Symmetric coverings and the Bruck-Ryser-Chowla theorem

Daniel Horsley (Monash University, Australia)

Joint work with
Darryn Bryant, Melinda Buchanan, Barbara Maenhaut and Victor Scharaschkin
and with
Nevena Francetić and Sara Herke

Part 1:

The Bruck-Ryser-Chowla theorem

Symmetric designs

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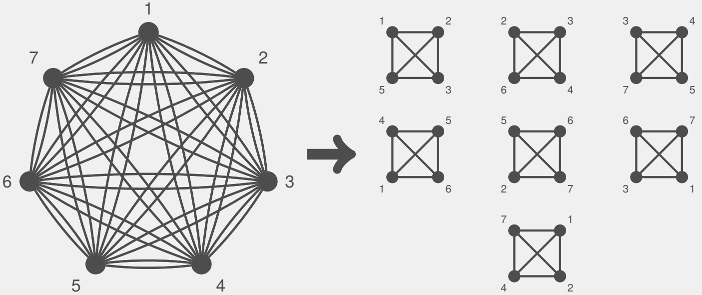

A symmetric $(7, 4, 2)$-design

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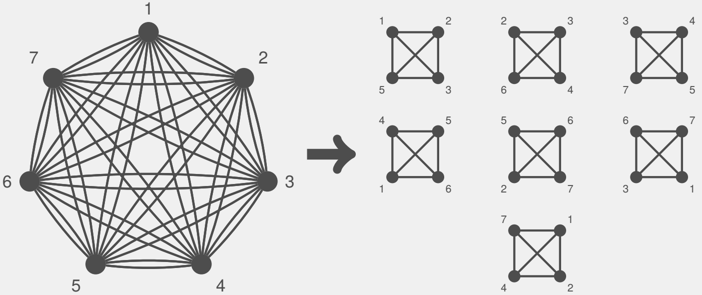

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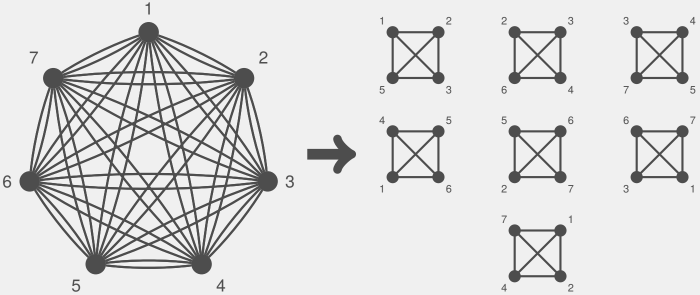

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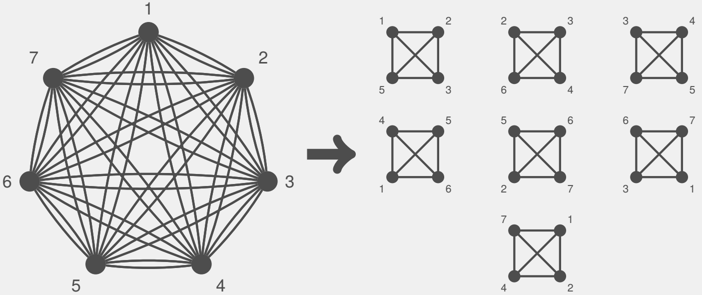

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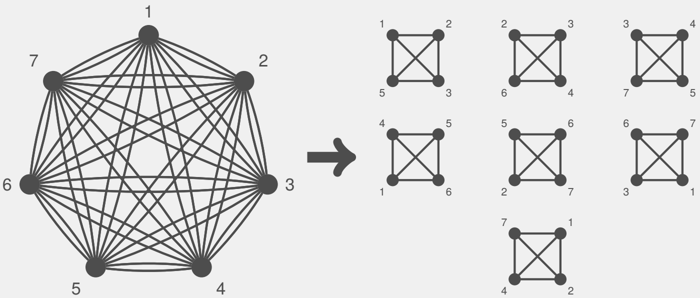

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A symmetric (v, k, λ)-design has $v = \frac{k(k-1)}{\lambda} + 1$.

