The game EUCLID, its variants, and continued fractions

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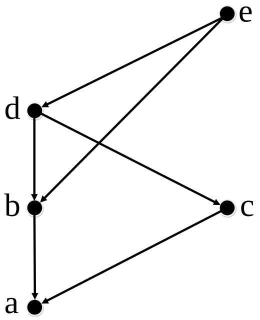
Outline

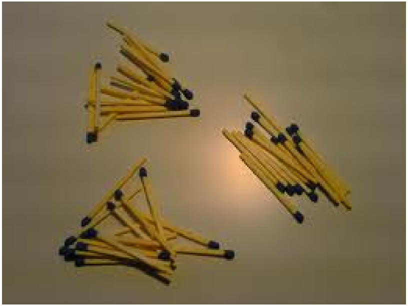
- Quick theory of impartial combinatorial games
- The game EUCLID
- The first variant: Grossman's variant
- Relation of two games
- One more variant: same approach
- Common questions in games

What is an impartial combinatorial game

- A game is a directed graph: acyclic
- A position is a vertex
- A move is an arc
- Two players move alternately, no skip
- Game terminates
- The player who makes the last move wins



NIM: the simplest?



- piles of objects
- choose a pile and remove as many as you want
- what is the graph corresponding to NIM(1,2)?

$\mathcal{N}$-positions v.s. $\mathcal{P}$-positions

Every position is either an

- $\mathcal{N}$-positions: from there the player about to move can win, or a
- $\mathcal{P}$-positions: otherwise.

Lemma

1. *For every vertex p in $\mathcal{P}$, all the moves from p terminate in $\mathcal{N}$,*
2. *For every vertex p in $\mathcal{N}$, there is a move from p that terminates in $\mathcal{P}$.*

Strategy for players: leaving the game in a $\mathcal{P}$-position.

Sprague-Grundy function $\mathcal{G}$

Definition (MEX: minimum excluded value)

Let S be a finite set of nonnegative integers. $\mathrm{mex}(S)$ *is the smallest nonnegative integer not in S.*

Example:

$$\mathrm{mex}\{0, 1, 2, 4, 7\} = 3$$

Definition (Sprague-Grundy function)

$$\mathcal{G}(v) = 0 \text{ if } v \text{ is the final position},$$

$$\mathcal{G}(v) = mex\{\mathcal{G}(u) : \text{ if there exists a move from } v \text{ to } u\}.$$

$\mathcal{G}(v)$ is also called NIM-value of v.

WHY $\mathcal{G}$?

Let's play sums.

- Given two games G_1 and G_2
- Choose one game, either, and play there, following the rule of that game
- The next move can be made in any game
- The sum ends when both games end.

Theorem

The position (v_1, v_2) in the sum $G_1 + G_2$ has NIM-value $\mathcal{G}(v_1) \oplus \mathcal{G}(v_2)$ where $\oplus =$ XOR, also called NIM-sum. It is a $\mathcal{P}$-position if and only if $\mathcal{G}(v_1) = \mathcal{G}(v_2)$.

The game EUCLID [Cole & Davie, 1969]

- **Set up:** a pair (a, b) of positive integers.
- Two players move alternately, subtracting from the greater entry a positive integer multiple of the smaller one without making the result negative:

$$(a, b) \to (a, b - ia) \quad \text{where} \quad i \in \mathbb{N}, b - ia \geq 0.$$

- The game ends when one of entries becomes zero, i.e. $(0, c)$.
- winner: making the last move: $(a, b) \to (a, 0)$.

The $\mathcal{P}$-positions

Proposition (Cole & Davie, 1969)

$$\mathcal{P} = \{(0, b), (a, b) | 0 < a < b < \phi a\}$$

$\phi = \frac{\sqrt{5}+1}{2} = 1.6180\ldots$ *is the Golden ratio.*

So:

$$\mathcal{P} = \{(2,3), (3,4), (4,5), (4,6), (5,6), (5,7), (5,8), \ldots\} \rightarrow \textbf{DONE}.$$

Example:

$$(8, 21) \rightarrow (5, 8) \rightarrow (3, 5) \rightarrow (2, 3) \rightarrow (1, 2) \rightarrow (0, 1).$$

Playing the game without ϕ (Spitznagel, Jr, 1973)

Your year-9 daughter: Hey Mummy, I want to play but I don't like that ϕ, can I win?

Umhhh!!!.

Playing the game without ϕ (Spitznagel, Jr, 1973)

Given $b \geq 2a, b = qa + r$,

$$(a, b) = (a, \mathbf{q}a + r) \to (a, \mathbf{p}a + r)$$

We need $p \leq 1$ and so either $(r, a) \in \mathcal{P}$ or $(a, a + r) \in \mathcal{P}$, but not both.

Proposition (Spitznagel, Jr, 1973)

Strategy: $\frac{a}{r}$ *v.s* $\frac{a+r}{a}$. *The position with smaller ratio is in* $\mathcal{P}$.

Examples

$(7, 25) \to ?$, $(7, 4)$ *or* $(7, 11)$. *Since* $7 \times 7 = 49 > 4 \times 11$, $\frac{7}{4} > \frac{11}{7}$ *and so* $(7, 11)$ *is* $\mathcal{P}$.

