Subtraction games with expandable subtraction sets

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Outline

- The game of Nim
- Nim-values and Nim-sequences
- Subtraction games
- Periodicity of subtraction games
- Expansion of subtraction sets
The game of Nim

- a row of piles of coins,
- two players move alternately, choosing one pile and removing an arbitrary number of coins from that pile,
- the game ends when all piles become empty,
- the player who makes the last move wins.
Nim-addition

Nim-addition, denoted by $\oplus$, is the addition in the binary number system without carrying.

For example, $5 = 101_2$, $3 = 11_2$ and so $5 \oplus 3$ is 6 obtained as follows

\[
\begin{array}{c c c}
  & 1 & 1_2 \\
+ & 1 & 0 & 1_2 \\
\hline
  & 1 & 1 & 0_2
\end{array} = 6
\]
Winning strategy in Nim

You can win in Nim if you can force your opponent to move from a position of the form \((a_1, a_2, \ldots, a_k)\) such that

\[ a_1 \oplus a_2 \oplus \ldots \oplus a_k = 0^1. \]

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\(^1\)C.L. Bouton, Nim, a game with a complete mathematical theory, *Ann. of Math.* (2) \textbf{3} (1901/02), no. 1/4, 35–39.
Example

\[(2, 3, 6) \rightarrow (2, 3, 1) \rightarrow (3, 1) \rightarrow (1, 1) \rightarrow (1) \rightarrow \emptyset.\]

**Homework:** Find a winning move from position \((1, 2, 3, 4)\).

\[(1, 2, 3, 4) \rightarrow ?\]
One-pile Nim-like games: example 1

From a pile of coins, remove any number of coins strictly smaller than half the size of the pile.

**Strategy:** You can win if and only if you can leave the game in a pile of size $2^k$. 
One-pile Nim-like games: example 2

Given a pile of coins, remove at most $m$ coins, for some given $m$.

**Strategy:** You can win if and only if you can leave the game in a pile of size $n$ such that $\mod (n, m + 1) = 0$. 

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Games as directed graphs

A game $\equiv$ finite directed acyclic graph without multiple edges
in which

- vertices $\equiv$ positions,
- downward edges $\equiv$ moves,
- source $\equiv$ initial position,
- sinks $\equiv$ final positions.

We can assume that such a graph have exactly one sink.
**mex value**

Let $S$ be a set of nonnegative integers.

The **minimum excluded value** of the set $S$ is the least nonnegative integer which is not included in $S$ and is denoted $mex(S)$.

$$mex(S) = \min\{k \in \mathbb{Z}, k \geq 0 | k \notin S\}.$$ 

We define $mex\{\} = 0$.

**Example:**

$$mex\{0, 1, 3, 4\} = 2.$$
The Sprague-Grundy function for a game is the function

\[ G : \{ \text{positions of the game} \} \rightarrow \{ n \in \mathbb{Z}; n \geq 0 \} \]

defined inductively from the final position (sink of graph) by

\[ G(p) = \text{mex}\{ G(q) \mid \text{if there is one move from } p \text{ to } q \}. \]

The value \( G(p) \) is also called **nim-value**.
Subtraction games

A subtraction game is a variant of Nim involving a finite set $S$ of positive integers:

- the set $S$ is called subtraction set,
- the two players alternately remove some $s$ coins provided that $s \in S$.

The subtraction game with subtraction set $\{a_1, a_2, \ldots, a_k\}$ is denoted by $S(a_1, a_2, \ldots, a_k)$. 
Nim-sequence

For each non-negative integer $n$, we denote by $G(n)$ the nim-value of the single pile of size $n$ of a subtraction game.

The sequence

$$\{G(n)\}_{n\geq 0} = G(0), G(1), G(2), \ldots$$

is called nim-sequence.
Nim-sequences of some subtraction games

\[ S(1, 2, 3) : 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, \ldots \]
\[ S(2, 3, 5) : 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, \ldots \]
\[ S(1, 5, 7) : 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots \]
\[ S(3, 5, 9) : 0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 0, 2, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots \]
Periodicity of nim-sequences

A nim-sequence is said to be ultimately periodic if there exist $N, p$ such that $G(n + p) = G(n)$ for all $n \geq N$. The smallest such number $p$ is called the period.

