Extending Fisher's inequality to coverings

Daniel Horsley (Monash University)

(v, k, λ)-designs

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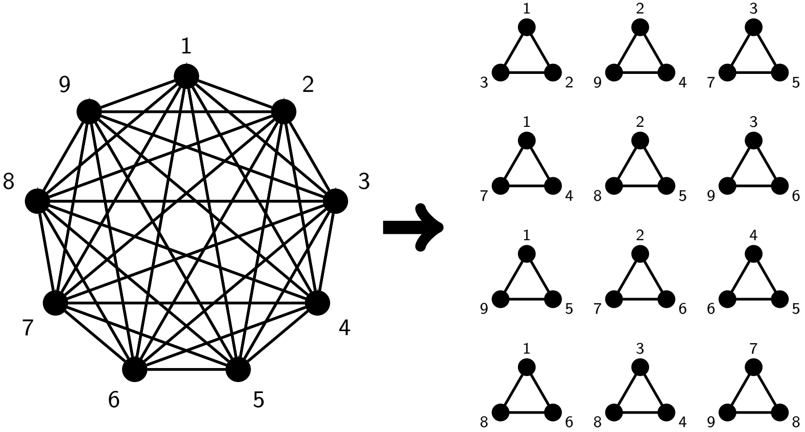

A $(9, 3, 1)$-design with 12 blocks

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(1) $r = \dfrac{\lambda(v-1)}{k-1}$ is an integer;

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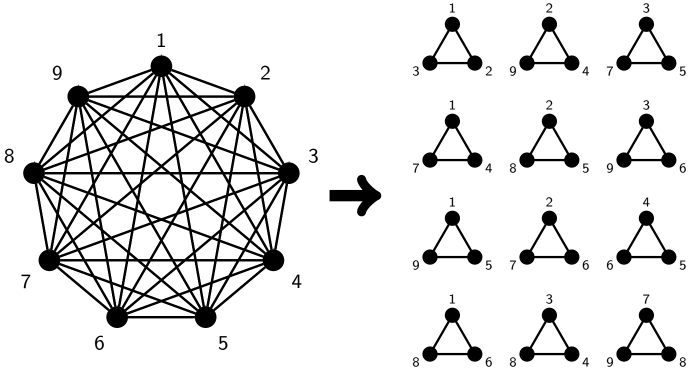

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- has $r \geq k$; or
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- has $r \geq k$; or
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We say parameter sets (v, k, λ) with $v < \frac{k(k-1)}{\lambda} + 1$ are *subsymmetric*.

Incidence matrix arithmetic

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Consider the incidence matrix of our $(9, 3, 1)$-design.

12 blocks

9 points

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{matrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{matrix}$$

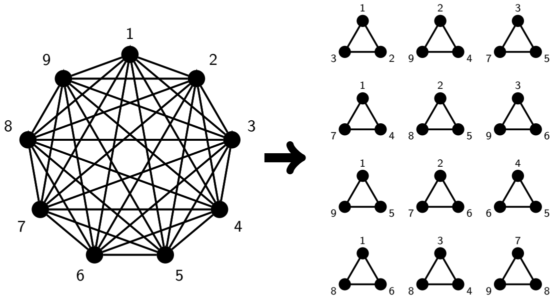
$$\begin{matrix} 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{matrix}$$

Incidence matrix arithmetic

Consider the incidence matrix of our $(9, 3, 1)$-design.

12 blocks (columns) × 9 points (rows)

1	0	0	1	0	0	1	0	0	1	0	0	4
1	1	0	0	1	0	0	1	0	0	0	0	4
1	0	1	0	0	1	0	0	0	0	1	0	4
0	1	0	1	0	0	0	0	1	0	1	0	4
0	0	1	0	1	0	1	0	1	0	0	0	4
0	0	0	0	0	1	0	1	1	1	0	0	4
0	0	1	1	0	0	0	1	0	0	0	1	4
0	0	0	0	1	0	0	0	0	1	1	1	4
0	1	0	0	0	1	1	0	0	0	0	1	4
3	3	3	3	3	3	3	3	3	3	3	3	



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1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
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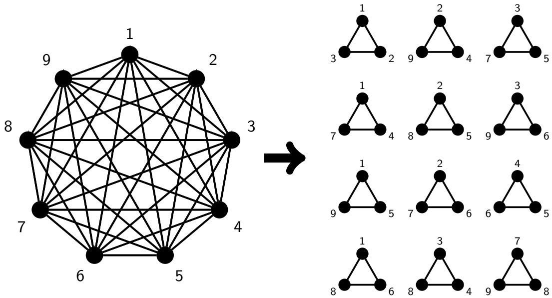

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1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
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$$AA^T = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

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AA^T is 9×9 and has rank 9.

