# Another family of Steiner triple systems without almost parallel classes

Daniel Horsley (Monash University)

Joint work with Darryn Bryant (University of Queensland)

# Steiner triple systems

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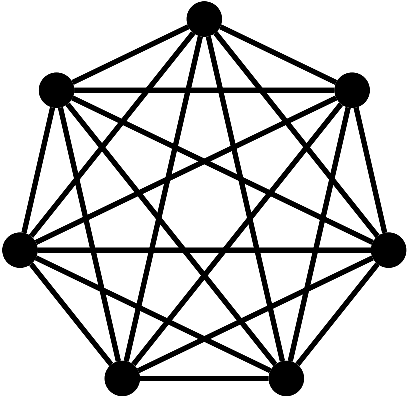

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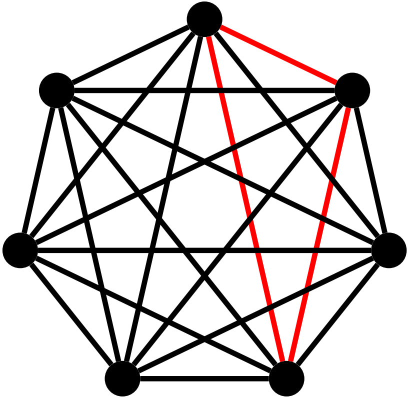

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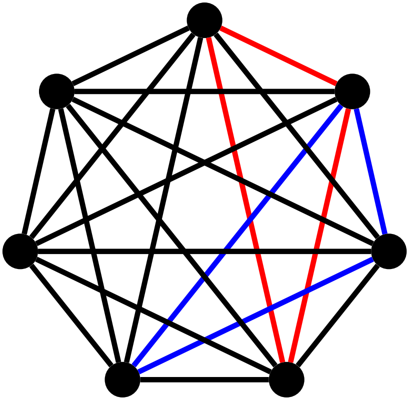

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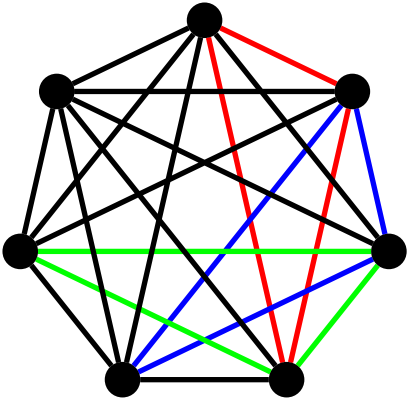

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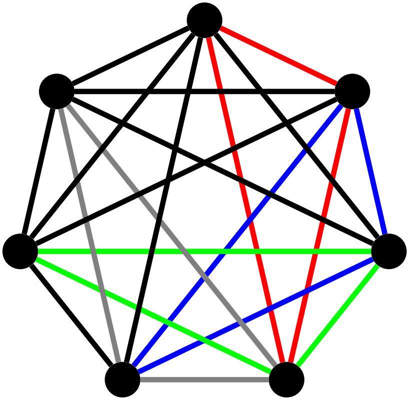

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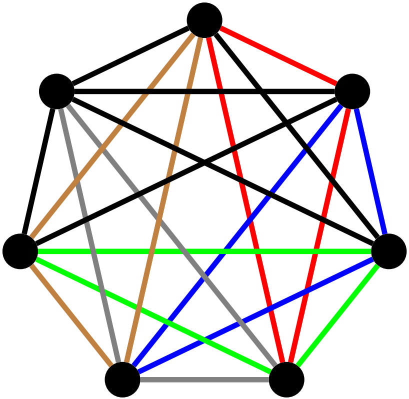

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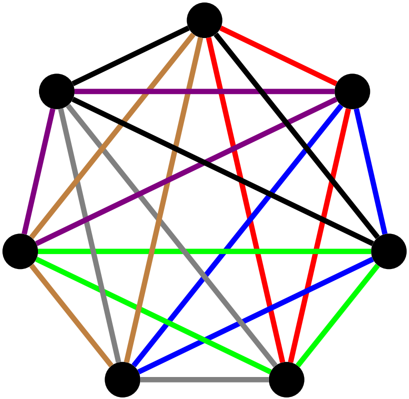