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Bruck-Ryser-Chowla theorem (1950)

If a symmetric (v, k, λ)-design exists then

- if v is even, then $k - \lambda$ is square; and
- if v is odd, then $x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$ has a solution for integers x, y, z, not all zero.

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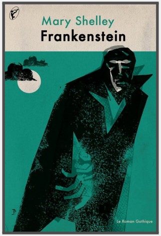

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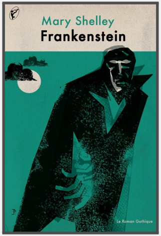

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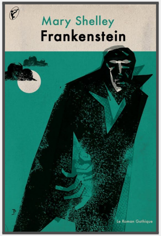

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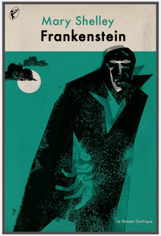

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\lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
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- $|MM^T| = |M|^2$ is square; and
- $MM^T \sim I$ (MM^T is *rationally congruent* to I).

($A \sim B$ if $A = QBQ^T$ for an invertible rational matrix Q.)

Part 2:

Extending BRC to coverings

Pair coverings

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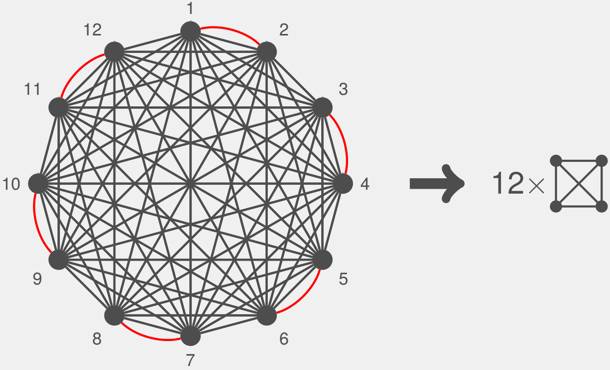

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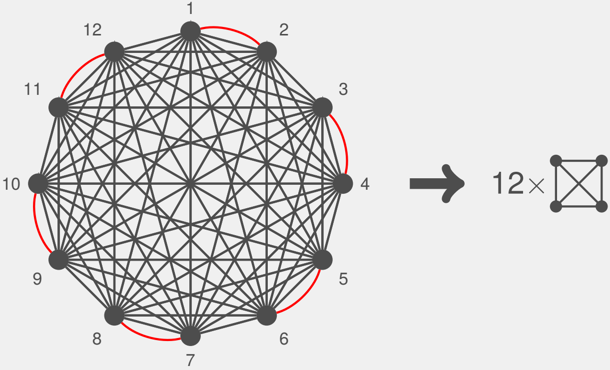

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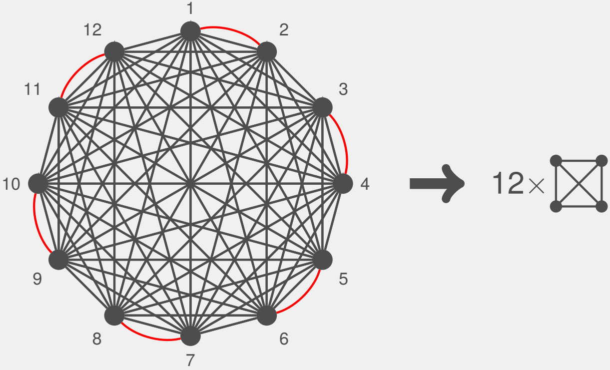

