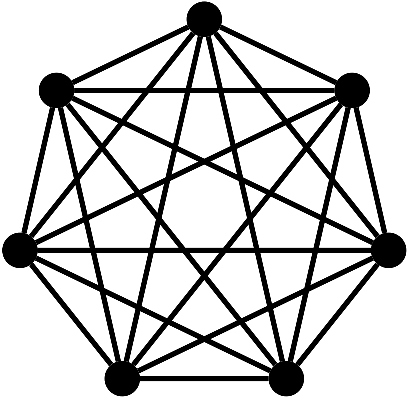
Embeddings of partial Steiner triple systems: best-possible and better

Daniel Horsley (Monash University)

Steiner triple systems

Steiner triple systems



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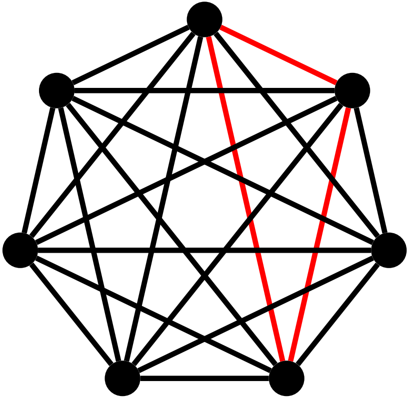

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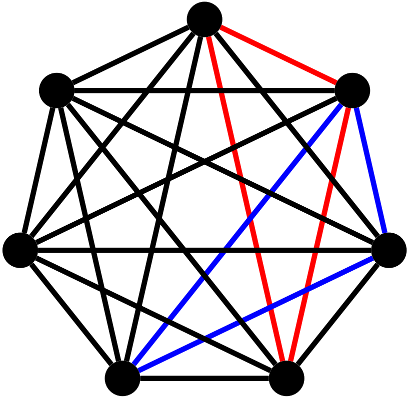

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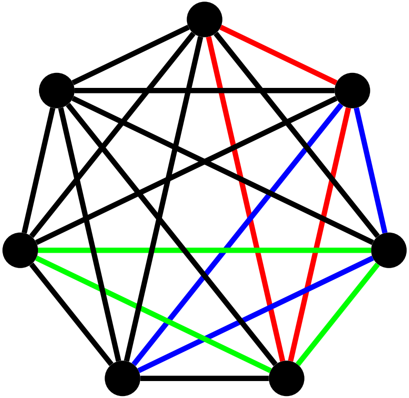

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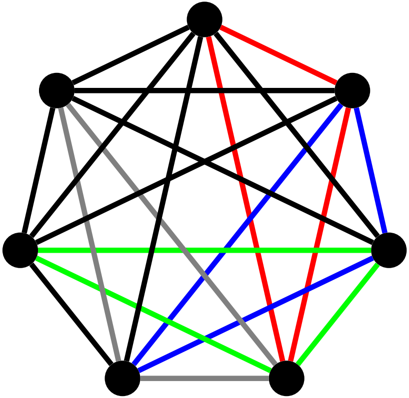

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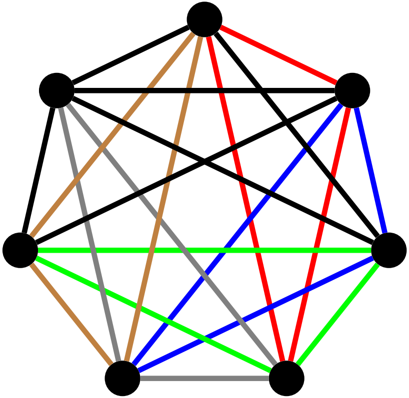

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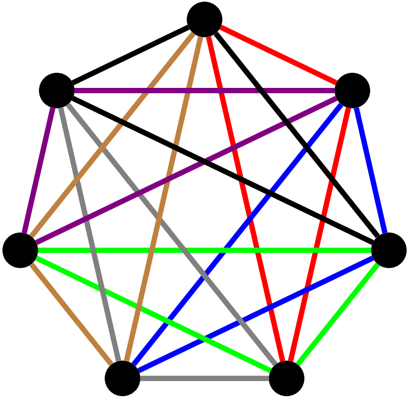

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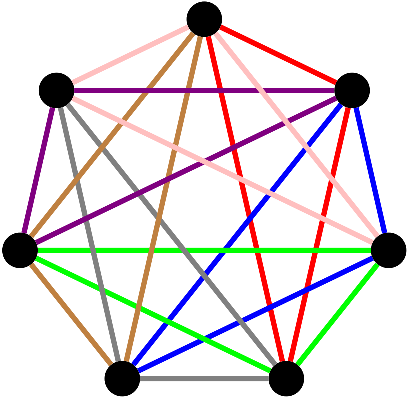

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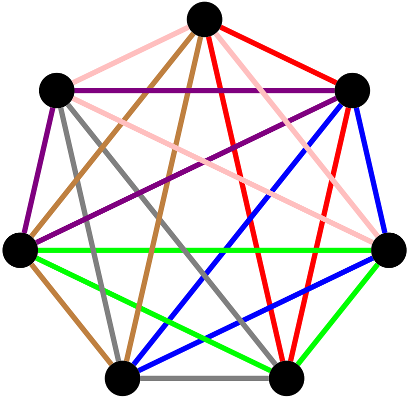

An STS(7)

Existence of Steiner triple systems

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Theorem (Kirkman 1847) An STS(v) exists if and only if $v \geq 1$ and $v \equiv 1, 3 \pmod{6}$.

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Call these integers *admissible*.