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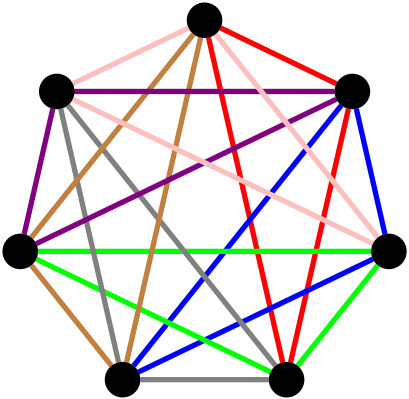

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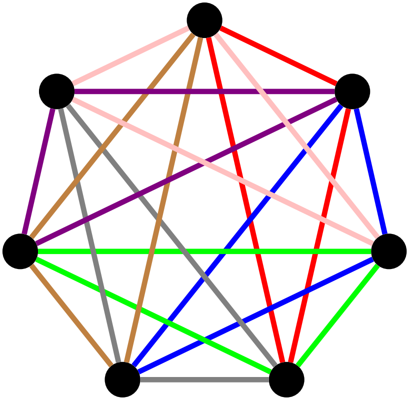

An STS(7)

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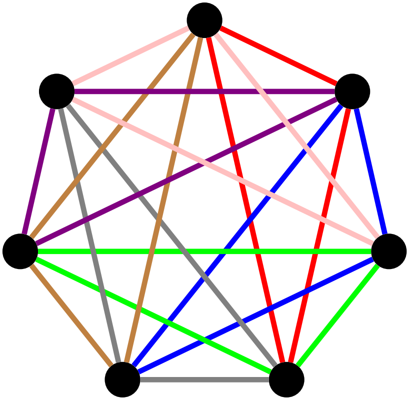

An STS(7)

**Theorem** (Kirkman 1847) An STS($v$) exists if and only if $v \geq 1$ and $v \equiv 1$ or $3 \pmod 6$.

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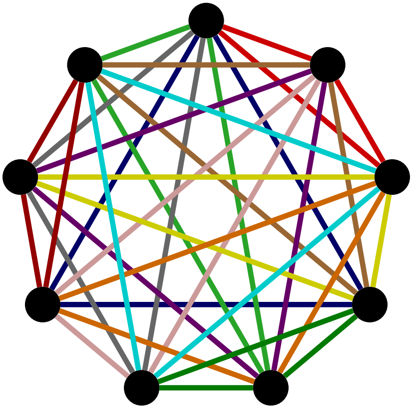

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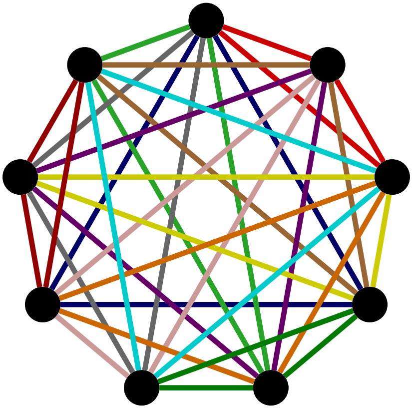

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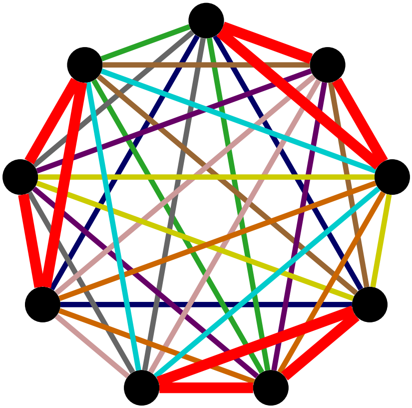

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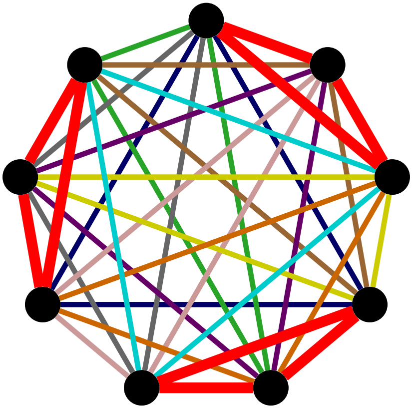

