Another family of Steiner triple systems without almost parallel classes

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Joint work with Darryn Bryant (University of Queensland)
Steiner triple systems

An STS(7)

Theorem

(Kirkman 1847) An STS(v) exists if and only if v ≥ 1 and v ≡ 1 or 3 (mod 6).
Steiner triple systems

Theorem (Kirkman 1847) An STS($v$) exists if and only if $v \geq 1$ and $v \equiv 1$ or $3 \pmod{6}$. 
Steiner triple systems

A Steiner triple system (STS) of order \( v \) exists if and only if \( v \geq 1 \) and \( v \equiv 1 \) or \( 3 \pmod{6} \).
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Theorem (Kirkman 1847) An STS(\(v\)) exists if and only if \(v \geq 1\) and \(v \equiv 1 \text{ or } 3 \pmod{6}\).
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Parallel classes
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An STS(9)
Parallel classes

An STS(9)
Parallel classes

An STS(9) with a PC
Almost parallel classes
Almost parallel classes

An STS(13)
Almost parallel classes

An STS(13)
Almost parallel classes

An STS(13) with an APC
What’s known: small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.
- Both STS(13)s have an APC.
- All but 10 of the 80 STS(15)s have PCs.
- All but 2 of the 11,084,874,829 STS(19)s have an APC. (Colbourn et al.)
- 12 STS(21)s are known to have no parallel class. (Mathon, Rosa)
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What’s guessed

Conjecture (Mathon, Rosa) There is an STS\((v)\) with no PC for all \(v \equiv 3 \pmod{6}\) except \(v = 3, 9\).

Conjecture (Rosa, Colbourn) There is an STS\((v)\) with no APC for all \(v \equiv 1 \pmod{6}\) except \(v = 13\).

No example is known of an STS which does not contain a set of \(\lfloor v/3 \rfloor - 1\) disjoint triples.
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No example is known of an STS which does not contain a set of \(\left\lfloor \frac{v}{3} \right\rfloor - 1\) disjoint triples.
What’s known: in general

\[ v \equiv 3 \pmod{6} \]

No infinite family of STSs of order 3 (mod 6) without PCs is known.

\[ v \equiv 1 \pmod{6} \]

**Theorem (Wilson, 1992)**
For each odd \( n \) there is an STS(2\(^n-1\)) with no almost parallel class.

Wilson’s examples are projective triple systems.

**Theorem (Bryant, Horsley)**
For each \( n \) there is an STS(2(3\(^n\)) + 1) with no almost parallel class.
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**Theorem (Wilson, 1992)** For each odd \( n \) there is an STS\((2^n - 1)\) with no almost parallel class.

Wilson’s examples are projective triple systems.

**Theorem (Bryant, Horsley)** For each \( n \) there is an STS\((2(3^n) + 1)\) with no almost parallel class.
Affine triple systems

An STS(3^n).

Points are $\mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$.

Triples are the 1-dimensional subspaces and their translates.

The affine STS(9):

\[
\begin{align*}
0, & 0, 0 \\
1, & 0, 0 \\
2, & 0, 0 \\
0, & 1, 1 \\
1, & 1, 1 \\
2, & 1, 1 \\
0, & 2, 2 \\
1, & 2, 2 \\
2, & 2, 2
\end{align*}
\]

Note that:

- Each triple adds to zero.
- The whole point set adds to zero.

1-dimensional subspaces have the form \(\{0, a, 2a\}\).

Translates of these have the form \(\{b, a+b, 2a+b\}\).
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The affine $\text{STS}(9)$:

- $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$
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The affine STS(9):