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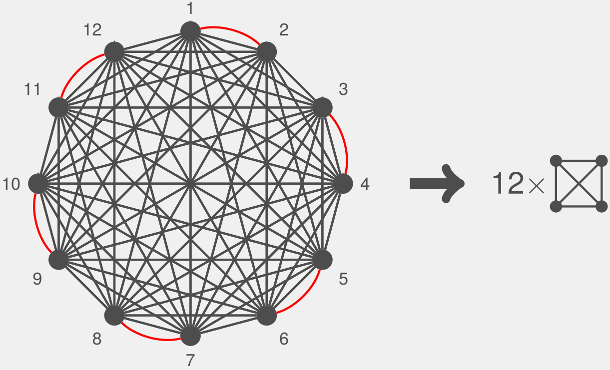

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BRC results for coverings

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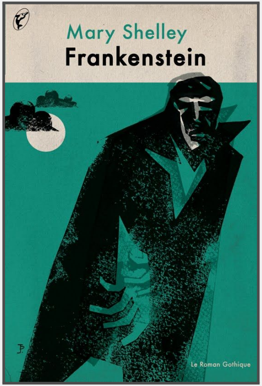
The Bruck-Ryser-Chowla theorem establishes the non-existence of certain symmetric coverings with empty excesses.



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Bose and Connor (1952) used similar methods to establish the non-existence of certain symmetric coverings with 1-regular excesses.

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k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda+1 & k & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
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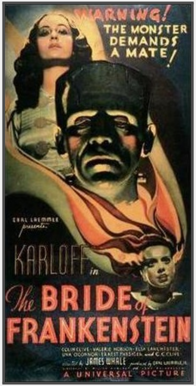

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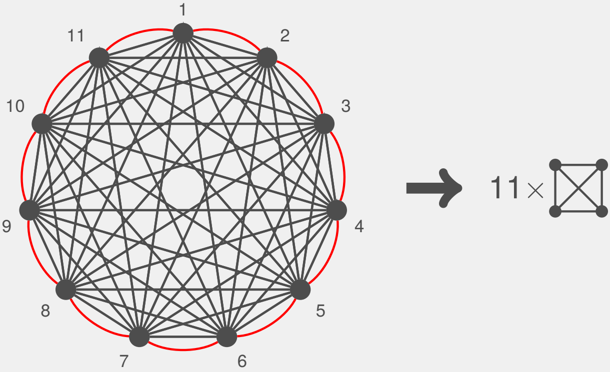

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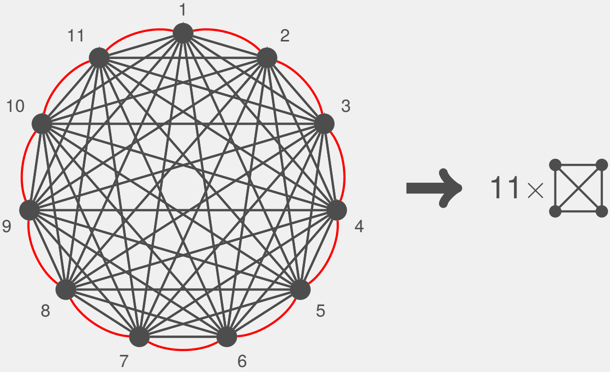

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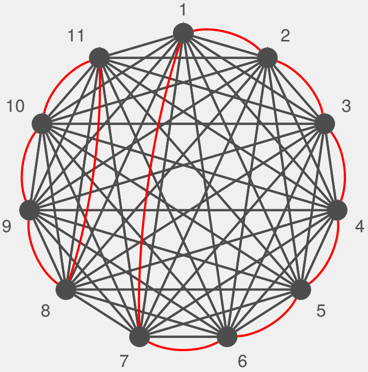

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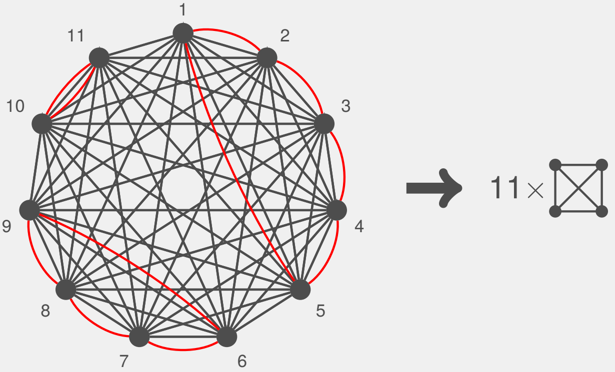