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- So AA^T is the 21×21 matrix

$$AA^T = \begin{pmatrix} 4 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 4 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 4 & & 1 & 1 & 1 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 1 & 1 & 1 & & 4 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 4 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 4 \end{pmatrix}.$$

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- So AA^T has rank 21. Contradiction.

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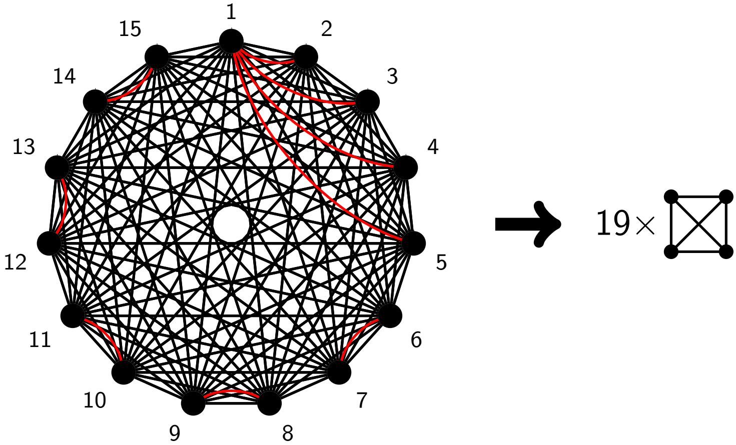

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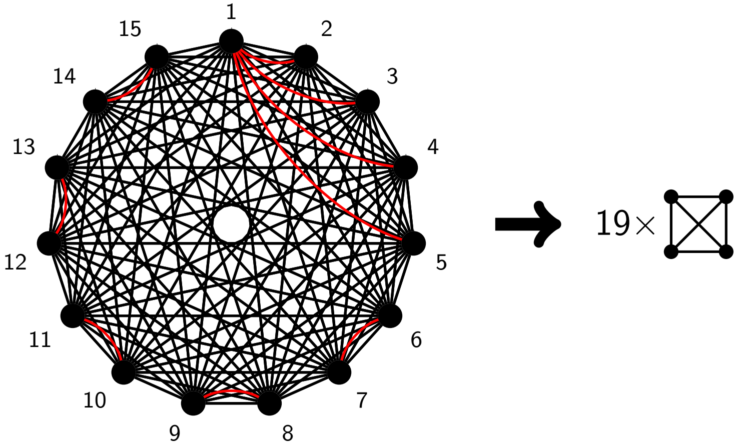

A $(15, 4, 1)$-covering with 19 blocks

The degree of a point in the excess is determined by the number of blocks on that point.

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Obvious Necessary Conditions. If there exists an (v, k, λ)-covering then

(1) for each point x the number of blocks containing x satisfies

$$r_x \geq r \quad \text{where} \quad r = \left\lceil \frac{\lambda(v-1)}{k-1} \right\rceil ;$$

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These conditions are probably sufficient for $v \gg k$.

Improvements to the Schönheim bound

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Improvements to the Schönheim bound

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- Fisher's inequality and the Bruck-Ryser-Chowla theorem improve the Schönheim bound by 1 in some cases where a covering meeting the bound would be a design.
- Bose and Connor improved the Schönheim bound by 1 in some cases where $\lambda(v-1) = -1 \pmod{k-1}$.
- For $\lambda = 1$, Todorov has improved the Schönheim bound in various cases:
  - $v = rk + 1$;
  - some cases where $v - 1 = r(k-1)$;
  - some cases where $k > O(v^{\frac{5}{7}})$.
- Bryant, Buchanan, Horsley, Maenhaut and Scharaschkin improved the Schönheim bound by 1 in some cases where $\lambda(v-1) = -1, -2 \pmod{k-1}$.

