Trails of triples in Steiner triple systems

Daniel Horsley (Monash University, Australia)

Joint work with Charles Colbourn and Chengmin Wang

Steiner triple systems and block colourings

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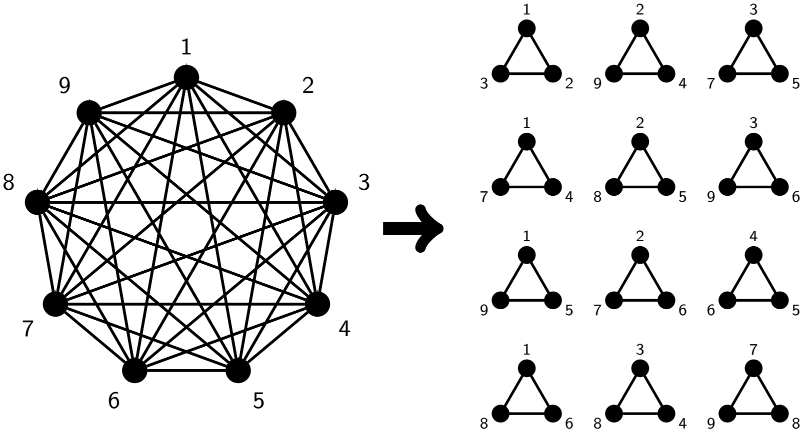
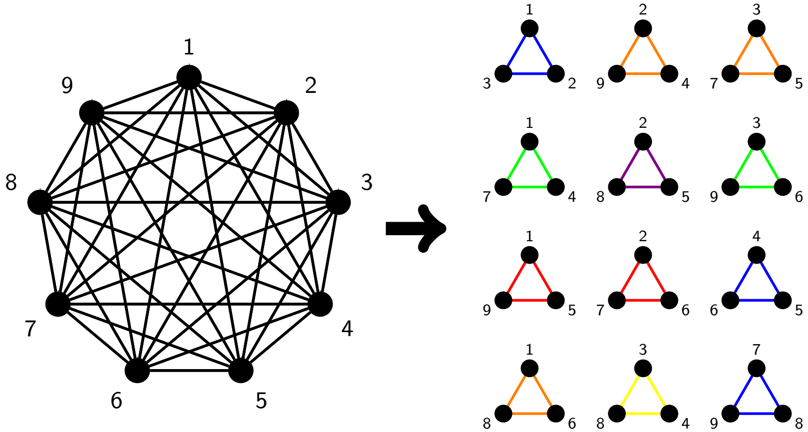


An STS(9)

Steiner triple systems and block colourings



An STS(9) admitting a colouring of type $(3, 3, 2, 2, 1, 1)$

Theorem [Kirkman (1847)] An STS(v) exists if and only if $v \geq 1$ and $v \equiv 1, 3 \pmod 6$.

Partial Steiner triple systems and block colourings

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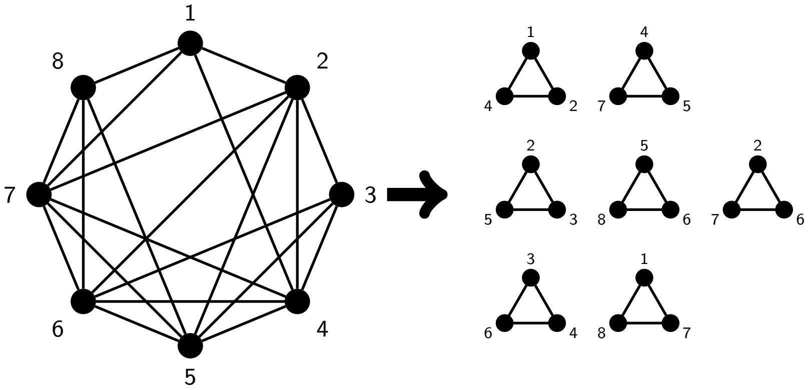
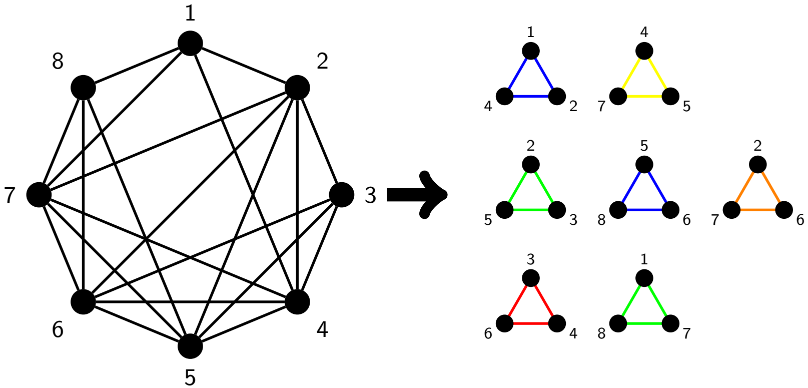


A PSTS(8)

Partial Steiner triple systems and block colourings



A PSTS(8) admitting a colouring of type $(2, 2, 1, 1, 1)$

Necessary conditions and a conjecture

For a PSTS(v) which admits a colouring of type $(c_1, c_2, \ldots, c_t)$ to exist we must have

(i) $c_i \leq \lfloor \frac{v}{3} \rfloor$ for $i = 1, 2, \ldots, t$; and

(ii) $c_1 + c_2 + \cdots + c_t \leq \mu(v)$, where $\mu(v)$ is the maximum number of triples in a PSTS(v).

