Subtraction games with expandable subtraction sets

Bao Ho

Department of Mathematics and Statistics
La Trobe University

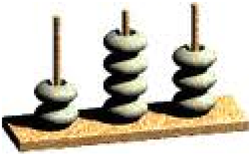
Monash University
April 11, 2012

Outline

- The game of Nim
- Nim-values and Nim-sequences
- Subtraction games
- Periodicity of subtraction games
- Expansion of subtraction sets

The game of Nim

- a row of piles of coins,
- two players move alternately, choosing one pile and removing an arbitrary number of coins from that pile,
- the game ends when all piles become empty,
- the player who makes the last move wins.



Nim-addition

Nim-addition, denoted by $\oplus$, is the addition in the binary number system without carrying.

For example, $5 = 101_2$, $3 = 11_2$ and so $5 \oplus 3$ is 6 obtained as follows

$$\begin{array}{ccccc} & 1 & 1_2 & & \\ 1 & 0 & 1_2 & & \\ \hline 1 & 1 & 0_2 & = & 6 \end{array}$$

Winning strategy in Nim

You can win in Nim if you can force your opponent to move from a position of the form $(a_1, a_2, \ldots, a_k)$ such that

$$a_1 \oplus a_2 \oplus \ldots \oplus a_k = 0^{1}.$$

[1] C.L. Bouton, Nim, a game with a complete mathematical theory, *Ann. of Math.* (2) **3** (1901/02), no. 1/4, 35–39.

Example

$$(2,3,6) \rightarrow (2,3,1) \rightarrow (3,1) \rightarrow (1,1) \rightarrow (1) \rightarrow \emptyset.$$

Homework: Find a winning move from position (1,2,3,4).

$$(1,2,3,4) \rightarrow ?$$

One-pile Nim-like games: example 1

From a pile of coins, remove any number of coins strictly smaller than half the size of the pile.

Strategy: You can win if and only if you can leave the game in a pile of size 2^k.

One-pile Nim-like games: example 2

Given a pile of coins, remove at most m coins, for some given m.

Strategy: You can win if and only if you can leave the game in a pile of size n such that $\mod(n, m+1) = 0$.

Games as directed graphs

A game $\equiv$ finite directed acyclic graph without multiple edges in which

- vertices $\equiv$ positions,
- downward edges $\equiv$ moves,
- source $\equiv$ initial position,
- sinks $\equiv$ final positions.

We can assume that such a graph have exactly one sink.

mex value

Let S be a set of nonnegative integers.

The minimum excluded value of the set S is the least nonnegative integer which is not included in S and is denoted $mex(S)$.

$$mex(S) = min\{k \in \mathbb{Z}, k \geq 0 | k \notin S\}.$$

We define $mex\{\} = 0$.

Example:

$$mex\{0, 1, 3, 4\} = 2.$$

Sprague-Grundy function

The Sprague-Grundy function for a game is the function

$$\mathcal{G} : \{\text{positions of the game}\} \to \{n \in \mathbb{Z}; n \geq 0\}$$

defined inductively from the final position (sink of graph) by

$$\mathcal{G}(p) = mex\{\mathcal{G}(q) | \text{if there is one move from } p \text{ to } q\}.$$

The value $\mathcal{G}(p)$ is also called nim-value.

Subtraction games

A subtraction game is a variant of Nim involving a finite set S of positive integers:

- the set S is called subtraction set,
- the two players alternately remove some s coins provided that $s \in S$.

The subtraction game with subtraction set $\{a_1, a_2, \ldots, a_k\}$ is denoted by $\mathcal{S}(a_1, a_2, \ldots, a_k)$.

Nim-sequence

For each non-negative integer n, we denote by $\mathcal{G}(n)$ the nim-value of the single pile of size n of a subtraction game.

The sequence

$$\{\mathcal{G}(n)\}_{n\geq 0} = \mathcal{G}(0), \mathcal{G}(1), \mathcal{G}(2), \ldots$$

is called nim-sequence.