Partial Steiner triple systems

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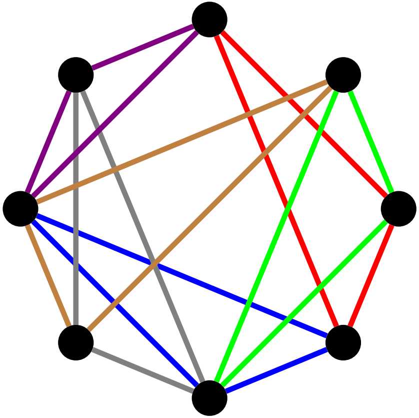
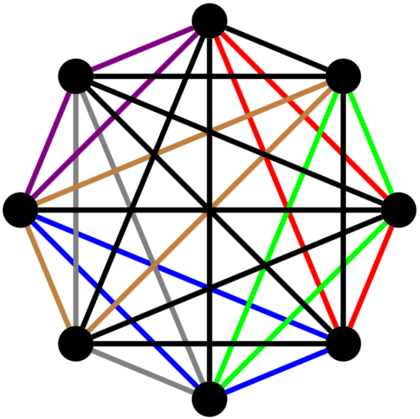


A PSTS(8)

Partial Steiner triple systems



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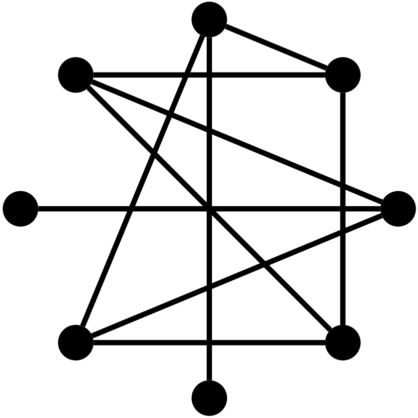

The leave of the PSTS(8)

Completing partial Steiner triple systems

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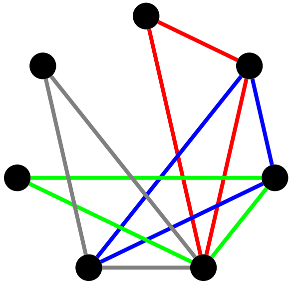

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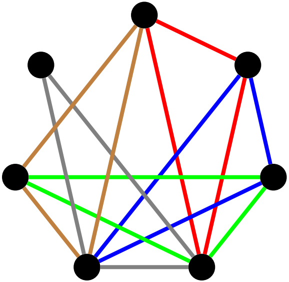

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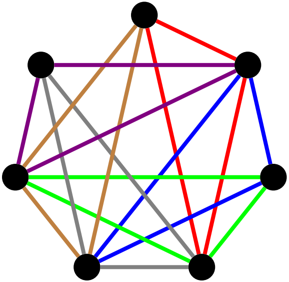

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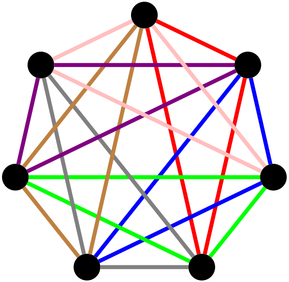

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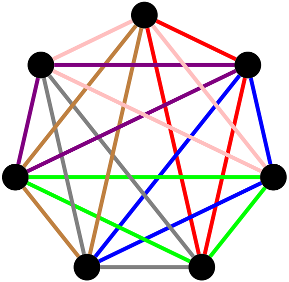
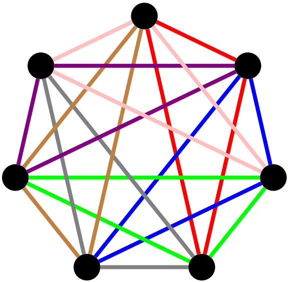


A completion of the PSTS(7)

Completing partial Steiner triple systems



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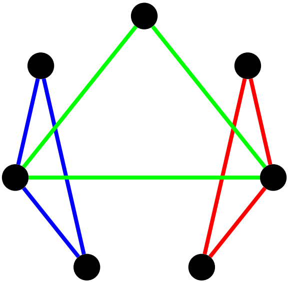

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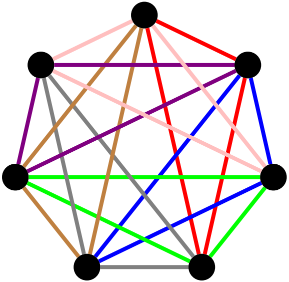

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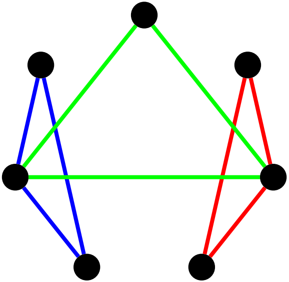

A PSTS(7) with no completion

Embeddings of partial Steiner triple systems

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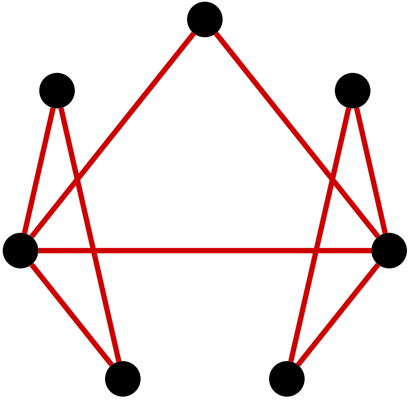

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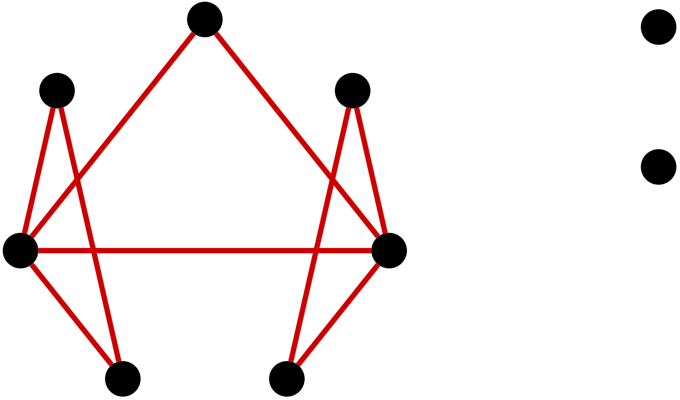