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Conjecture [Colbourn, Horsley, Wang (2011)] Let $v \geq 14$. For every v-feasible colour type $(c_1, c_2, \ldots, c_t)$ there exists a PSTS(v) admitting a colouring of type $(c_1, c_2, \ldots, c_t)$.

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- A nearly Kirkman triple system is a maximum PSTS$(6t)$ admitting a colouring of type $(2t, 2t, \ldots, 2t)$.
- A Hanani triple system is an STS$(6t+1)$ admitting a colouring of type $(2t, 2t, \ldots, 2t, t)$.
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- The conjecture is also related to 3-frames and to many other block colouring problems.
- A strong Kirkman signal set $\mathrm{SKSS}(v, m)$ is a maximum $\mathrm{PSTS}(v)$ admitting a colouring of type $(m, m, \ldots, m, r)$, where $1 \leq r \leq m$.

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The realisability of many colour types follows immediately from the realisability of others.

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But, for any large v, there are still vast numbers of feasible colour types which are not implied in this way.

For example, for $v = 15$, $(4,4,4,4,4,4,4,4,3)$ is not implied by $(5,5,5,5,5,5,5)$ (or any other colour type).

Trails of triples

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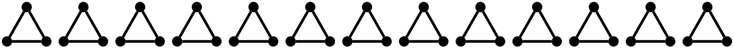

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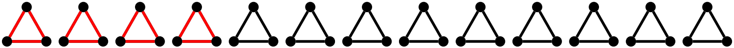

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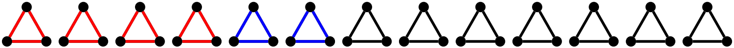

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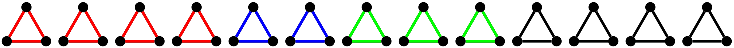

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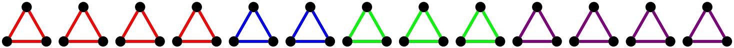

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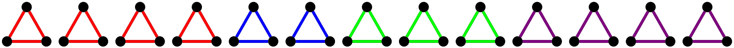

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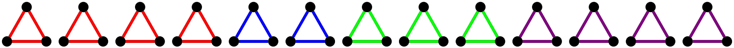

Such an ordering of the triples of a PSTS is called *m-pessimal*.

Note that m can be at most $\lfloor \frac{v}{3} \rfloor$.

Trails of triples

Theorem [Cohen, Colbourn (2000)] For each $v \geq 1$ such that $v \equiv 1, 3 \pmod 6$, there is an STS(v) admitting an $\lfloor \frac{v+6}{9} \rfloor$-pessimal ordering.

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In fact $m = \lfloor \frac{1}{3}(v - (9v)^{2/3}) \rfloor + \mathrm{O}(v^{1/3})$.

Corollary For each sufficiently large v, each v-feasible colour type $(c_1, c_2, \ldots, c_t)$ with $c_i \leq m$ for $i = 1, 2, \ldots, t$ is realisable.

Proof sketch

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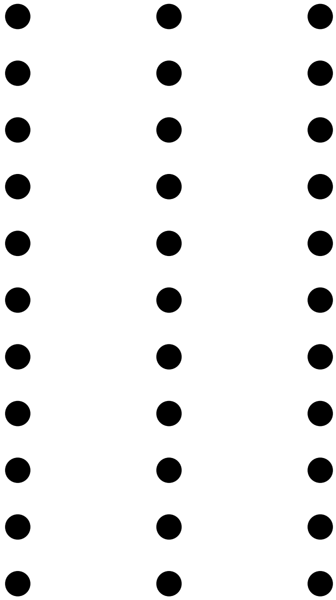
Lemma Let n be an odd integer. There exists an decomposition of the complete tripartite graph $K_{n,n,n}$ into triples which admits an $(n-2)$-pessimal ordering.

Proof of lemma



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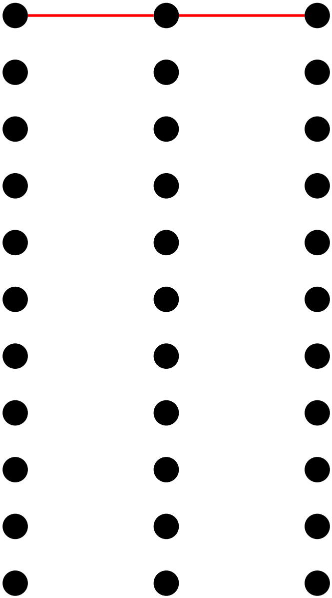

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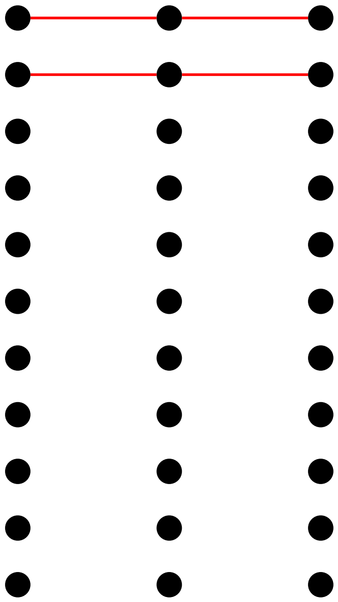