If $N = 0$, then the nim-sequence is said to be periodic.

$S(2, 3, 5) : 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, 2, 2, 3, 0, 0, 1, 1, \ldots$

$S(1, 5, 7) : 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$

$S(3, 5, 9) : 0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 0, 2, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots$
Periodicity of subtraction games

A game is said to be (ultimately) periodic if its nim-sequence is (ultimately) periodic.

Theorem

*a* Every subtraction game is (ultimately) periodic.

Open problem in the periodicity of subtraction games\(^2\)

**Problem**

*Given a subtraction set, describe the nim-sequence of the subtraction game.*

The question is still open for subtraction games with three element subtraction sets.

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Subtraction games agreeing nim-sequences: Examples

- $S(2, 3)$
  
  0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, ... 

- $S(2, 3, 7)$
  
  0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, ... 

- $S(2, 3, 7, 8)$
  
  0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, ... 

- $S(2, 3, 7, 8, 12)$
  
  0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, 1, 2, 0, 0, 1, ...
More examples

<table>
<thead>
<tr>
<th>Subtraction set (with optional extras)</th>
<th>nim-sequence</th>
<th>period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (3,5,7,9,\ldots)</td>
<td>010101\ldots</td>
<td>2</td>
</tr>
<tr>
<td>2 (6,10,14,18, \ldots)</td>
<td>00110011\ldots</td>
<td>4</td>
</tr>
<tr>
<td>1,2 (4,5,7,8,10,11, \ldots)</td>
<td>012012</td>
<td>3</td>
</tr>
<tr>
<td>3 (9,15,21,27, \ldots)</td>
<td>000111000111\ldots</td>
<td>6</td>
</tr>
<tr>
<td>2,3 (7,8,12,13,17,18,\ldots)</td>
<td>0011200112\ldots</td>
<td>5</td>
</tr>
<tr>
<td>2,3 (7,8,12,13,17,18,\ldots)</td>
<td>0011200112\ldots</td>
<td>5</td>
</tr>
<tr>
<td>3,6,7 (4,5,13,14,15,16,17,23,24\ldots)</td>
<td>0001112223\ldots</td>
<td>10</td>
</tr>
</tbody>
</table>
Problem

Let \( S = \{a_1, a_2, \ldots, a_k\} \) be a subtraction set. Find all integers \( a \) so that \( a \) can be added into \( S \) without changing the nim-sequence.

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Problem 3

Let $S = \{a_1, a_2, \ldots, a_k\}$ be a subtraction set. Find all integers $a$ so that $a$ can be added into $S$ without changing the nim-sequence. If we can find $a$ so that $G(n - a) \neq G(n)$ for every $n$ then $a$ can be added into the subtraction set without changing the nim-sequence.

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Problem

Let $S = \{a_1, a_2, \ldots, a_k\}$ be a subtraction set. Find all integers $a$ so that $a$ can be added into $S$ without changing the nim-sequence.

If we can find $a$ so that $G(n - a) \neq G(n)$ for every $n$ then $a$ can be added into the subtraction set without changing the nim-sequence.

For a given subtraction set $S$, we denote by $S^{\text{ex}}$ the set of all integers that can be added into $S$ without changing the nim-sequence.

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Periodic games

Let $S(s_1, s_2, \ldots, s_k)$ be a periodic subtraction game with period $p$. Then, for $1 \leq i \leq k$ and $m \geq 0$, $s_i + mp$ can be added into the subtraction set without changing the nim-sequence.

Let

$$S^p = \{ s_i + mp | 1 \leq i \leq k, m \geq 0 \}.$$ 

Then,

$$S = \{ s_1, s_2, \ldots, s_k \} \subseteq S^p \subseteq S^{ex}.$$
Periodic games

Let $S(s_1, s_2, \ldots, s_k)$ be a periodic subtraction game with period $p$. Then, for $1 \leq i \leq k$ and $m \geq 0$, $s_i + mp$ can be added into the subtraction set without changing the nim-sequence. Let

$$S^p = \{s_i + mp | 1 \leq i \leq k, m \geq 0\}.$$ 

Then,

$$S = \{s_1, s_2, \ldots, s_k\} \subseteq S^p \subseteq S^\text{ex}.$$ 

Definition

If

$$S^p = S^\text{ex}$$ 

then the subtraction set $S$ is said to be non-expandable. Otherwise, $S$ is expandable and $S^\text{ex}$ is called the expansion of $S$. 
The first simple case: $S = \{a\}$

The singleton subtraction set is non-expandable.
The second simple case: \( S = \{a, b\} \)

Let \( a < b \ (gcd(a, b) = 1, \ b \text{ is not an odd multiple of } a) \). Consider the subtraction set \( \{a, b\} \).