An STS(9) with a PC

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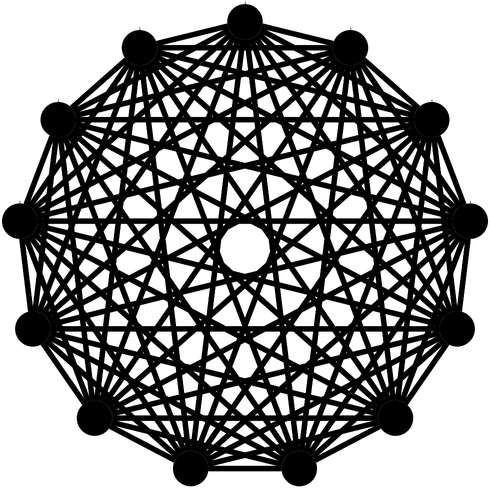

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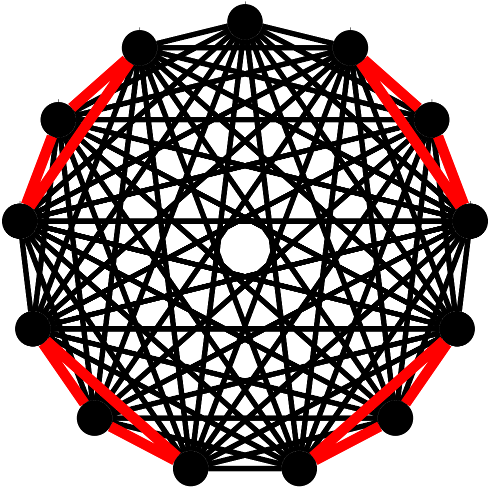

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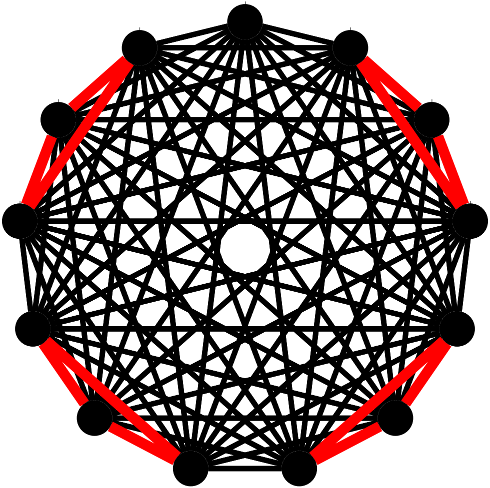

An STS(13) with an APC

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- 12 STS(21)s are known to have no parallel class. (Mathon, Rosa)

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**Conjecture** (Mathon, Rosa) There is an STS($v$) with no PC for all $v \equiv 3 \pmod 6$ except $v = 3, 9$.

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No example is known of an STS which does not contain a set of $\lfloor \frac{v}{3} \rfloor - 1$ disjoint triples.

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**Theorem** (Bryant, Horsley) For each $n$ there is an $\mathrm{STS}(2(3^n) + 1)$ with no almost parallel class.

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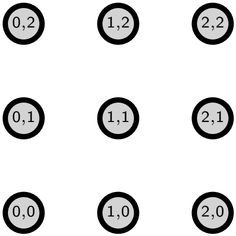

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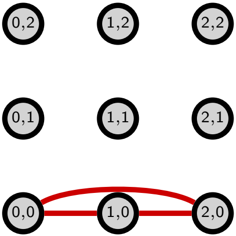

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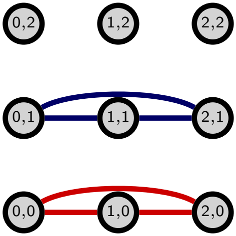

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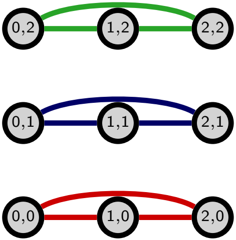

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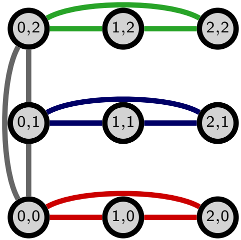