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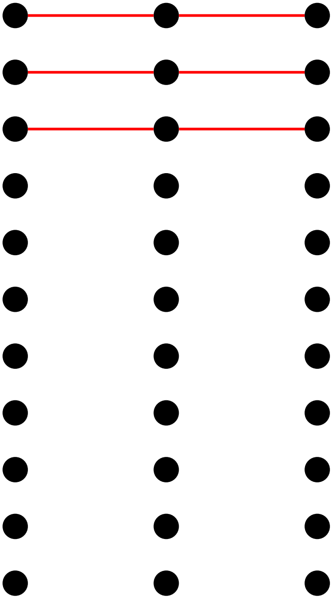

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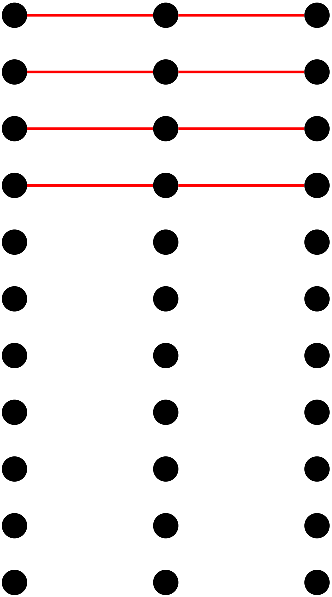

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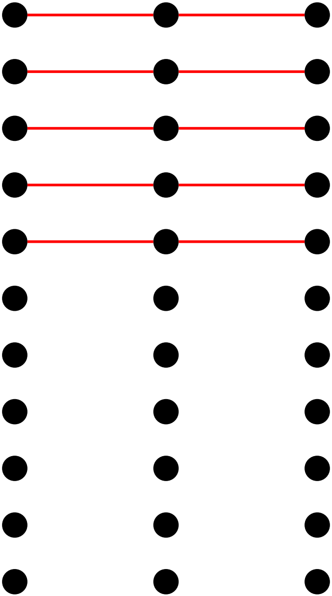

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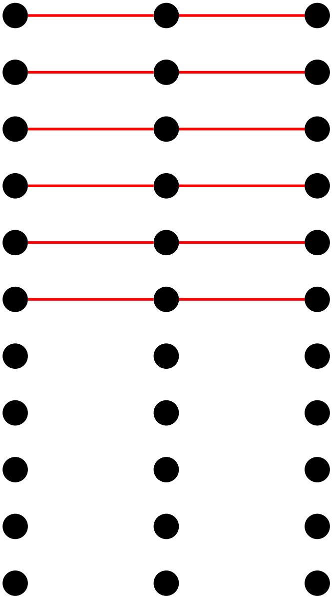

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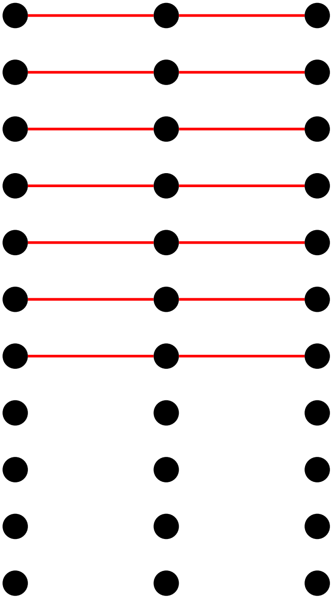

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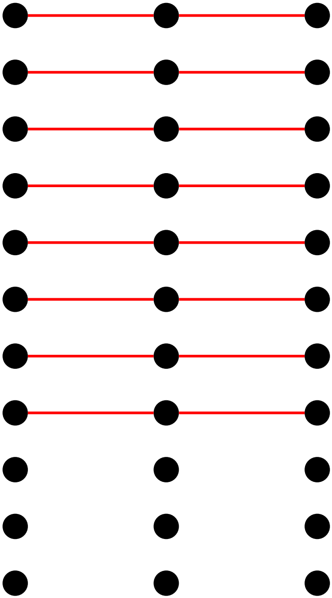

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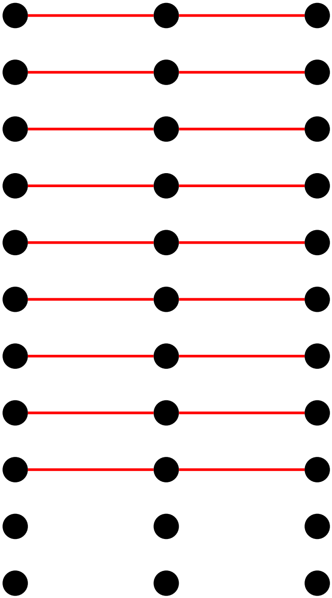

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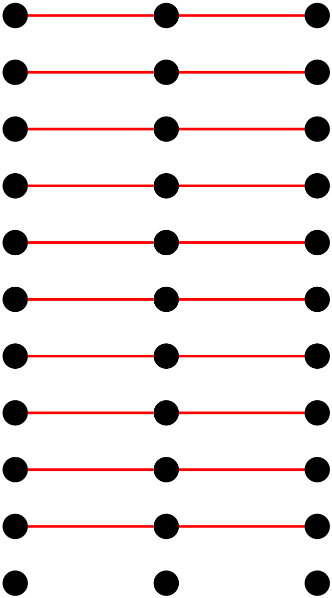

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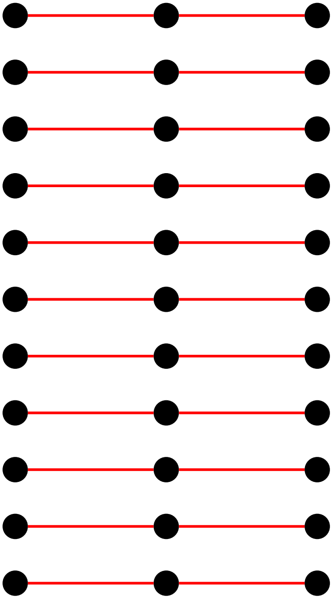

