A solution to Alspach's problem for complete graphs of large odd order

Daniel Horsley

Joint work with Darryn Bryant (University of Queensland)

Cycle Decompositions

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Cycle Decompositions of Complete Graphs

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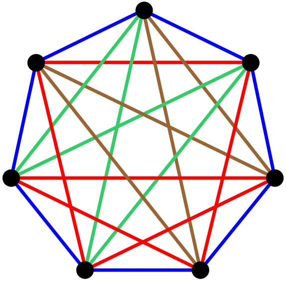

$(4, 4, 6, 7)$-decomposition of K_7

If there exists an $(m_1, m_2, \ldots, m_t)$-decomposition of K_n then

(1) n is odd;

(2) $3 \leq m_1, m_2, \ldots, m_t \leq n$; and

(3) $m_1 + m_2 + \cdots + m_t = |E(K_n)|$.

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If a list satisfies (2) and (3) for some odd n we call it *n-admissible*.

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- The cycles can have as many as n vertices.
- We have to find a very large number of decompositions for each value of n

Existing Results

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There exists a $(m_1, m_2, \ldots, m_t)$-decomposition of K_n when (1), (2), (3) and any of the following hold.

- $m_1 = m_2 = \cdots = m_t$ — Alspach, Gavlas, Šajna
- each $m_i \in \{3, 4, 5\}$ — Balister
- each $m_i \in \{3, n\}$ — Bryant, Maenhaut
- each $m_i \in \{2^k, 2^{k+1}\}$ for some $k \geq 2$ — Heinrich, Horák, Rosa
- n is large and each $m_i \leq \lfloor \frac{n-112}{20} \rfloor$ — Balister
- etc.

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- etc.

- each $m_i \geq \frac{n+5}{2}$ — Bryant, Horsley
- each $m_i \leq \frac{n-1}{2}$ and $m_t \leq 2m_{t-1}$ — Bryant, Horsley

Our Result

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Theorem: There is an integer N such that for all odd $n \geq N$ there exists an $(m_1, m_2, \ldots, m_t)$-decomposition of K_n if and only if

(1) $3 \leq m_1, m_2, \ldots, m_t \leq n$; and

(2) $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$.

Equalise Cycles Lemma

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Suppose there exists an $(m_1, m_2, \ldots, m_t, x, y)$-decomposition of K_n where $x < y$ and $x + y \geq n + 2$. Then there exists an $(m_1, m_2, \ldots, m_t, x + 1, y - 1)$-decomposition of K_n.

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Example:

$$\begin{aligned} & (m_1, m_2, \ldots, m_t, 10, 15)\text{-decomposition of } K_{23} \\ \Rightarrow \quad & (m_1, m_2, \ldots, m_t, 11, 14)\text{-decomposition of } K_{23} \\ \Rightarrow \quad & (m_1, m_2, \ldots, m_t, 12, 13)\text{-decomposition of } K_{23} \end{aligned}$$

Join Cycles Lemma

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Suppose there exists an $(m_1, m_2, \ldots, m_t, c, x, y)$-decomposition of K_n where

(1) $c \geq \frac{1}{2}(x + y)$; and

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Then there exists an $(m_1, m_2, \ldots, m_t, c, x + y)$-decomposition of K_n.

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Example: $(m_1, m_2, \ldots, m_t, 10, 7, 6)$-decomposition of K_{23}

$\Rightarrow$ $(m_1, m_2, \ldots, m_t, 10, 13)$-decomposition of K_{23}

A Reduction of Alspach's Problem

$m_1, m_2, \ldots, m_t, x, y$ $\quad$ $3 < x \leq y < n, \;\; x + y \geq n + 2$

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$$m_1, m_2, \ldots, m_t, x, y$$
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$$\vdots$$

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$$m_1, m_2, \ldots, m_{s-1}, m_s, c, n, n, \ldots, n \qquad m_i \le c,\ \ m_i + c \le n + 1$$