Continued fractions, the bridge from the game EUCLID to NIM

- Let $[a_0, a_1, \dots, a_n]$, with $a_n \geq 2$, be the continued fraction expansion of b/a.
- The move
$$(a, b) \to (a, b - ia)$$
is equivalent to the move (always from the left)
$$[a_0, a_1, \dots, a_n] \to [a_0 - i, a_1, \dots, a_n].$$

Theorem (Lengyel, 2003)

(a, b) is a $\mathcal{P}$-position iff there exists some even k s.t.

$$1 = a_0 = \cdots = a_k < a_{k+1}.$$

The game of SERIAL NIM

Definition (Levine, 2006)

Playing as NIM *but moving from the leftmost pile:*

$$(a_1, a_2 \ldots, a_n) \to (a_1 - i, a_2, \ldots, a_n).$$

Theorem (Sprague-Grundy function: Levine, 2006)

Set $a_{n+1} = 0$, $m = \min\{i | a_i \neq a_1\}$.

$$\mathcal{G}(a_1, a_2, \ldots, a_n) = \begin{cases} a_1 - 1, & \text{if } m \text{ is odd and } a_m < a_1; \\ a_1 - 1, & \text{if } m \text{ is even and } a_m > a_1; \\ a_1, & \text{otherwise.} \end{cases}$$

EUCLID: reformulation of Levine's formula

Theorem (G. Cairns, N.B. Ho, T. Lengyel, 2011)

Let $0 < a < b$, consider the continued fraction expansion $[a_0, a_1, \ldots, a_n]$ of b/a, and let $\mathcal{I}(a, b)$ be the largest nonnegative integer i such that

$$a_0 = \cdots = a_{i-1} \leq a_i.$$

Then the Sprague-Grundy value of the position (a, b) in the game EUCLID *is*

$$\mathcal{G}(a, b) = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & : \text{ if } \mathcal{I}(a, b) \text{ is even;} \\ 1 & : \text{ otherwise.} \end{cases}$$

Grossman's variant: Sprague-Grundy function

Definition (Grossman, 1997)

Playing as EUCLID, *the game ends when the two entries are equal:*

$$\cdots (a, b) \rightarrow (a, a) : \text{ END, I win.}$$

Proposition (Straffin, 1998)

$$\mathcal{P} = \{(0, b), (a, b) | 0 < a < b < \phi a\}.$$

The two games share the same $\mathcal{P}$-positions, and so strategy.

HOW COME? $\begin{cases} (8,21) \rightarrow (5,8) \rightarrow (3,5) \rightarrow (2,3) \rightarrow (1,2) \rightarrow (0,1). \\ (8,21) \rightarrow (5,8) \rightarrow (3,5) \rightarrow (2,3) \rightarrow (1,2) \rightarrow (1,1). \end{cases}$

Grossman's variant

Theorem (Nivasch, 2006)

$$\mathcal{G}_G(a, b) = \lfloor \frac{b}{a} - \frac{a}{b} \rfloor.$$

How are the Sprague-Grundy functions of the two games related

Theorem (G. Cairns, N.B. Ho, T. Lengyel, 2011)

$$G_{\mathcal{G}}(a,b) = \left\lfloor \left| \frac{b}{a} - \frac{a}{b} \right| \right\rfloor = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & : \text{ if } \mathcal{I}(a,b) \text{ is } \textbf{even}; \\ 1 & : \text{ otherwise,} \end{cases}$$

except in the special case where $a_0 = a_1 = \cdots = a_n$, *in which special case,*

$$G_{\mathcal{G}}(a,b) = \left\lfloor \left| \frac{b}{a} - \frac{a}{b} \right| \right\rfloor = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & : \text{ if } \mathcal{I}(a,b) \text{ is } \textbf{odd}; \\ 1 & : \text{ otherwise.} \end{cases}$$

How are the Sprague-Grundy functions of the two games related

Theorem

For $0 < a \leq b$, suppose that b/a has continued fraction expansion $[a_0, a_1, \ldots, a_n]$. Then

$$\mathcal{G}(a, b) = \mathcal{G}_G(a, b)$$

unless $a_0 = a_1 = \cdots = a_n$, in which special case,

$$\mathcal{G}(a, b) = \mathcal{G}_G(a, b) + (-1)^n$$

.

Recall that $\mathcal{G}_G(a, b) = \lfloor|\frac{b}{a} - \frac{a}{b}|\rfloor$.

The game M-EUCLID

Definition (Cairns & Ho, 2012)

The game ends when one entries is a positive multiple of the other:

$$\cdots \to (a,b) \to (a+ia) \quad \textit{game ENDS.}$$

Proposition ($\mathcal{P}$-positions without continued fractions)

$$\{?|??\}$$

Sprague-Grundy function for the game M-EUCLID

Theorem (Cairns & Ho, 2012)

Let $0 < a < b$ where b is not a multiple of a, consider the continued fraction expansion $[a_0, a_1, \ldots, a_n]$ of $\frac{b}{a}$, and let $\mathcal{J}(a, b)$ be the largest nonnegative integer $j < n$ such that $a_0 = \cdots = a_{j-1} \leq a_j$. Then

$$\mathcal{G}_M(a, b) = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0, & \text{if } \mathcal{J}(a, b) \text{ is even;} \\ 1, & \text{otherwise.} \end{cases}$$

Common questions in games

- What are $\mathcal{P}$-positions? Can they be represented in some algebraic characterization (An algebraic characterization likely gives a polynomial algorithm to decide whether or not a given position is a $\mathcal{P}$-position)
- What are the next optimal move from an $\mathcal{N}$-position?
- Is there a formula for Sprague-Grundy function (rather than using mex which requires exponential time)?

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