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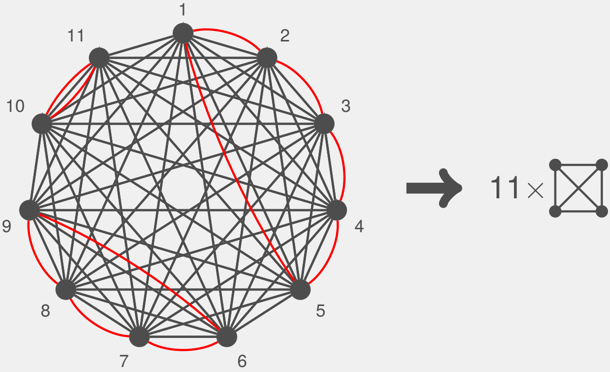

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When $v = \frac{k(k-1)-2}{\lambda} + 1$, a symmetric (v, k, λ)-covering must have a 2-regular excess.

The rest of this talk is about nonexistence of symmetric coverings with 2-regular excesses.

Degenerate coverings

Degenerate coverings

There is a $(v, v-2, v-4)$-symmetric covering with excess D for every $v \geqslant 5$ and every 2-regular graph D on v vertices.

(It has block set $\{V \setminus \{x, y\} : xy \in E(D)\}$.)

What does MM^T look like now?

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If M is the incidence matrix of a $(11, 4, 1)$-covering with excess [11],

$$MM^T = \begin{pmatrix} k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 \\ \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 \\ \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k \end{pmatrix}.$$

We call this matrix $X_{(11,4,1)}[11]$.

What does MM^T look like now?

If M is the incidence matrix of a $(11, 4, 1)$-covering with excess $[7, 4]$,

$$MM^T = \begin{pmatrix} k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & k & \lambda+1 & \lambda & \lambda+1 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & k & \lambda+1 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda+1 & k \end{pmatrix}.$$

We call this matrix $X_{(11,4,1)}[7, 4]$.

What does MM^T look like now?

If M is the incidence matrix of a $(11,4,1)$-covering with excess $[6,3,2]$,

$$MM^T = \begin{pmatrix}
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\end{pmatrix}.$$

We call this matrix $X_{(11,4,1)}[6,3,2]$.

Determinant results (with BBM&S)

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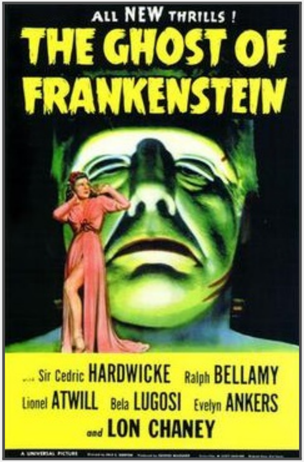

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Based around the observation that $|MM^T|$ is square.

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Lemma

$$|X_{(v,k,\lambda)}[c_1, \ldots, c_t]| = (k - \lambda + 2)^{t-1}(k - \lambda - 2)^e \quad \text{(up to a square)},$$

where e is the number of even c_i.

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Theorem

If there exists a nondegenerate symmetric (v, k, λ)-covering with a 2-regular excess, then

- v is even, $k-\lambda-2$ is square, and the excess has an odd number of cycles; or
- v is even, $k-\lambda+2$ is square, and the excess has an even number of cycles; or
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There does not exist a nondegenerate symmetric (v,k,λ)-covering with a 2-regular excess if v is even and neither $k-\lambda-2$ nor $k-\lambda+2$ is square.

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Can we say more (especially for odd v)?