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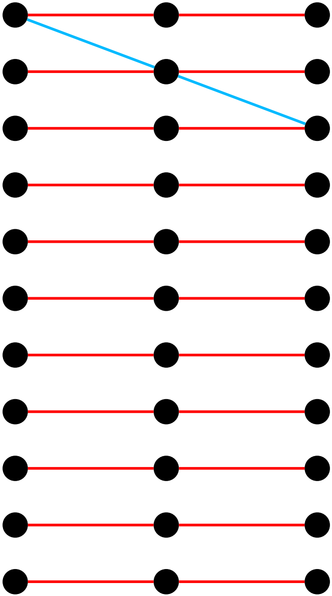

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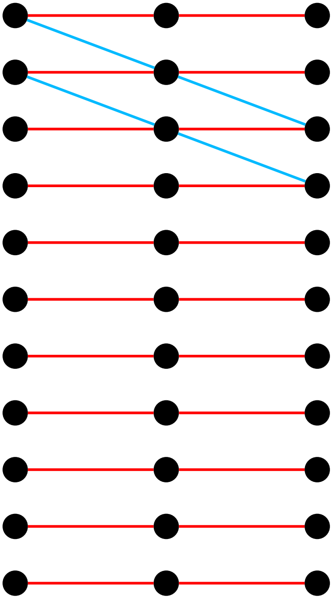

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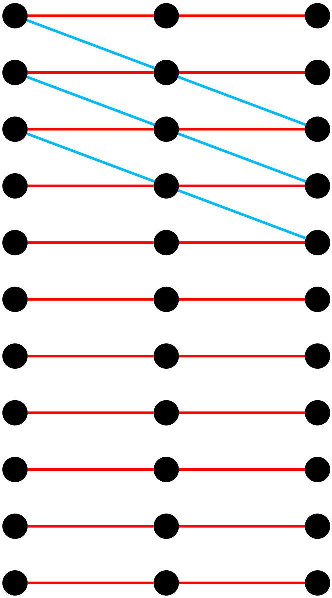

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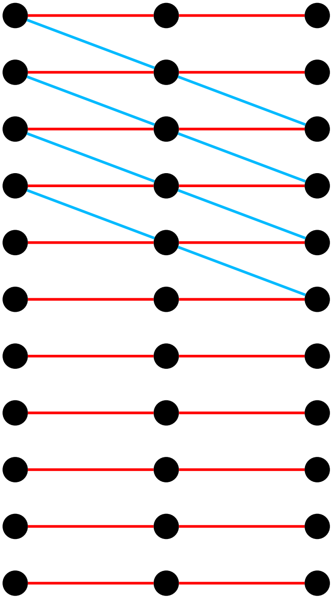

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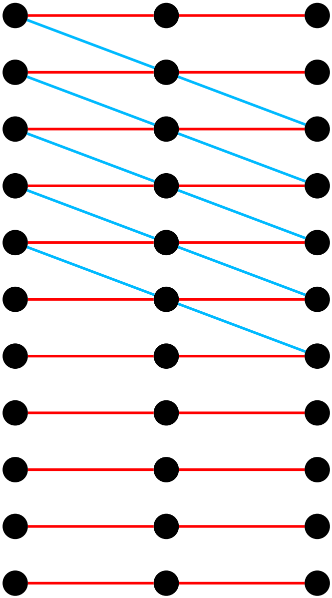

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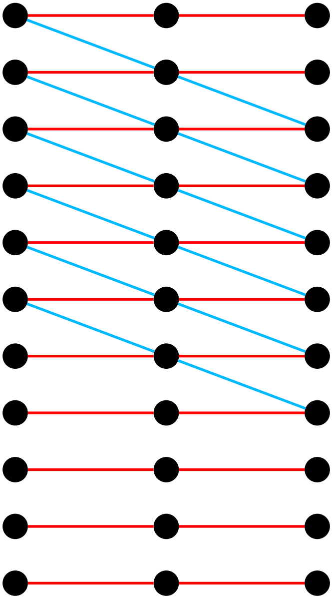

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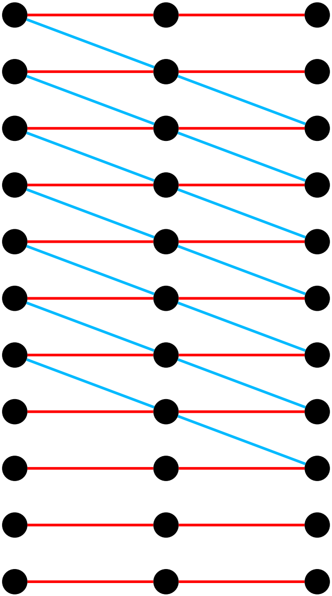

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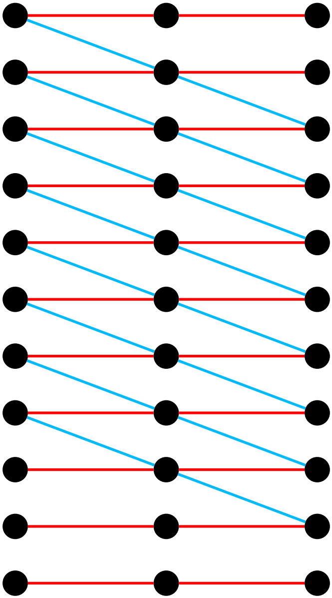

