 1, 2, ..., r-1$ and is zero otherwise.
The proof is inductive over r; the factorial moment convergence follows upon summation and the following identity:
Set

$$J_{r,m} := \{1 \leq j_1 < ... < j_r \leq m : \ j_k \leq j_{k+1} - 2, \ \forall k = 1, 2, ..., r-1\}.$$

Then

$$\sum_{J_{r,m}} \left(\prod_{k=1}^{r} j_{r-k+1} \right) = \frac{(m+1)_{2r}}{2^r r!}.$$