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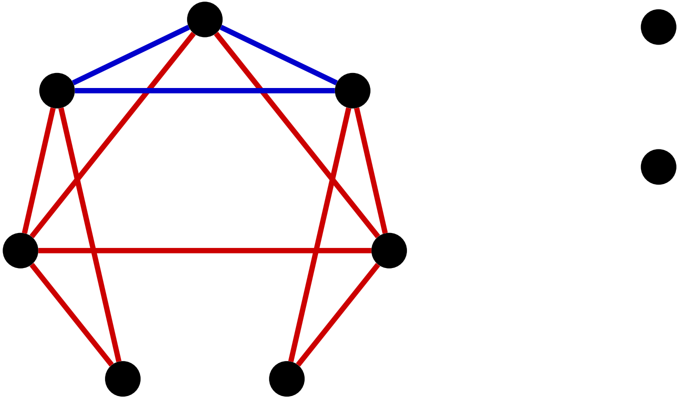

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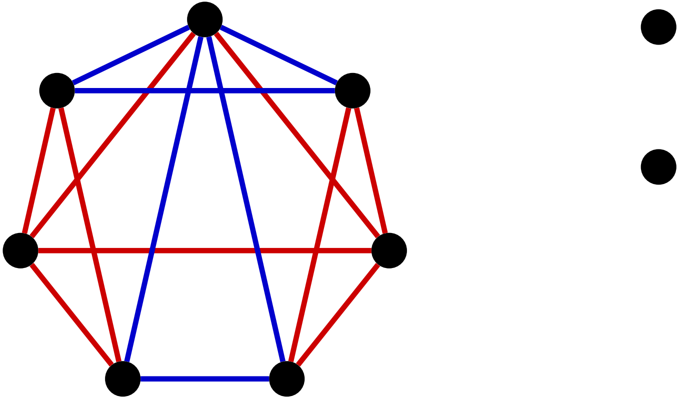

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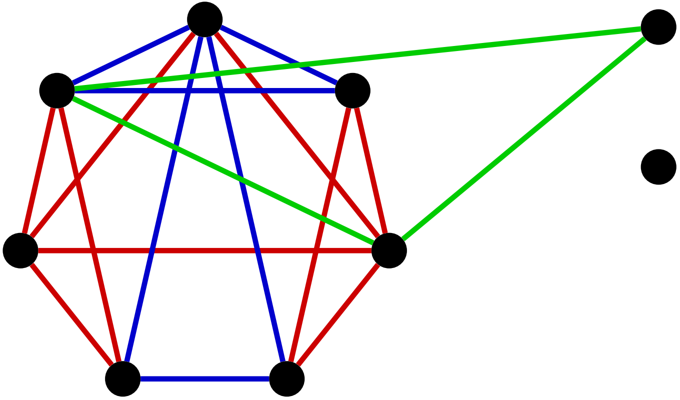

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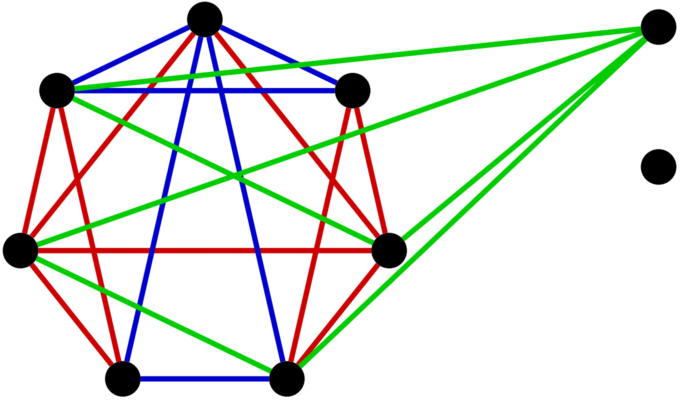

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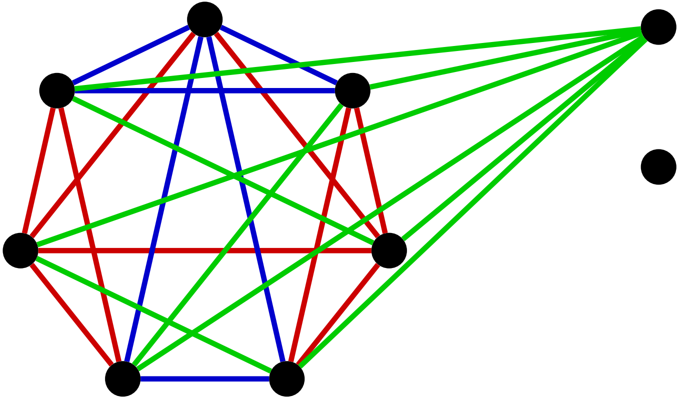

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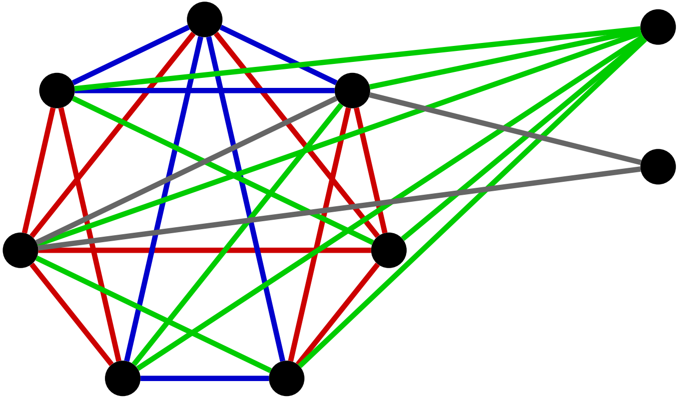