Nim-sequences of some subtraction games

$$\mathcal{S}(1,2,3): 0,1,2,3,0,1,2,3,0,1,2,3,0,1,2,3,0,1,2,3,0,1,2,3,0,\ldots$$

$$\mathcal{S}(2,3,5): 0,0,1,1,2,2,3,0,0,1,1,2,2,3,0,0,1,1,2,2,3,0,0,1,1,\ldots$$

$$\mathcal{S}(1,5,7): 0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,\ldots$$

$$\mathcal{S}(3,5,9): 0,0,0,1,1,1,2,2,0,3,3,1,0,2,0,1,0,1,0,1,0,1,0,1,0,\ldots$$

Periodicity of nim-sequences

A nim-sequence is said to be ultimately periodic if there exist N, p such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq N$. The smallest such number p is called the period.

If $N = 0$, then the nim-sequence is said to be periodic.

$S(2,3,5) : 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, \ldots$

$S(1,5,7) : 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$

$S(3,5,9) : 0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 0, 2, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$

Periodicity of subtraction games

A game is said to be (ultimately) periodic if its nim-sequence is (ultimately) periodic.

Theorem

[a] *Every subtraction game is (ultimately) periodic.*

[a] E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning ways for your mathematical plays. Vol. 1*, second ed., A K Peters Ltd., Natick, MA, 2001.

Open problem in the periodicity of subtraction games[2]

Problem

Given a subtraction set, describe the nim-sequence of the subtraction game.

The question is still open for subtraction games with three element subtraction sets.

[2]E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning ways for your mathematical play 1*, second ed., A K Peters Ltd., Natick, MA, 2001.

Subtraction games agreeing nim-sequences: Examples

- $\mathcal{S}(2,3)$

  $0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,\ldots$

- $\mathcal{S}(2,3,7)$

  $0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,\ldots$

- $\mathcal{S}(2,3,7,8)$

  $0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,\ldots$

- $\mathcal{S}(2,3,7,8,12)$

  $0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,1,2,0,0,1,\ldots$

More examples

Subtraction set (with optional extras)	nim-sequence	period
1 (3,5,7,9,. . .)	010101. . .	2
2 (6,10,14,18, . . .)	00110011. . .	4
1,2 (4,5,7,8,10,11, . . .)	012012	3
3 (9,15,21,27, . . .)	000111000111. . .	6
2,3 (7,8,12,13,17,18,. . .)	0011200112. . .	5
2,3 (7,8,12,13,17,18,. . .)	0011200112. . .	5
3,6,7 (4,5,13,14,15,16,17,23,24. . .)	0001112223	
	0001112223. . .	10

Problem[3]

Problem

Let $S = \{a_1, a_2, \ldots, a_k\}$ *be a subtraction set. Find all integers* a *so that* a *can be added into* S *without changing the nim-sequence.*

[3] E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning ways for your mathematical play* *1*, second ed., A K Peters Ltd., Natick, MA, 2001.

Problem[3]

Problem

Let $S = \{a_1, a_2, \ldots, a_k\}$ be a subtraction set. Find all integers a so that a can be added into S without changing the nim-sequence.

If we can find a so that $\mathcal{G}(n-a) \neq \mathcal{G}(n)$ for every n then a can be added into the subtraction set without changing the nim-sequence.

[3] E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning ways for your mathematical play[s] 1*, second ed., A K Peters Ltd., Natick, MA, 2001.

Problem[3]

Problem

Let $S = \{a_1, a_2, \ldots, a_k\}$ be a subtraction set. Find all integers a so that a can be added into S without changing the nim-sequence.

If we can find a so that $\mathcal{G}(n-a) \neq \mathcal{G}(n)$ for every n then a can be added into the subtraction set without changing the nim-sequence.

For a given subtraction set S, we denote by S^{ex} the set of all integers that can be added into S without changing the nim-sequence.

[3] E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning ways for your mathematical play[s] 1*, second ed., A K Peters Ltd., Natick, MA, 2001.

Periodic games

Let $\mathcal{S}(s_1, s_2, \ldots, s_k)$ be a periodic subtraction game with period p. Then, for $1 \leq i \leq k$ and $m \geq 0$, $s_i + mp$ can be added into the subtraction set without changing the nim-sequence.

Let

$$S^{*p} = \{s_i + mp | 1 \leq i \leq k, m \geq 0\}.$$

Then,

$$S = \{s_1, s_2, \ldots, s_k\} \subseteq S^{*p} \subseteq S^{ex}.$$

Periodic games

Let $\mathcal{S}(s_1, s_2, \ldots, s_k)$ be a periodic subtraction game with period p. Then, for $1 \leq i \leq k$ and $m \geq 0$, $s_i + mp$ can be added into the subtraction set without changing the nim-sequence.