- $\{0,2\}$
- $\{1,2\}$
- $\{2,2\}$
- $\{0,1\}$
- $\{1,1\}$
- $\{2,1\}$
- $\{0,0\}$
- $\{1,0\}$
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The affine STS(9):

```
<table>
<thead>
<tr>
<th>0,2</th>
<th>1,2</th>
<th>2,2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,1</td>
<td>1,1</td>
<td>2,1</td>
</tr>
<tr>
<td>0,0</td>
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```
0,2  1,2  2,2
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The affine STS(9):

![Diagram of the affine STS(9)](attachment:affine_STS_9.png)
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The affine STS(9):

![Graph showing the affine STS(9) with points $0,0$, $0,1$, $0,2$, $1,0$, $1,1$, $1,2$, $2,0$, $2,1$, $2,2$ and connections indicating the structure of the system.](image-url)
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```
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0,1  1,1  2,1
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Our Proof

Affine STS(3^n) (2, . . . , 2) (0, . . . , 0) (2, . . . , 2) (0, . . . , 0) ⊥

STS(2^{(3^n)} + 1)

Suppose this STS has an APC. The missing vertex must be in the subsystem.

a So one triple in the APC contains ∞.

b Adding the labels of the black vertices:

0 + ··· + 0 = 2 (∑ x ∈ Z^3 × ··· × Z^3 x) − a − 2 b

a = b

Contradiction.
Our Proof

Suppose this STS has an APC. The missing vertex must be in the subsystem. So one triple in the APC contains \( \infty \).

Adding the labels of the black vertices:

\[
0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}^n} x \right) - a - 2b = 0 - a - 2b = b
\]

Contradiction.
Our Proof

Suppose this STS has an APC. The missing vertex must be in the subsystem. So one triple in the APC contains $\infty$. Adding the labels of the black vertices:

$$0 + \cdots + 0 = 2\left(\sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x - a - 2b\right)$$

Contradiction.
Our Proof

Affine STS(3^n)

Suppose this STS has an APC. The missing vertex must be in the subsystem. So one triple in the APC contains \( \infty \).

Adding the labels of the black vertices:

\[
\sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x - a - 2b = 0 - a - 2b = b
\]

Contradiction.
Our Proof

Affine STS\((3^n)\)

Suppose this STS has an APC. The missing vertex must be in the subsystem.

Adding the labels of the black vertices:

\[
0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x \right) - a - 2 b = 0 - a - 2 b = a \Rightarrow \text{Contradiction.}
\]
Our Proof

Suppose this STS has an APC. The missing vertex must be in the subsystem.

Adding the labels of the black vertices:

\[ 0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}^n} x - a - 2b \right) \]

Contradiction.
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Affine STS(3^n)

Suppose this STS has an APC. The missing vertex must be in the subsystem. So one triple in the APC contains \( \infty \).

Adding the labels of the black vertices:

\[
0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x \right) - a - 2b
\]

Contradiction.
Our Proof

Affine STS($3^n$)

Suppose this STS has an APC. The missing vertex must be in the subsystem.

Adding the labels of the black vertices:

\[ \sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x - a - 2b = 0 \]

Contradiction.
Our Proof

Suppose this STS has an APC. The missing vertex must be in the subsystem. So one triple in the APC contains $\infty$. Adding the labels of the black vertices:

$$0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x \right) - a - 2 b$$

$$a = b$$

Contradiction.
Our Proof

Suppose this STS has an APC. The missing vertex must be in the subsystem.

Adding the labels of the black vertices:

Contradiction.
Our Proof

Affine $STS(3^n)$

$STS(2(3^n) + 1)$

Suppose this STS has an APC. The missing vertex must be in the subsystem. So one triple in the APC contains $\infty$.

Adding the labels of the black vertices:

$$0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x \right) - a - 2b$$

$$a = b$$

Contradiction.
Our Proof

Suppose this STS has an APC.

Affine $\text{STS}(3^n)$

$\text{STS}(2(3^n) + 1)$
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Affine STS(3^n)

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Suppose this STS has an APC.

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Affine STS($3^n$)

STS($2(3^n) + 1$)

Suppose this STS has an APC.

The missing vertex must be in the subsystem.

The labels of the black vertices:

\[ x + \cdots + x = 2 \sum_{x \in \mathbb{Z}^3 \times \cdots \times \mathbb{Z}^3} x - a - 2b \]

Contradiction.
Our Proof

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Adding the labels of the black vertices:

$$
\sum_{x \in \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3} x - a - 2b = 0
$$

Contradiction.
Suppose this STS has an APC.

The missing vertex must be in the subsystem.

So one triple in the APC contains $\infty$.

Adding the labels of the black vertices:

$$0 + \cdots + 0$$
Suppose this STS has an APC.

The missing vertex must be in the subsystem.

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Adding the labels of the black vertices:

$$0 + \cdots + 0 = 2 \left( \sum_{x \in \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3} x \right) - a - 2b$$
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Adding the labels of the black vertices:

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$$0 = 0 - a - 2b$$
Our Proof

Suppose this STS has an APC.

The missing vertex must be in the subsystem.

So one triple in the APC contains $\infty$.

Adding the labels of the black vertices:

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Contradiction.
Our Results

Theorem

For each $n$ there is an STS($2(3^n) + 1$) with no almost parallel class.

Theorem

If $q$, $n$, and $d$ are integers with $1 \leq d \leq n$ such that both $q$ and $q^d - 1$ are prime powers, then there exists a $(q^n(q^d - 1) + 1, q^d, (n - 1)d - 1)q)$-design with no almost parallel class.
Our Results

**Theorem**  For each $n$ there is an STS($2(3^n) + 1$) with no almost parallel class.
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The End