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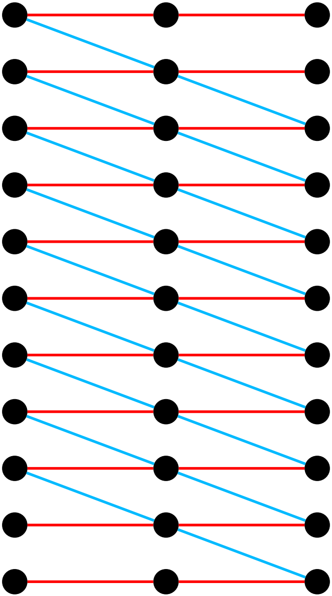

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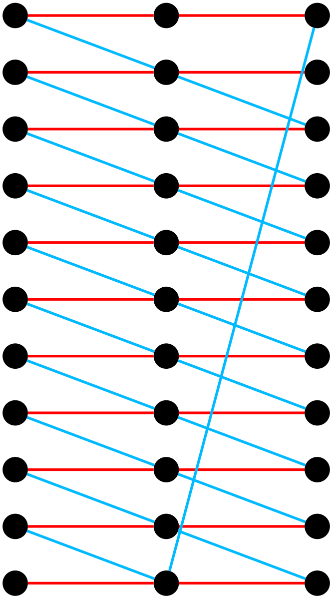

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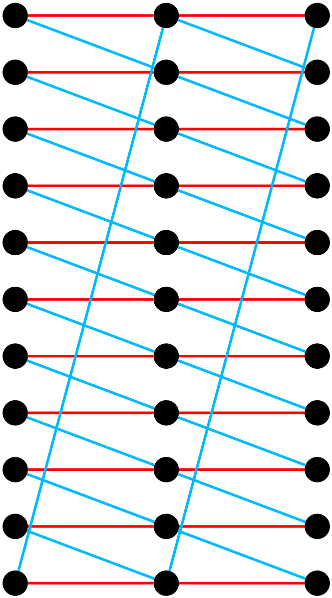

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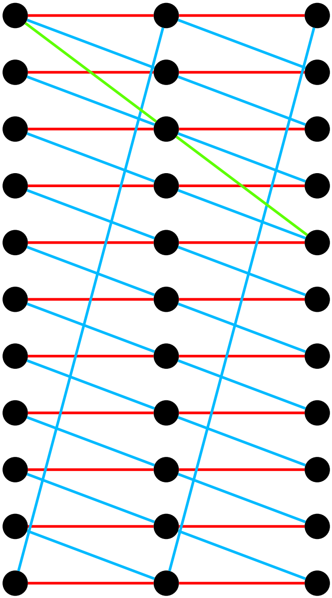

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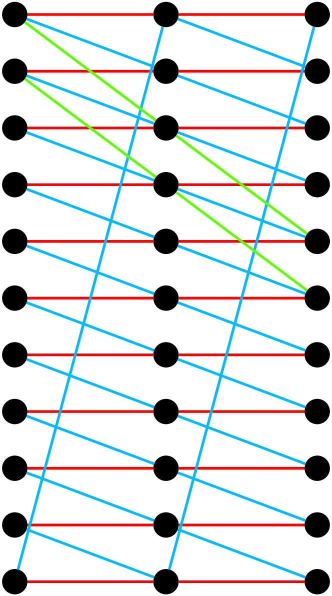

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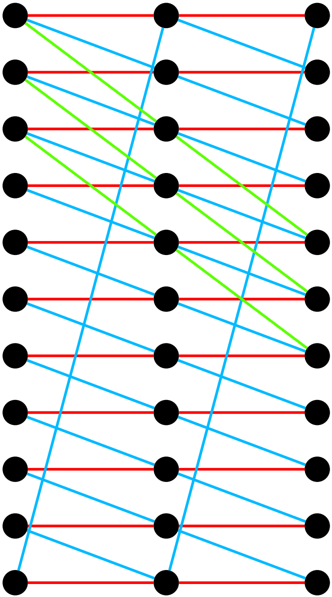

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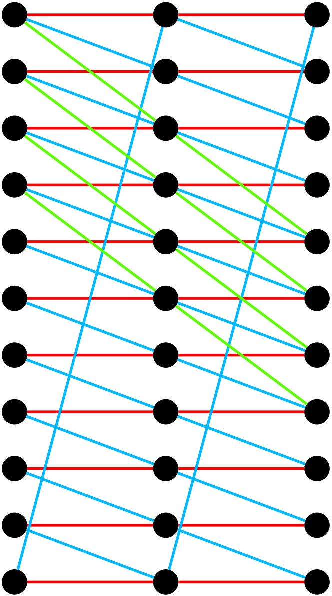

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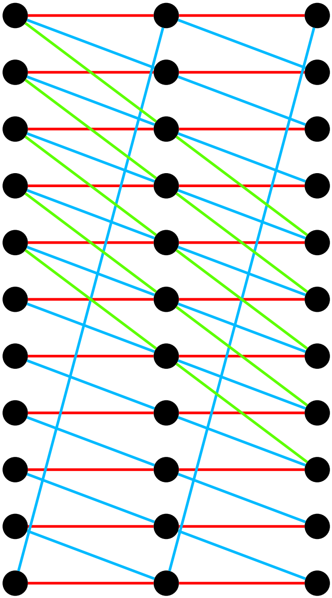