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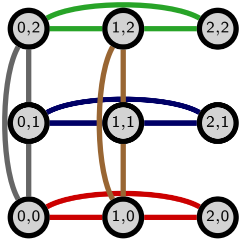

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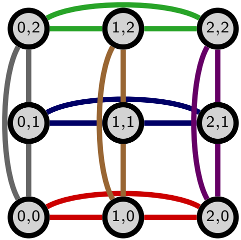

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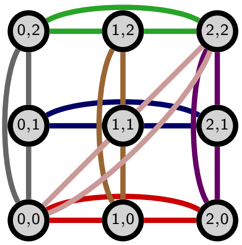

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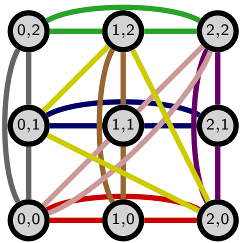

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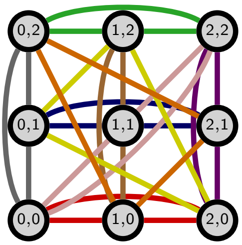

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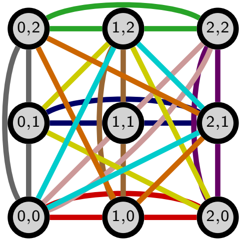

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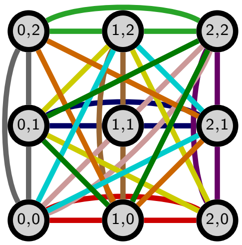

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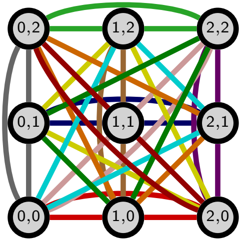

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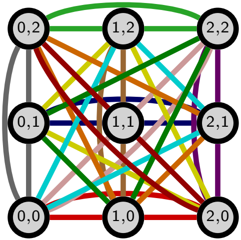


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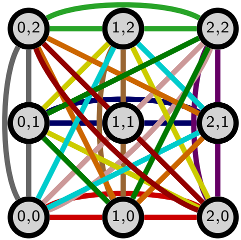


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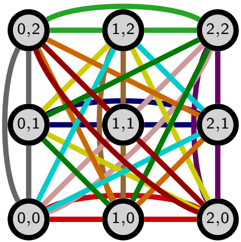


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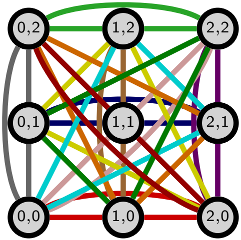


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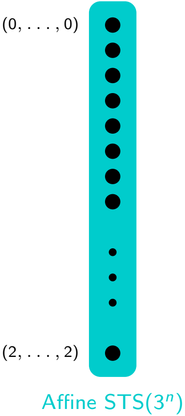

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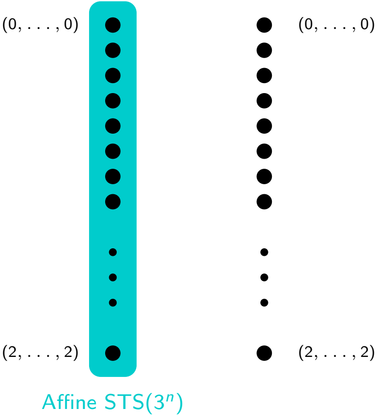

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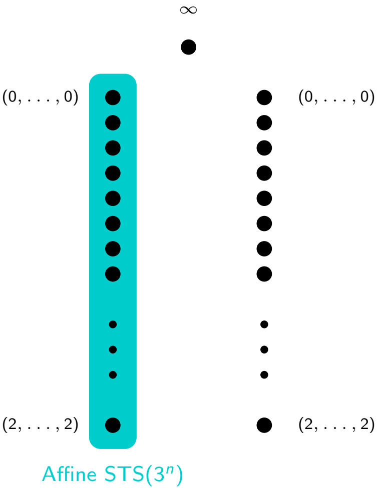