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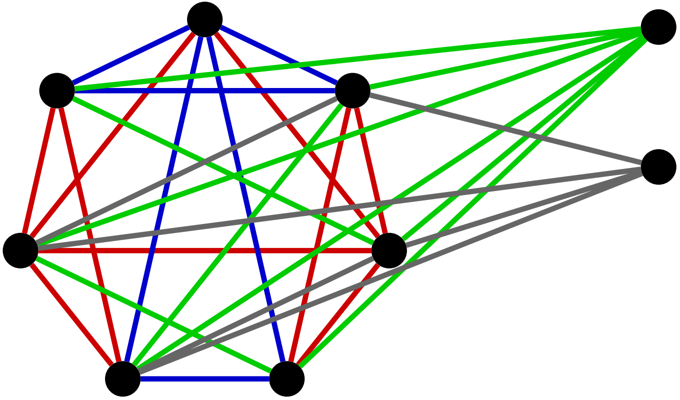

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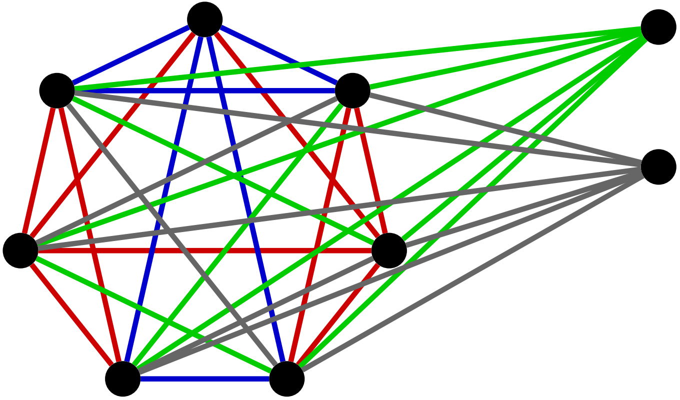

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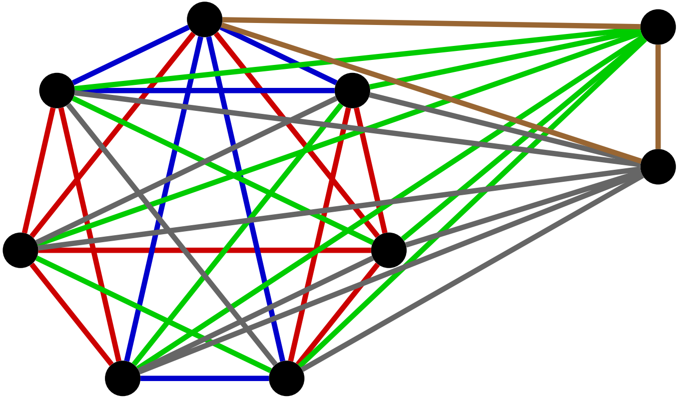

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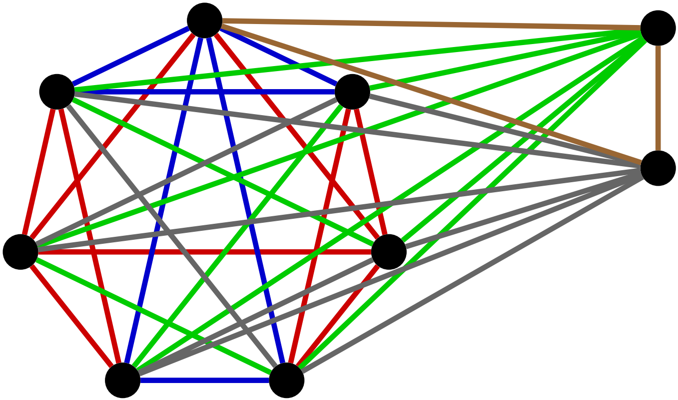
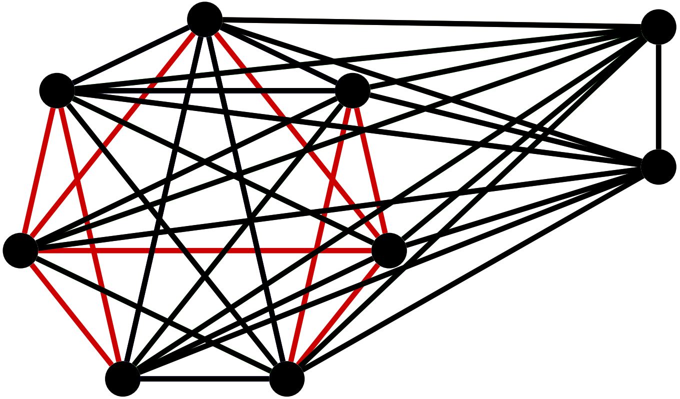


An embedding of the PSTS(7) of order 9

Embeddings of partial Steiner triple systems



$L \vee K_2$ was decomposed into triangles

The blind problem

The blind problem

Problem For what values of v does *every* $\mathrm{PSTS}(u)$ have an embedding of order v?