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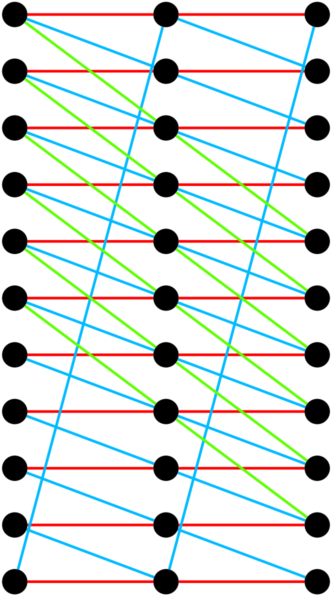

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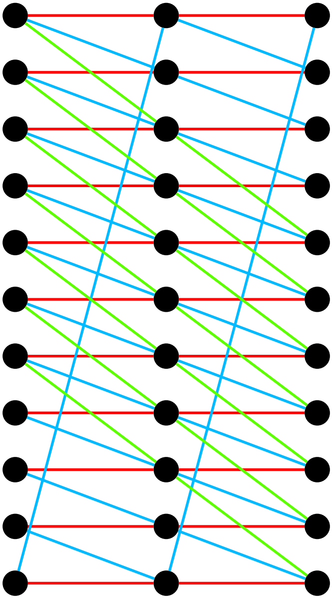

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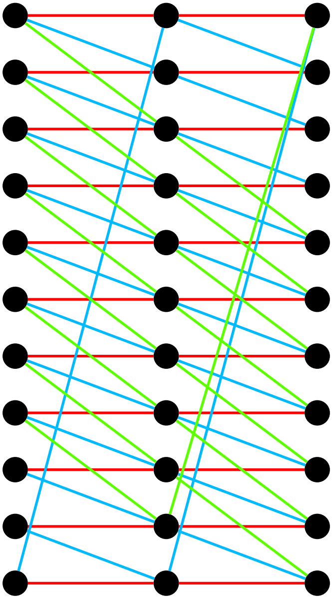

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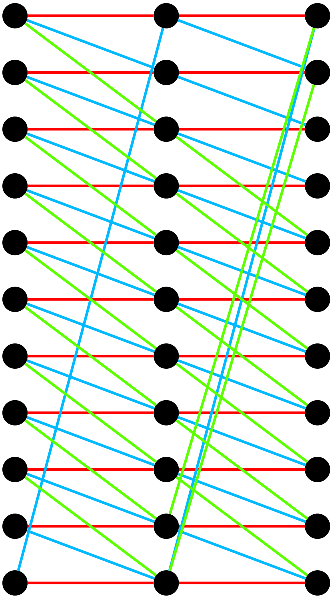

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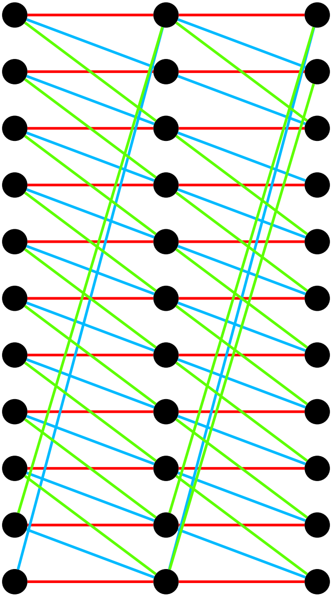

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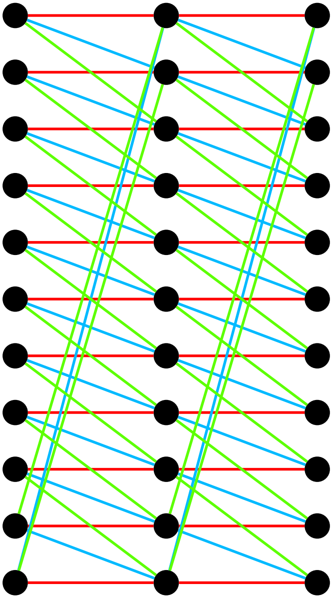

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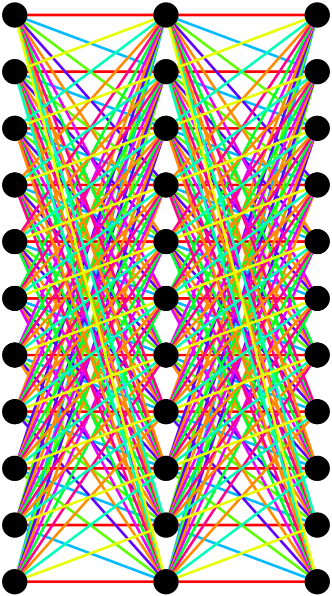

Example: an STS(73)

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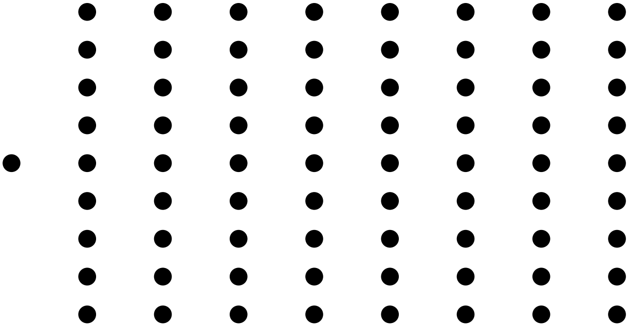