A Reduction of Alspach's Problem

$$
\begin{array}{cl}
m_1, m_2, \ldots, m_t, x, y & 3 < x \leq y < n, \;\; x + y \geq n + 2 \\
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\Uparrow & \\
m_1, m_2, \ldots, m_{s-1}, 3, m_s - 3, c, n, n, \ldots, n & (\text{if } m_s \notin \{3, 4, 5\})
\end{array}
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A Reduction of Alspach's Problem

n-ancestor lists: n-admissible lists of the form

$$3, 3, \ldots, 3, 4, 4, \ldots, 4, 5, 5, \ldots, 5, c, n, n, \ldots, n$$

Theorem: If an $(m'_1, m'_2, \ldots, m'_t)$-decomposition of K_n exists for each n-ancestor list $m'_1, m'_2, \ldots, m'_t$, then an $(m_1, m_2, \ldots, m_t)$-decomposition of K_n exists for each n-admissible list $m_1, m_2, \ldots, m_t$.

A Reduction of Alspach's Problem

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Note: Everything I've said so far is also true for $K_n - I$ when n is even.

Main Result

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Lemma: There is an integer N such that for all odd $n \geq N$ there exists an $(m'_1, m'_2, \ldots, m'_t)$-decomposition of K_n for each n-ancestor list $m'_1, m'_2, \ldots, m'_t$.

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Packings and Leaves

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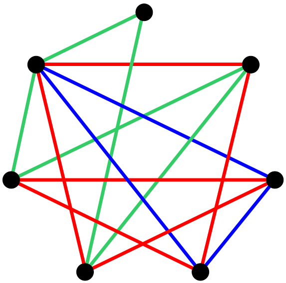

$(6, 5, 3)-$packing of K_7

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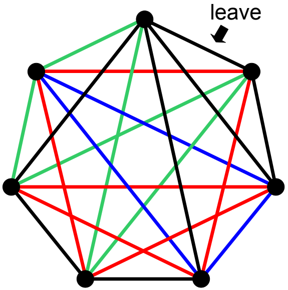

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Proof of Equalise Lemma

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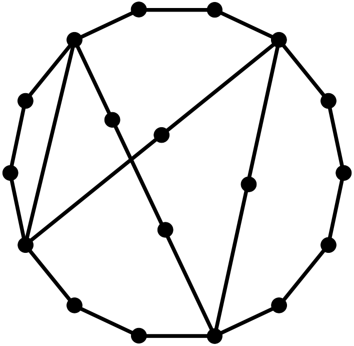
Proof of Equalise Lemma

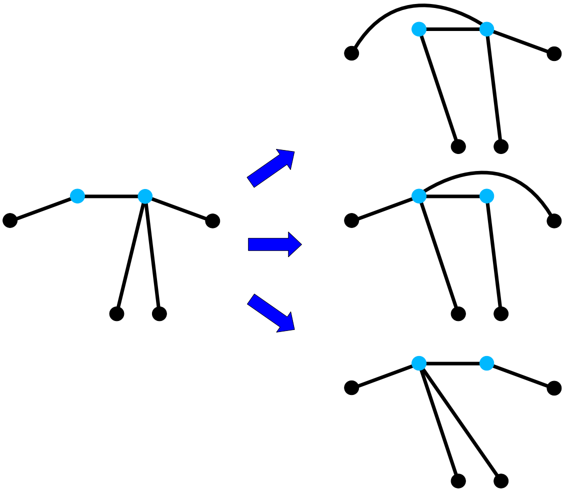
Omit the $x-$cycle and $y-$cycle to form an $(m_1, m_2, \ldots, m_t)-$packing of K_n. Picture the leave with the (long) $y-$cycle "outside" and the (short) $x-$cycle "inside".