Small embeddings may not exist

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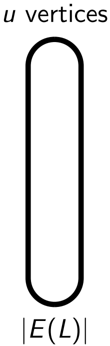

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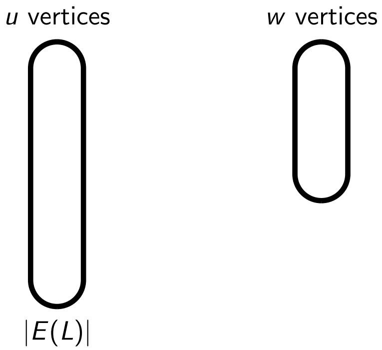

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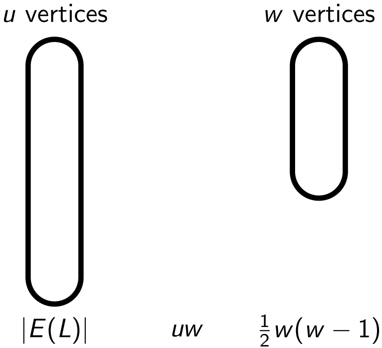

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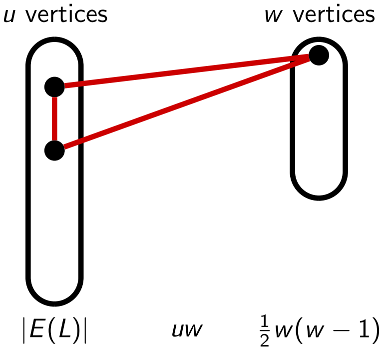

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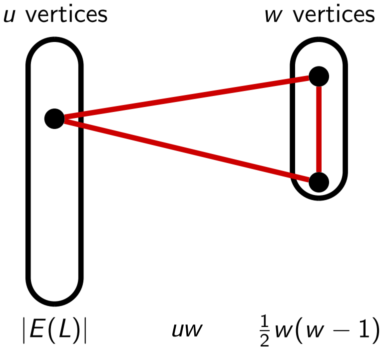

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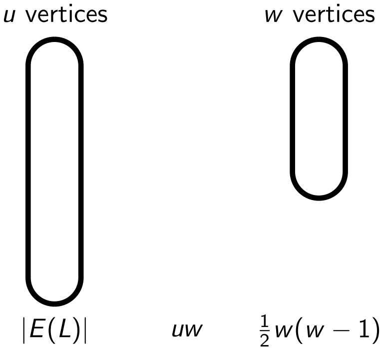

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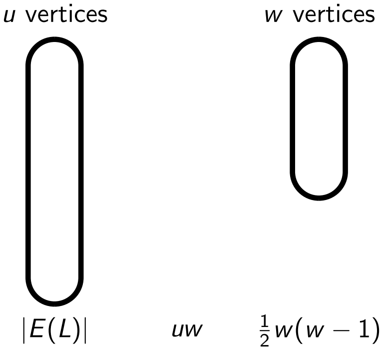

$uw \leq 2(|E(L)| + \frac{1}{2}w(w-1))$

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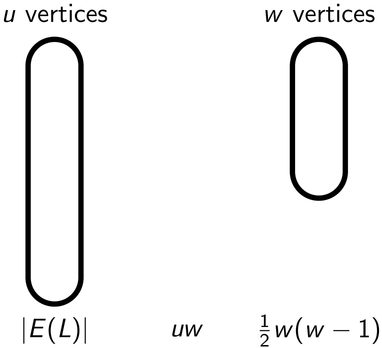

$$uw \leq 2(|E(L)| + \tfrac{1}{2}w(w-1)) \quad \Rightarrow \quad |E(L)| \geq \tfrac{1}{2}w(u-w+1)$$

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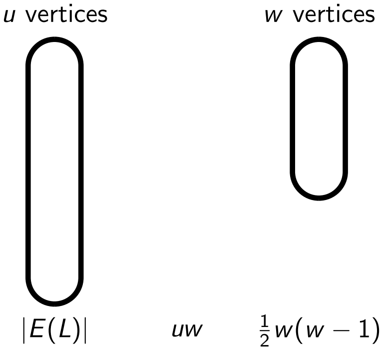

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If $|E(L)| = 6$, then we must have $w = 0$ or $w \geq u + 1$ (for decent size u).

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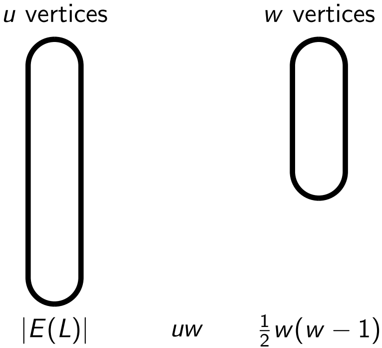

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Result For each $u > 9$ there is a PSTS(u) which does not have any embedding of order smaller than $2u + 1$.

Results on the blind problem

Treash (1971) Every PSTS has an embedding (of order at most 2^{2u}).

Lindner (1975) Every PSTS(u) has an embedding of order $6u+3$.

Conjecture (Lindner 1977) Every PSTS(u) has an embedding of order v for each admissible $v \geq 2u+1$.

Andersen, Hilton, Mendelsohn (1980) Every PSTS(u) has an embedding of order v for each admissible $v \geq 4u+1$.

Bryant (2004) Every PSTS(u) has an embedding of order v for each admissible $v \geq 3u-2$.

Bryant, Horsley (2009) Every PSTS(u) has an embedding of order v for each admissible $v \geq 2u+1$.

The sighted problem

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Problem Given a PSTS, for what values of v does it have an embedding of order v?

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Call embeddings of order less than $2u+1$ *small*.