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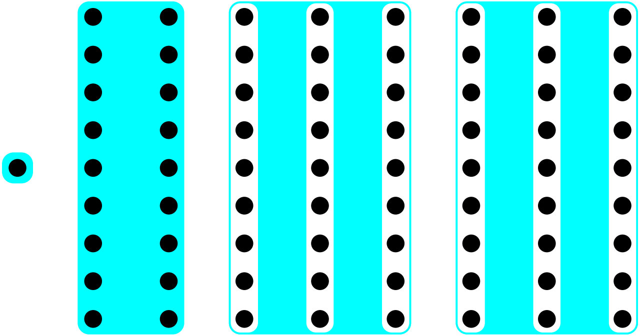

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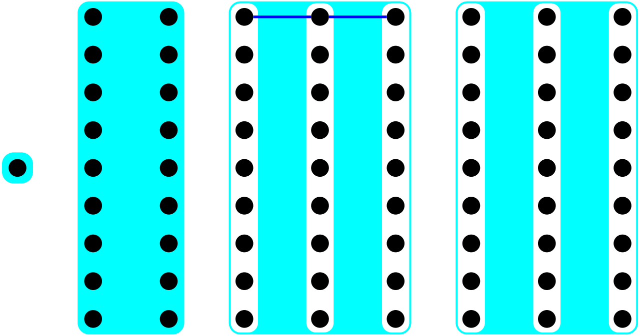

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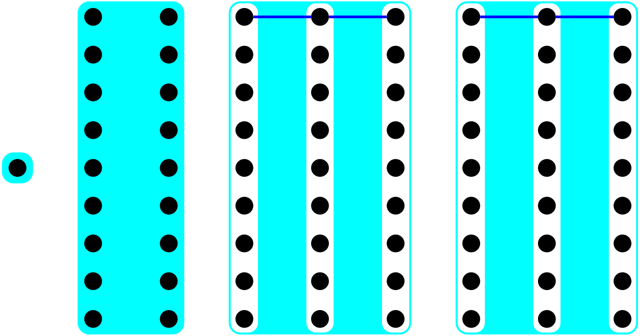

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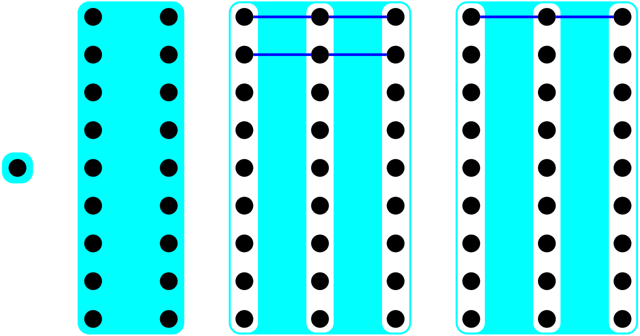

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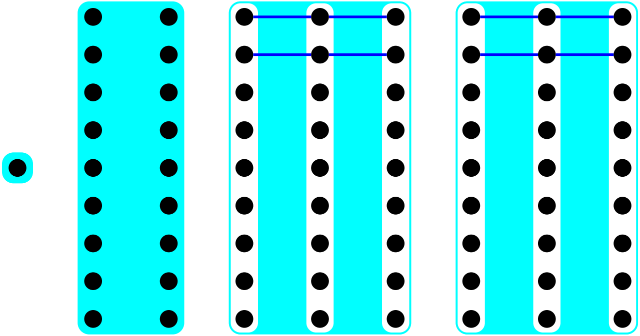

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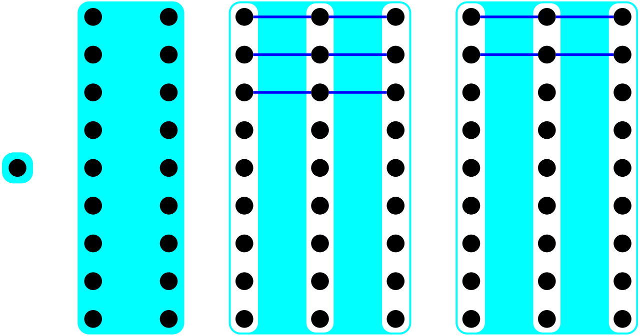

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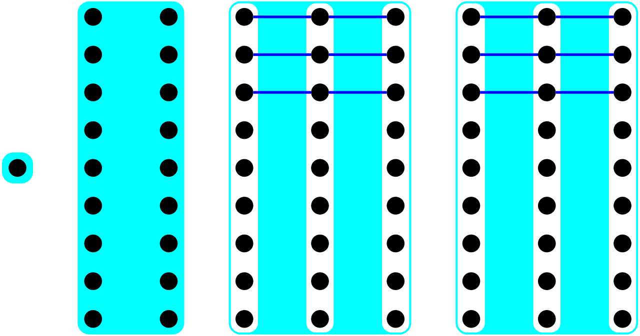

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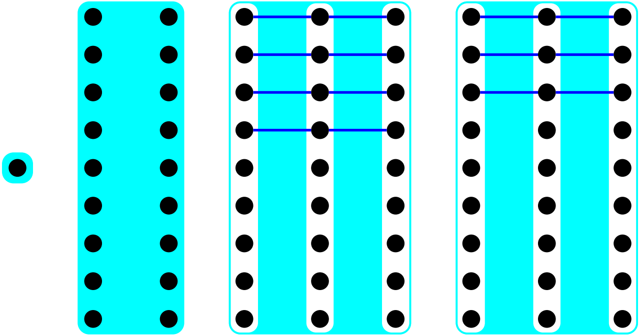