Rational congruence results (with F&H)

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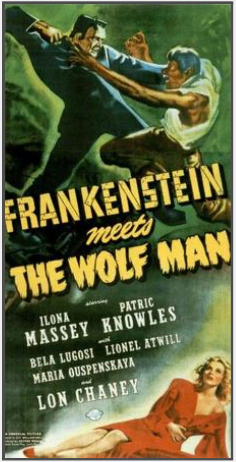

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Rational, nondegenerate $n \times n$ matrices X, Y are rationally congruent if and only if

$$C_p(X) = C_p(Y) \quad \text{for all primes } p \text{ and for } p = \infty,$$

where

- a matrix is nondegenerate if all of its principal minors are invertible, and
- $C_p(X) \in \{-1, 1\}$ is the *Hasse-Minkowski invariant of X with respect to p*.

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- X_i is the ith principal minor of X
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tl;dr

- If $C_p(X) \neq C_p(Y)$ for some p, then $X \nsim Y$.
- The hard part of computing $C_p(X)$ is taking a determinant of every principal minor of X.

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If a (v, k, λ)-covering with excess $[c_1, \ldots, c_t]$ exists then, for all p,

$$C_p(X_{(v,k,\lambda)}[c_1, \ldots, c_t]) = C_p(I) = \begin{cases} -1, & \text{if } p \in \{2, \infty\} \\ +1, & \text{if } p \text{ is an odd prime.} \end{cases}$$

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- We ruled out the existence of many more symmetric coverings with specified excesses.
- We ruled out the existence of *cyclic* symmetric coverings for some entire parameter sets.

Computational rational congruence results

Computational rational congruence results

Example: $(v, k, \lambda) = (11, 4, 1)$

Possible excess types:

$[11]$,
$[9, 2]$, $[8, 3]$, $[7, 4]$, $[6, 5]$,
$[7, 2, 2]$, $[6, 3, 2]$, $[5, 4, 2]$, $[5, 3, 3]$, $[4, 4, 3]$,
$[5, 2, 2, 2]$, $[4, 3, 2, 2]$, $[3, 3, 2, 2]$,
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It turns out $[11]$ and $[6, 3, 2]$ are realisable and $[5, 3, 3]$ is not.

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(v, k, λ)	# of excess types	# ruled out by det results	# ruled out by RC results ($p < 10^3$)	# which may exist
$(11, 4, 1)$	14	7	4	3
$(19, 5, 1)$	105	52	43	10
$(29, 6, 1)$	847	423	393	31
$(41, 7, 1)$	7245	3621	3376	248
$(55, 8, 1)$	65121	32555	30746	1820
$(71, 9, 1)$	609237	304604	292475	12158

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- Using $p < 1000$ we can rule out cyclic symmetric coverings with the following parameter sets for $v < 200$.

v	k	λ	v	k	λ	v	k	λ	v	k	λ
153	18	2	111	32	9	95	49	25	199	98	48
37	11	3	157	38	9	53	38	27	199	101	51
169	23	3	63	30	14	81	47	27	137	87	55
23	10	4	81	34	14	123	60	29	111	79	56
53	15	4	63	33	17	123	63	32	117	86	63
27	12	5	37	26	18	135	66	32	157	119	90
23	13	7	121	47	18	135	69	35	199	134	90
161	34	7	137	50	18	171	84	41	161	127	100
27	15	8	199	65	21	171	87	44	153	135	119
117	31	8	95	46	22	121	74	45	169	146	126

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53	15	4	63	33	17	123	63	32	117	86	63
27	12	5	37	26	18	135	66	32	157	119	90
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- The red entries correspond to $(v, \frac{v-3}{2}, \frac{v-7}{4}, v-3)$-almost difference sets which can be used to produce sequences with desirable autocorrelation properties.

Theoretical rational congruence results

Theoretical rational congruence results

Theorem

There does not exist a symmetric $(\frac{1}{2}p^\alpha(p^\alpha - 1), p^\alpha, 2)$-covering with Hamilton cycle excess when $p \equiv 3 \pmod 4$ is prime, α is odd and $(p, \alpha) \neq (3, 1)$.

The end.


ALL TOGETHER!
FRANKENSTEIN'S
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HUNCHBACK!
MAD DOCTOR!
The
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