Some necessary conditions

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Result Suppose there is an embedding of a PSTS(u) of order $u + w$. If L is the leave of the PSTS, then there is a subgraph L' of L such that

- $L - L'$ has a decomposition into triangles;
- $\chi'(L') \leq w$; and
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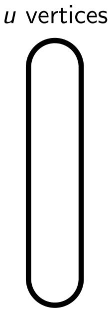

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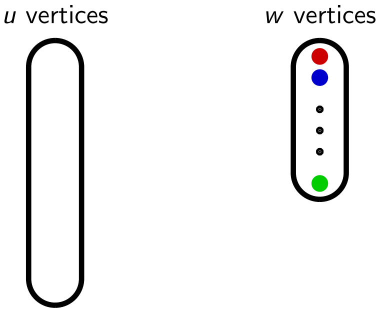

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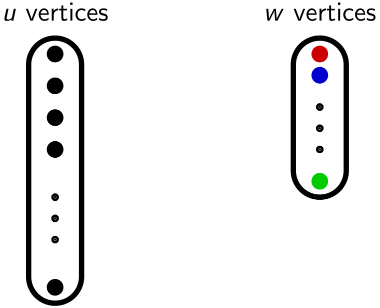

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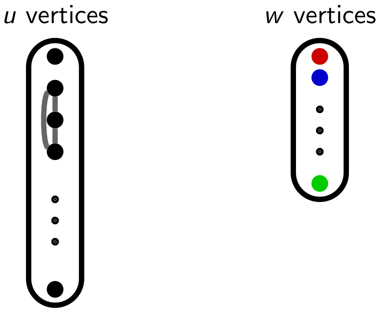

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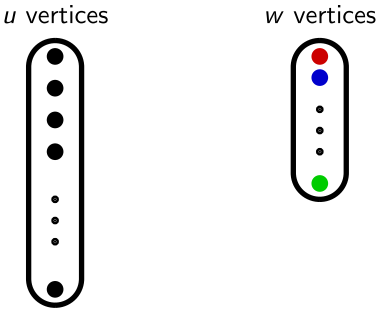

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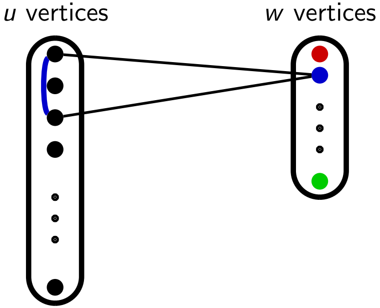

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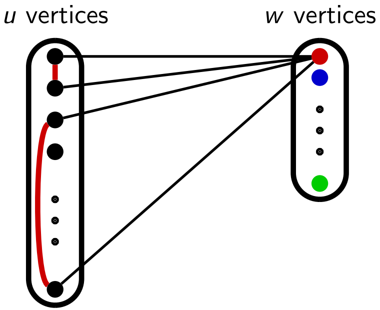

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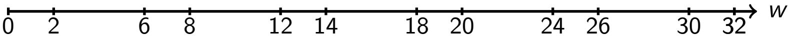

$u = 31$

A closer look at those conditions

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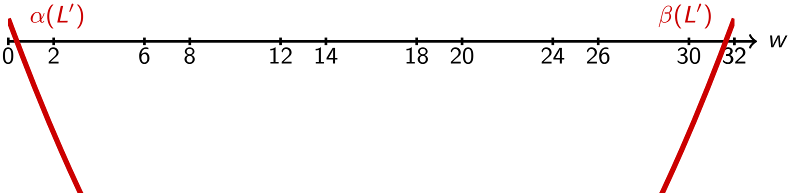

$u = 31$ $|E(L')| = 6$

A closer look at those conditions

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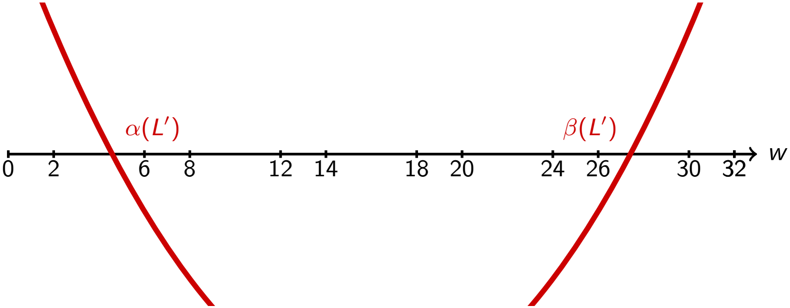

$u = 31$ $|E(L')| = 63$

A closer look at those conditions

Result Suppose there is an embedding of a PSTS(u) of order $u + w$. If L is the leave of the PSTS, then there is a subgraph L' of L such that

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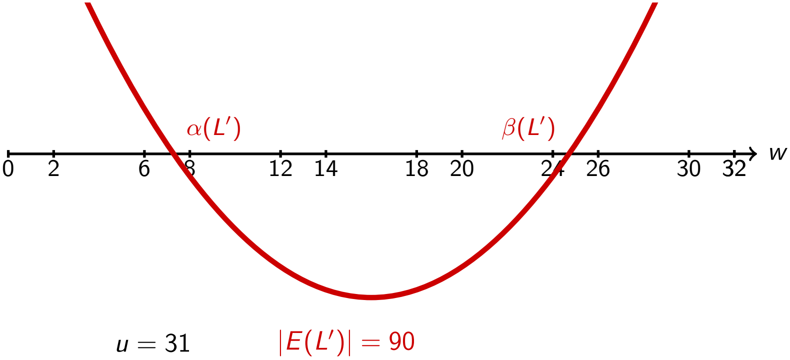

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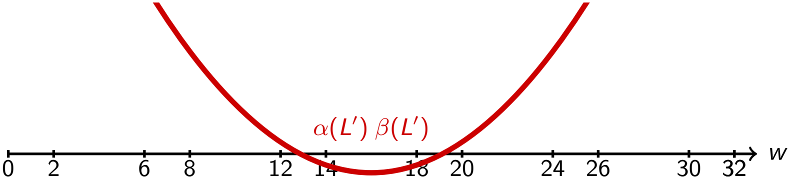