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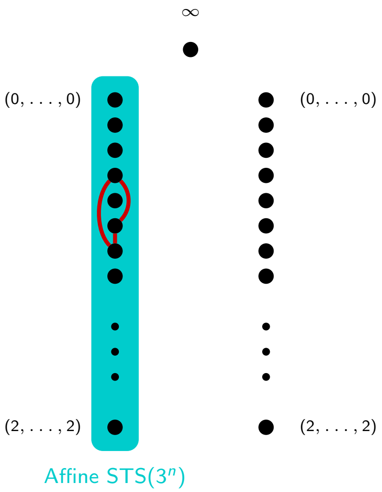

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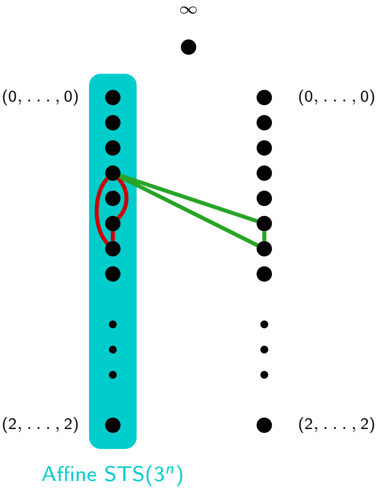

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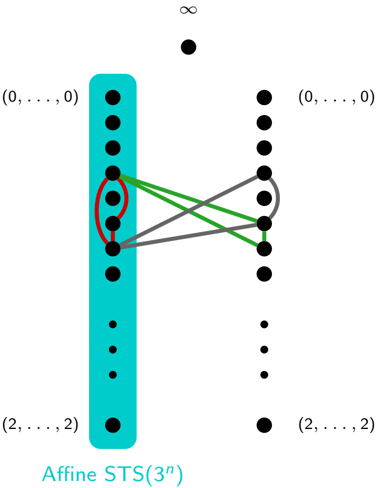

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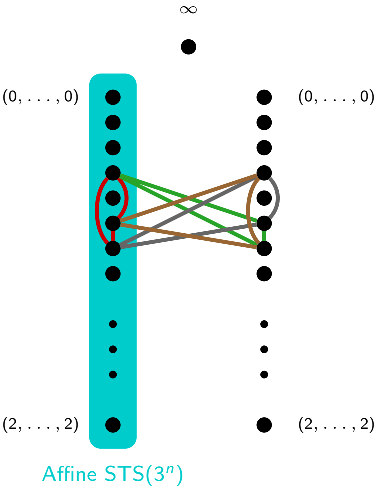

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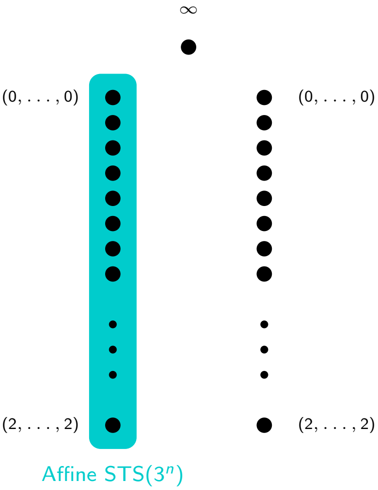

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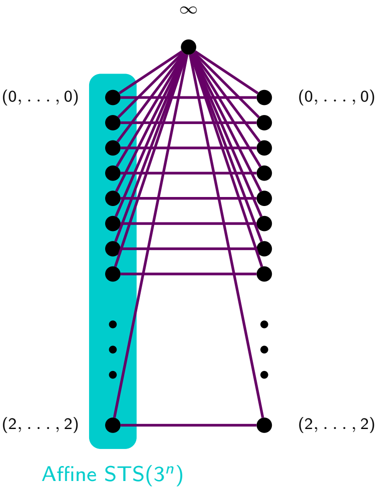

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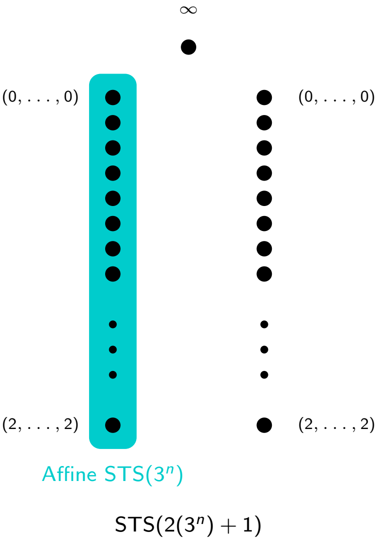