Example:



Proof of Equalise Lemma

A sublemma:



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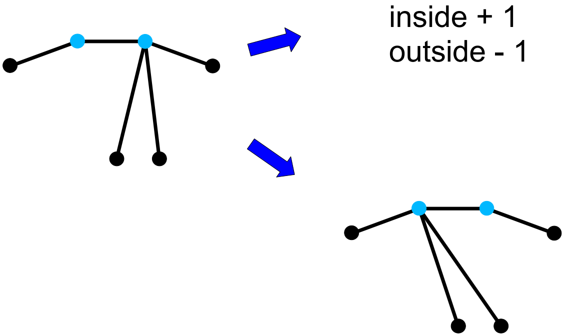

Proof of Equalise Lemma

Theorem (Thomason): If a graph G has a decomposition into two cycles which share at least two vertices then there is another decomposition of G into two cycles which is distinct from the first.

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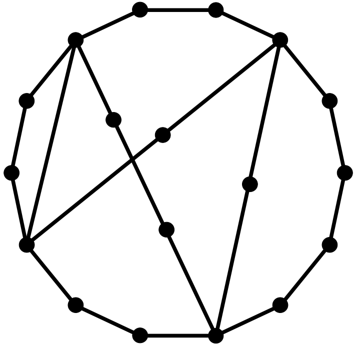

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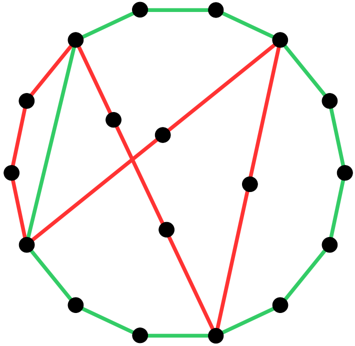

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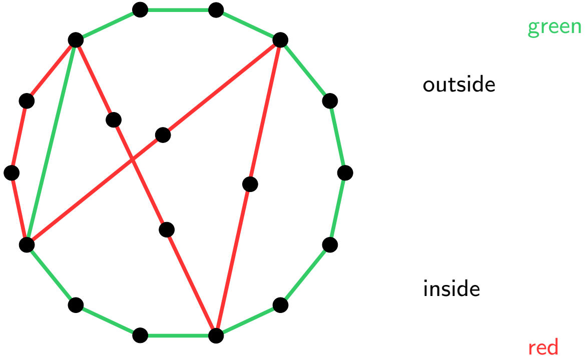

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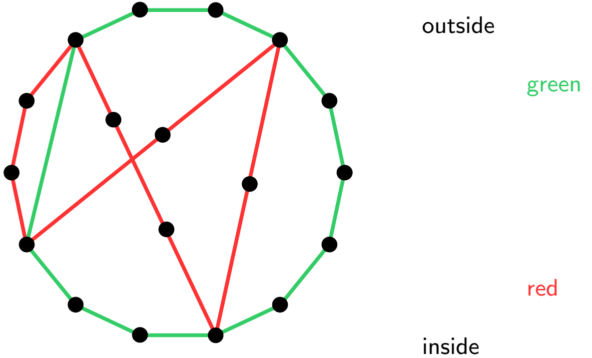

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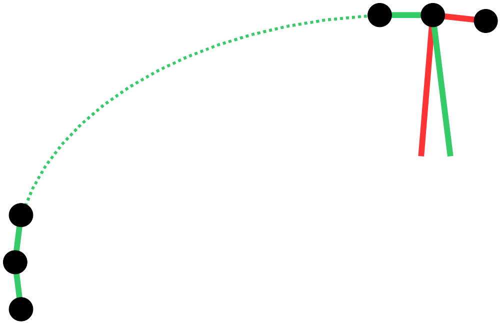

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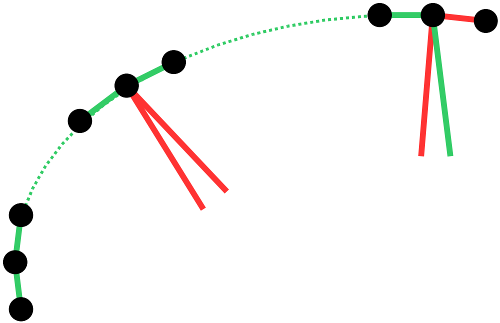