$u = 31$ $|E(L')| = 123$

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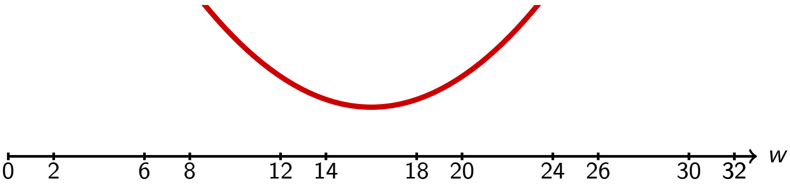

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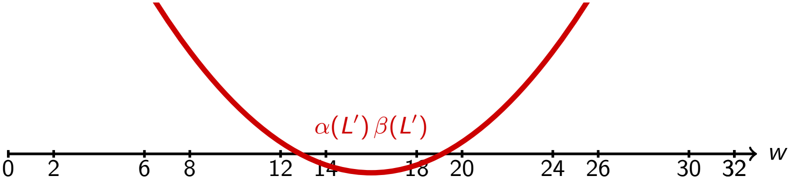

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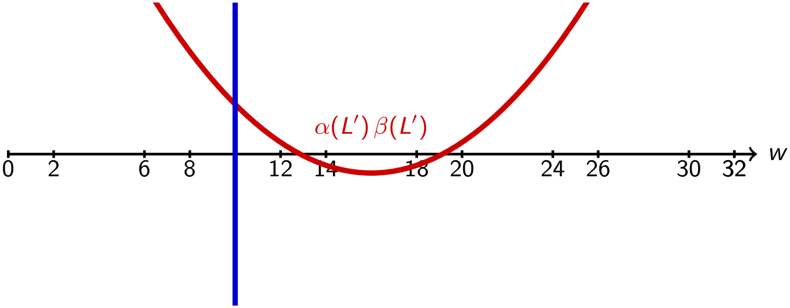

$u = 31$ $|E(L')| = 123$ $\chi'(L') = 10$

A closer look at those conditions

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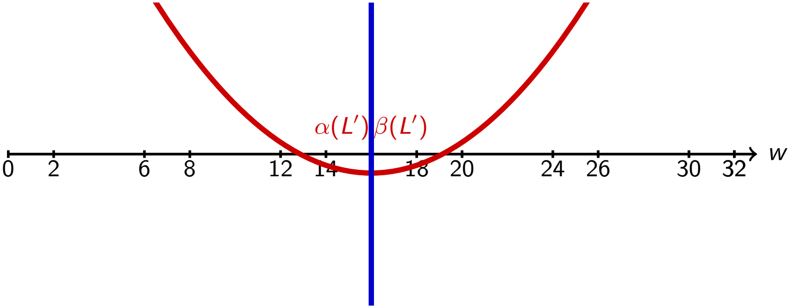

$u = 31$ $|E(L')| = 123$ $\chi'(L') = 16$

A closer look at those conditions

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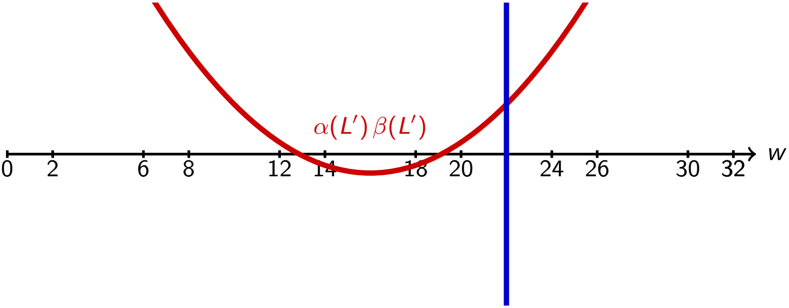

$u = 31$ $|E(L')| = 123$ $\chi'(L') = 22$

Finding embeddings is NP-complete

Finding embeddings is NP-complete

Colbourn (1983) constructed a class C of PSTSs such that

- it is NP-complete to determine whether a member of C has a completion;
- no member of C can be embedded in an STS(v) for any $u < v < 2u + 1$.

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- it is NP-complete to determine whether a member of C has a completion;
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This means that determining whether a PSTS has an embedding of order less than $2u + 1$ is NP-complete.

Results on the sighted problem

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Doyen, Wilson (1973) A (complete) STS(u) has an embedding of order v if and only if $v = u$ or $v \geq 2u + 1$.

Colbourn, Colbourn, Rosa (1983) Showed “double stars” are completable.

Bryant (2002) Gave necessary and sufficient conditions for a PSTS with a $\{0, d\}$-regular leave to have an embedding of order $u + d$. Determined the embedding spectrum in the case $d = 2$.

Bryant, Maenhaut, Quinn, Webb (2004) Determined the embedding spectrum for all PSTS(10)s with 3-regular leaves.

Bryant, Horsley (2006) Determined the embedding spectrum for PSTSs with complete bipartite leaves.

A new result

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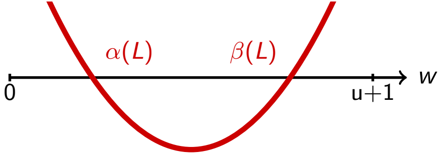
Theorem Any PSTS(u) with a leave L such that $\Delta(L) \leq \frac{1}{4}(u-9)$ and $|E(L)| < \frac{1}{32}(u-5)(u-11)+2$ has an embedding of order $u+w$ for each integer w such that $w \geq \beta(L)$ and $u+w$ is admissible.