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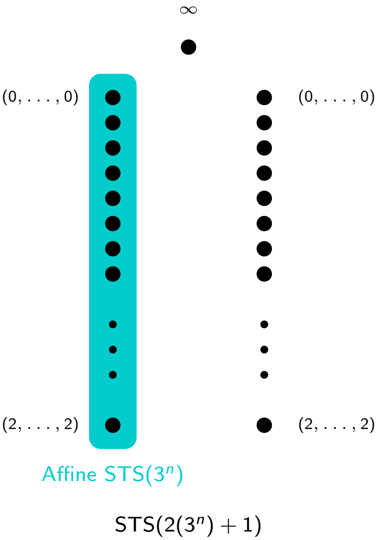


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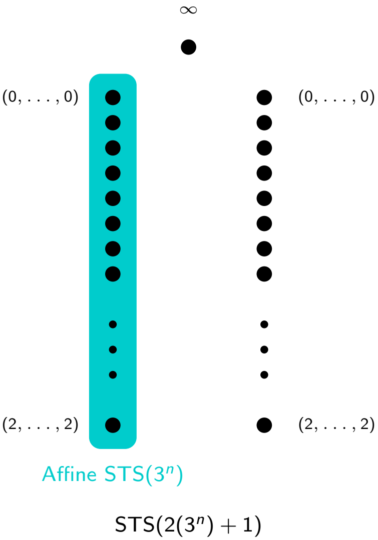


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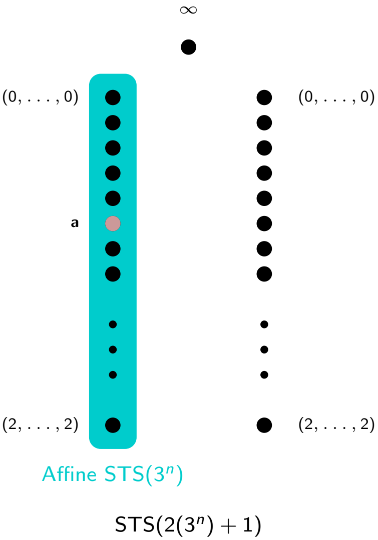


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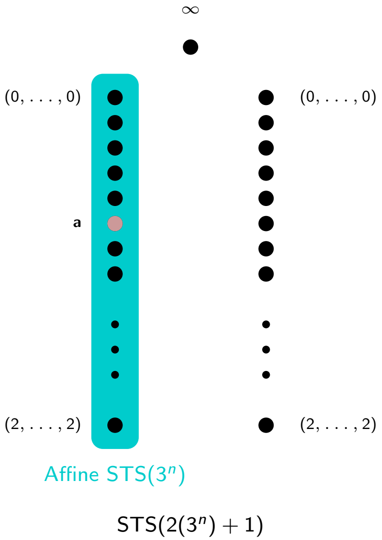


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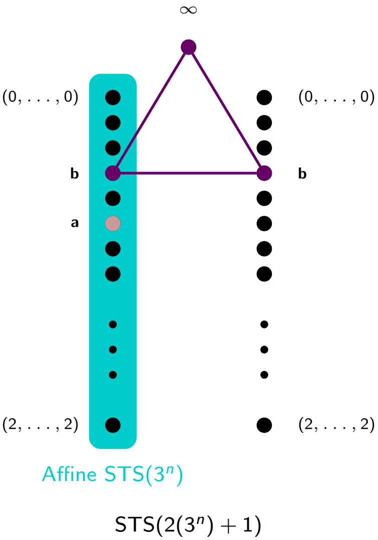


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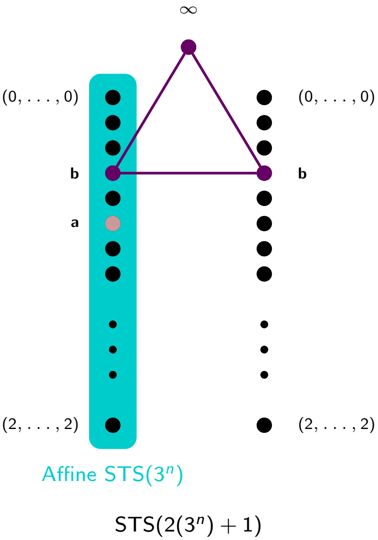