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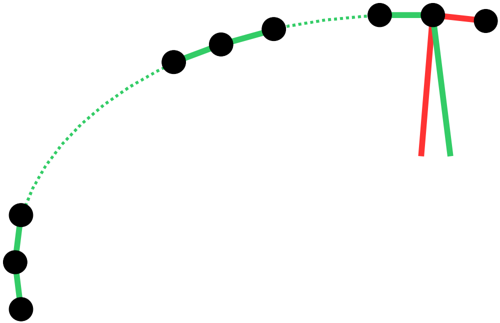

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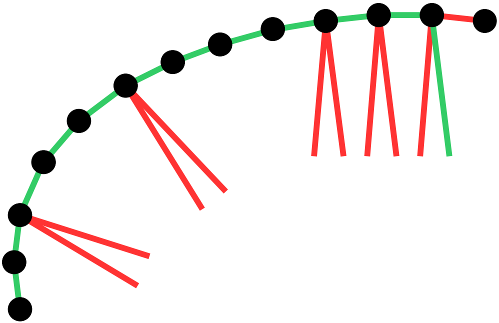

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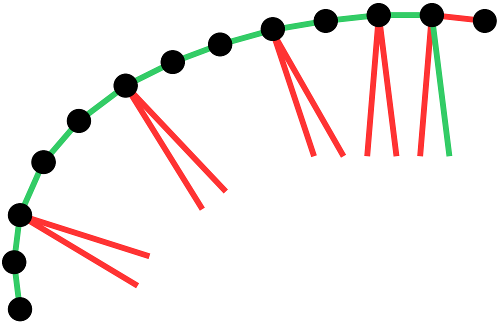

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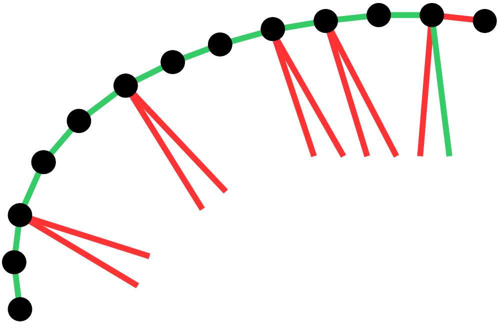

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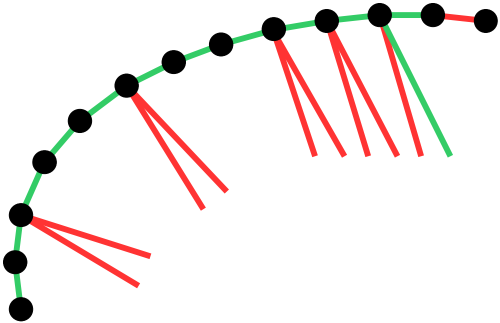

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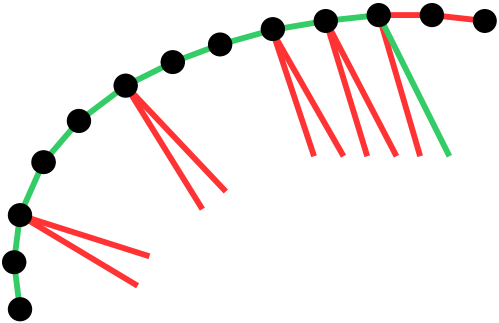

Proof of Equalise Lemma

So if red $<$ outside we can get either

- inside $+ 1$, outside $- 1$; or
- red $+ 1$, green $- 1$.

Proof of Equalise Lemma

green

outside

So if red $<$ outside we can get either

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inside

red

Proof of Equalise Lemma

green

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Equalise Lemma

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Proved!

The End