- If \( a + 1 < b \leq 2a \), then the subtraction set has expansion
  \[
  \{a, a + 1, \ldots, b\}^{(a+b)}. 
  \]

- If either \( a = 1 \), or \( b = a + 1 \), or \( b > 2a \), then the subtraction set is non-expandable.
More examples: $S = \{1, a, b\}$

**Example 1:**
Let $a \geq 2$ be an even integer. The subtraction game $S(1, a, 2a + 1)$ is periodic and the subtraction set is non-expandable.

**Example 2:**
Let $a < b$ such that $a$ is odd, $b$ is even. The subtraction set $\{1, a, b\}$ is expandable with the expansion

$$\{\{1, 3, \ldots, a\} \cup \{b, b + 2, \ldots, b + a - 1\}\}^{(a+b)}.$$
Ultimately periodic games

Let $S(s_1, s_2, \ldots, s_k)$ be an ultimately periodic subtraction game with period $p$.

Note that the inclusion $S^p \subseteq S^\text{ex}$ does not necessarily hold.

**Definition**

If $S^\text{ex} = S$ then the subtraction set $S$ is non-expandable. Otherwise, $S^\text{ex}$ is called the expansion of $S$. 
An example

Let $a \geq 4$ be an even integer. The subtraction game $S(1, a, 3a - 2)$ is ultimately periodic with period $3a - 1$.

- If $a = 4$ then the subtraction set is non-expandable,
- otherwise, the subtraction set has expansion

$$\{1, a, 3a - 2, 3a\} \cup \{4a - 1, 6a - 1\}^{3a-1}.$$
Ultimately bipartite subtraction games

A subtraction game is said to be **ultimately bipartite** if its nim-sequence is ultimately periodic with period 2 with, for sufficiently large $n$, alternating nim-values $0, 1, 0, 1, 0, 1, \ldots$. 

Some ultimately bipartite subtraction games:

- $S(3, 5, 9, \ldots, 2^k + 1)$, for $k \geq 3$,
- $S(3, 5, 2^k + 1)$, for $k \geq 3$,
- $S(a, a + 2, 2a + 3)$, for odd $a \geq 3$,
- $S(a, 2a + 1, 3a)$, for odd $a \geq 5$.

Example: $S(3, 5, 9)$:

0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 0, 2, 0, 1, 0, 1, 0, 1, \ldots
Ultimately bipartite subtraction games

A subtraction game is said to be **ultimately bipartite** if its nim-sequence is ultimately periodic with period 2 with, for sufficiently large \( n \), alternating nim-values 0, 1, 0, 1, 0, 1, . . . .

Some ultimately bipartite subtraction games:

- \( S(3, 5, 9, \ldots, 2^k + 1) \), for \( k \geq 3 \),
- \( S(3, 5, 2^k + 1) \), for \( k \geq 3 \),
- \( S(a, a + 2, 2a + 3) \), for odd \( a \geq 3 \),
- \( S(a, 2a + 1, 3a) \), for odd \( a \geq 5 \).
Ultimately bipartite subtraction games

A subtraction game is said to be **ultimately bipartite** if its nim-sequence is ultimately periodic with period 2 with, for sufficiently large $n$, alternating nim-values $0, 1, 0, 1, 0, 1, \ldots$.

Some ultimately bipartite subtraction games:

- $S(3, 5, 9, \ldots, 2^k + 1)$, for $k \geq 3$,
- $S(3, 5, 2^k + 1)$, for $k \geq 3$,
- $S(a, a + 2, 2a + 3)$, for odd $a \geq 3$,
- $S(a, 2a + 1, 3a)$, for odd $a \geq 5$.

**Example:**

$S(3, 5, 9): 0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 0, 2, 0, 1, 0, 1, 0, 1, \ldots$
A conjecture

The subtraction set of an ultimately bipartite game is non-expandable.
For more details