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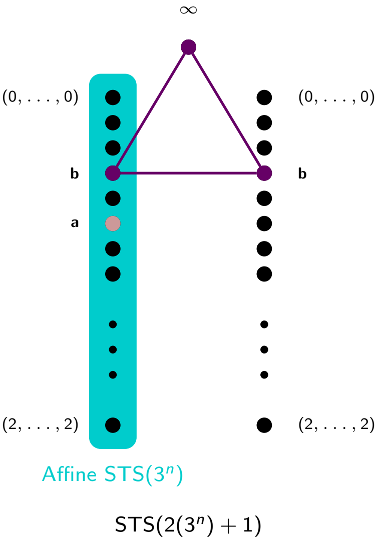


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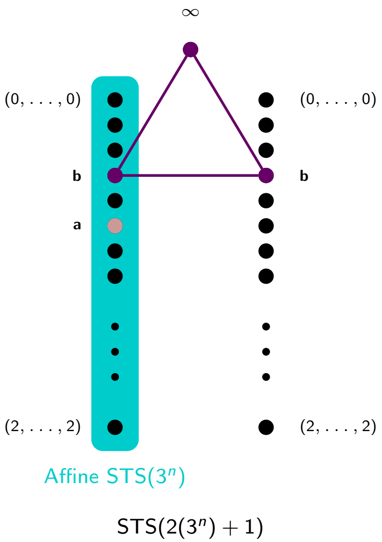


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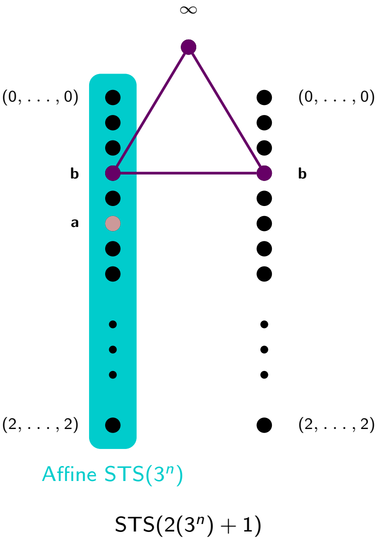


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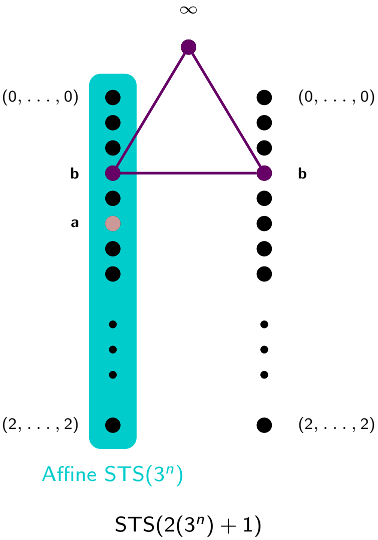


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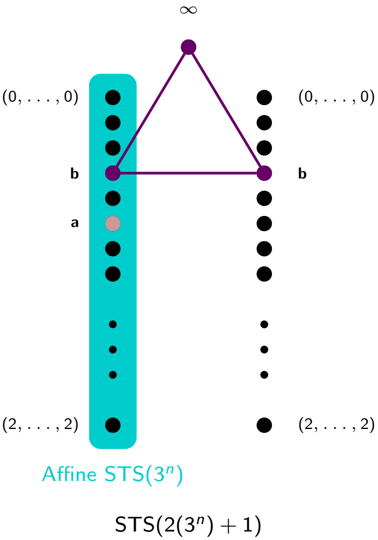


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Contradiction.

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**Theorem** If $q$, $n$ and $d$ are integers with $1 \le d \le n$ such that both $q$ and $q^d - 1$ are prime powers, then there exists a $\left(q^n(q^d-1)+1, q^d, \binom{n-1}{d-1}_q\right)$-design with no almost parallel class.

The End