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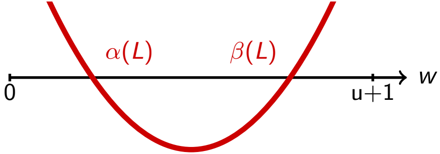

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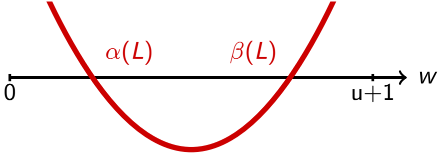

The condition on $|E(L)|$ always holds when $\Delta(L) \leq \frac{u}{16} - 1$.

When $|E(L)| \approx \frac{1}{32}(u-5)(u-11)+2$, we have
$\alpha(L) \approx \frac{1}{4}(2-\sqrt{3})(u+1) \approx 0.07(u+1)$,
$\beta(L) \approx \frac{1}{4}(2+\sqrt{3})(u+1) \approx 0.93(u+1)$.

Finding embedding spectra: neighbourhoods

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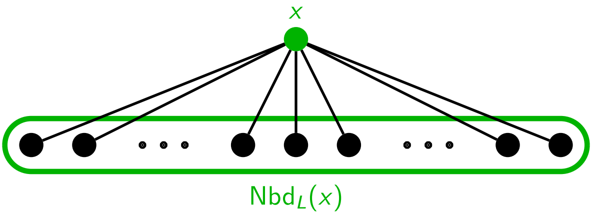

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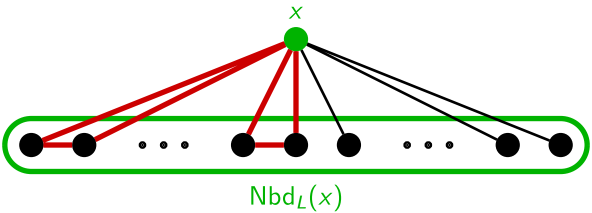

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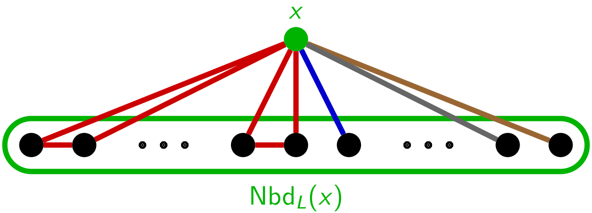

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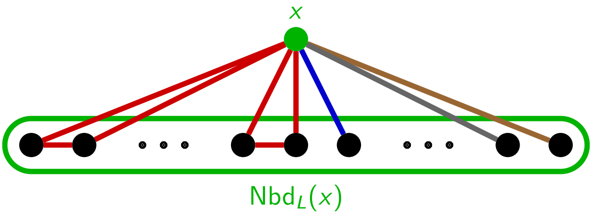

Example A PSTS(201) with $|E(L)| = 1165$, $\Delta(L) = 40$, and a vertex x such that $\deg_L(x) = 40$ and a largest matching in $\mathrm{Nbd}_L(x)$ has 13 edges.

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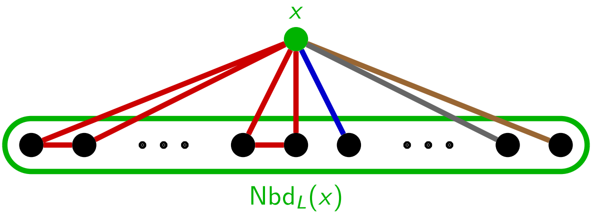

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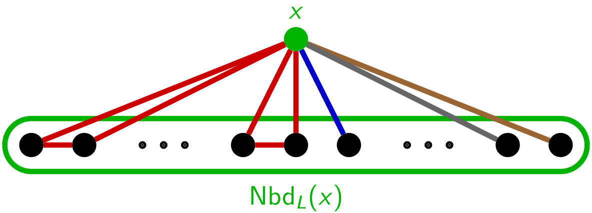

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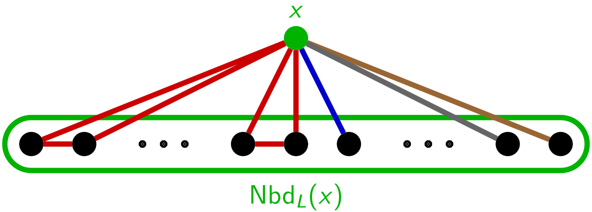

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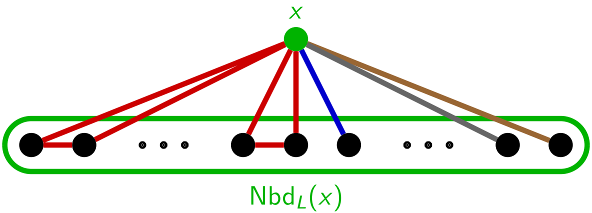

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Lemma A PSTS(u) has no embeddings of order at most $u + \alpha(L)$ if $|E(L)| < \frac{1}{2}c(u - c + 1)$, where $c = \max\{\deg_L(x) - 2\psi_L(x) : x \in V(L)\}$ and $\psi_L(x)$ is the size of a largest matching in $\text{Nbd}_L(x)$.

Finding embedding spectra: isolated vertices

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A more general result covering leaves with many vertices of small degree can be obtained.

Final thoughts

- isolating NP-completeness
- partially sighted questions

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Not a theorem yet Any PSTS(u) with $u \geq 57$ and at most $\frac{1}{60}u(u-1)$ triples has an embedding of order $u + w$ for each integer w such that $w \geq \frac{1}{5}(3u+17)$ and $u + w$ is admissible.

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There is a PSTS(u) with about $\frac{1}{60}u(u-1)$ triples which does not have any embedding of order $u+w$ for $w \leq 0.31(u+1)$.

The End