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New bound. If $r(k-2) < \lambda(v-2)$, then $b \geq \left\lceil \frac{v(r+1)}{k+1} \right\rceil$.

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The new bound is at least as good as the Schönheim bound for subsymmetric parameter sets, and never an improvement otherwise.

Comparison of bounds for $k = 16$, $\lambda = 1$

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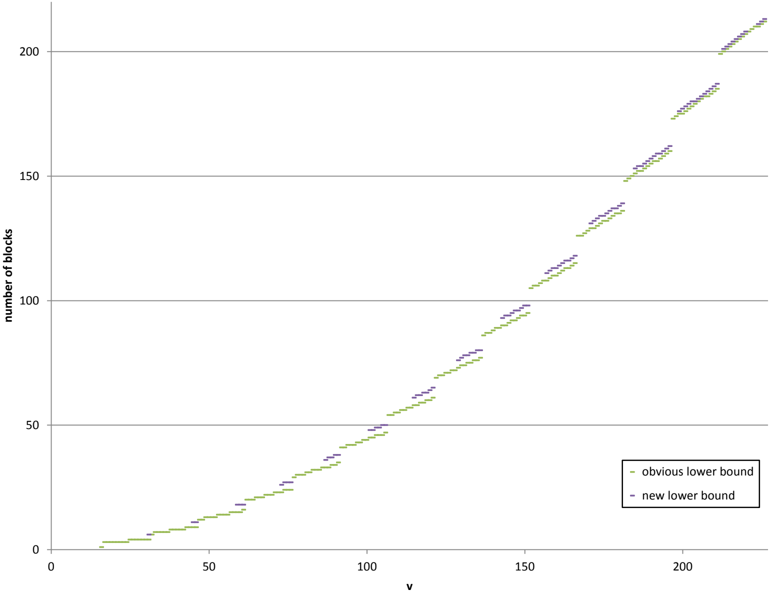

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- Note $r = 13$. It must be that $r_x = 13$ for at least 169 points.

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Suppose there exists a $(176, 15, 1)$-covering with 153 blocks.

- Note $r = 13$. It must be that $r_x = 13$ for at least 169 points.
- The incidence matrix A is 176×153. Row x adds to r_x. Distinct rows x and y have dot product $1 + \mu_E(xy)$.

New bound proof example

Suppose there exists a $(176, 15, 1)$-covering with 153 blocks.

- Note $r = 13$. It must be that $r_x = 13$ for at least 169 points.
- The incidence matrix A is 176×153. Row x adds to r_x. Distinct rows x and y have dot product $1 + \mu_E(xy)$.
- So $AA^T - J$ is 176×176, symmetric, looks like

$$\left(\begin{array}{ccccc} \geqslant 13 & & & & \\ & \ddots & & & \\ & & \geqslant 13 & & \\ \hline & & 12 & & \\ & & & \ddots & \\ & & & & \ddots \\ & & & & 12 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \geqslant 20 \\ \vdots \\ \geqslant 20 \end{array}\right\} \leqslant 7 \text{ rows} \\ \left.\begin{array}{c} 7 \\ \vdots \\ \vdots \\ 7 \end{array}\right\} \geqslant 169 \text{ rows} \end{array}$$

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Suppose there exists a $(176, 15, 1)$-covering with 153 blocks.

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$$
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\geqslant 13 & & & & \\
& \ddots & & & \\
& & \geqslant 13 & & \\
\hline
& & 12 & & \\
& & & \ddots & \\
& & & & \ddots \\
& & & & 12
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- So $AA^T - J$ has rank at least 169 and thus AA^T has rank at least 168. Contradiction.

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- So $AA^T - J$ has rank at least 169 and thus AA^T has rank at least 168. Contradiction.
- The condition $r(k-2) < \lambda(v-2)$ corresponds to $12 > 7$.

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More results:

- Some improvements to the new bounds for $r(k-2) < \lambda(v-2)$.
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Future work:

- Can coverings that meet these bounds be constructed?

That's all.