Let

$$S^{*p} = \{s_i + mp | 1 \leq i \leq k, m \geq 0\}.$$

Then,

$$S = \{s_1, s_2, \ldots, s_k\} \subseteq S^{*p} \subseteq S^{ex}.$$

Definition

If

$$S^{*p} = S^{ex}$$

then the subtraction set S is said to be non-expandable. Otherwise, S is expandable and S^{ex} is called the expansion of S.

The first simple case: $S = \{a\}$

The singleton subtraction set is non-expandable.

The second simple case: $S = \{a, b\}$

Let $a < b$ ($gcd(a, b) = 1$, b is not an odd multiple of a). Consider the subtraction set $\{a, b\}$.

- If $a + 1 < b \leq 2a$, then the subtraction set has expansion

$$\{a, a+1, \ldots, b\}^{*(a+b)}.$$

- If either $a = 1$, or $b = a + 1$, or $b > 2a$, then the subtraction set is non-expandable.

More examples: $S = \{1, a, b\}$

Example 1:
Let $a \geq 2$ be an even integer. The subtraction game $\mathcal{S}(1, a, 2a+1)$ is periodic and the subtraction set is non-expandable.

Example 2:
Let $a < b$ such that a is odd, b is even. The subtraction set $\{1, a, b\}$ is expandable with the expansion

$$\{\{1, 3, \dots, a\} \cup \{b, b+2, \dots, b+a-1\}\}^{*(a+b)}.$$

Ultimately periodic games

Let $\mathcal{S}(s_1, s_2, \ldots, s_k)$ be an ultimately periodic subtraction game with period p.

Note that the inclusion $S^{*p} \subseteq S^{ex}$ does not necessarily hold.

Definition

If $S^{ex} = S$ then the subtraction set S is non-expandable. Otherwise, S^{ex} is called the expansion of S.

An example

Let $a \geq 4$ be an even integer. The subtraction game $\mathcal{S}(1, a, 3a-2)$ is ultimately periodic with period $3a-1$.

- If $a = 4$ then the subtraction set is non-expandable,
- otherwise, the subtraction set has expansion

$$\{1, a, 3a-2, 3a\} \cup \{4a-1, 6a-1\}^{*(3a-1)}.$$

Ultimately bipartite subtraction games

A subtraction game is said to be ultimately bipartite if its nim-sequence is ultimately periodic with period 2 with, for sufficiently large n, alternating nim-values 0, 1, 0, 1, 0, 1, ...

Ultimately bipartite subtraction games

A subtraction game is said to be ultimately bipartite if its nim-sequence is ultimately periodic with period 2 with, for sufficiently large n, alternating nim-values 0, 1, 0, 1, 0, 1, ...

Some ultimately bipartite subtraction games:

- $\mathcal{S}(3, 5, 9, \ldots, 2^k + 1)$, for $k \geq 3$,
- $\mathcal{S}(3, 5, 2^k + 1)$, for $k \geq 3$,
- $\mathcal{S}(a, a + 2, 2a + 3)$, for odd $a \geq 3$,
- $\mathcal{S}(a, 2a + 1, 3a)$, for odd $a \geq 5$.

Ultimately bipartite subtraction games

A subtraction game is said to be ultimately bipartite if its nim-sequence is ultimately periodic with period 2 with, for sufficiently large n, alternating nim-values 0, 1, 0, 1, 0, 1, ...

Some ultimately bipartite subtraction games:

- $\mathcal{S}(3, 5, 9, \ldots, 2^k + 1)$, for $k \geq 3$,
- $\mathcal{S}(3, 5, 2^k + 1)$, for $k \geq 3$,
- $\mathcal{S}(a, a + 2, 2a + 3)$, for odd $a \geq 3$,
- $\mathcal{S}(a, 2a + 1, 3a)$, for odd $a \geq 5$.

Example:
$\mathcal{S}(3, 5, 9)$: 0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 0, 2, 0, 1, 0, 1, 0, 1, . . .

A conjecture

The subtraction set of an ultimately bipartite game is non-expandable.

For more details

G. Cairns, and N. B. Ho, Ultimately bipartite subtraction games. *Australas. J. Combin.* **48** (2010), 213–220

N. B. Ho, Subtraction games with three-element subtraction sets, submitted, arXiv:1202.2986v1.