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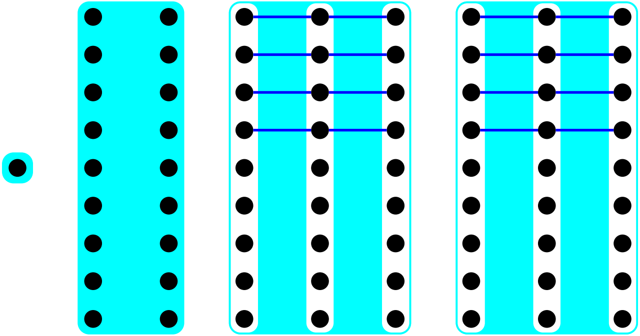

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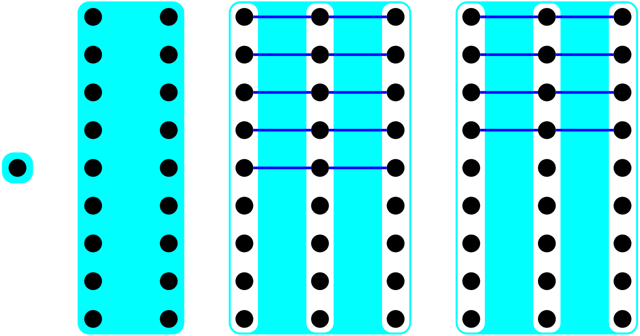

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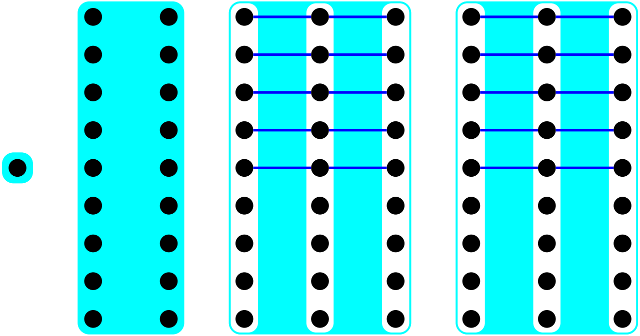

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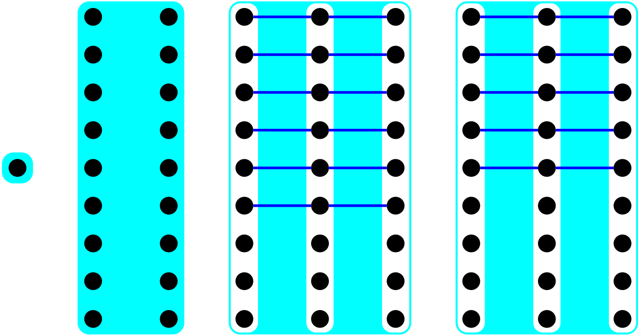

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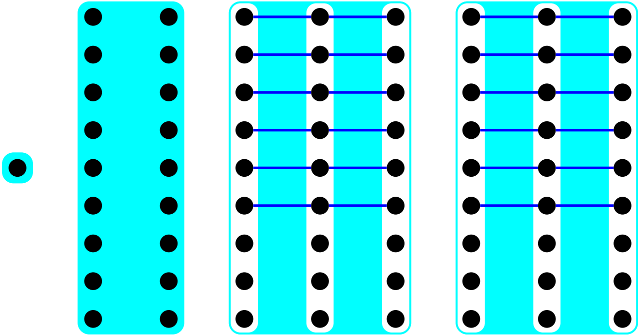

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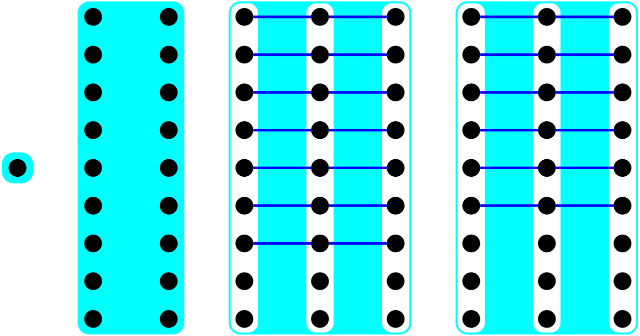

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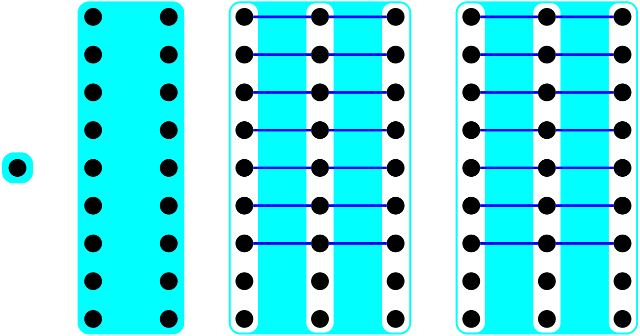

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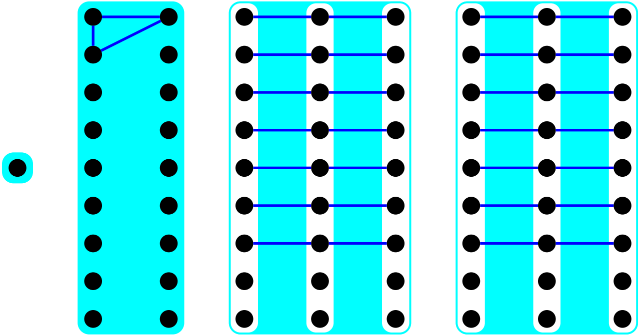

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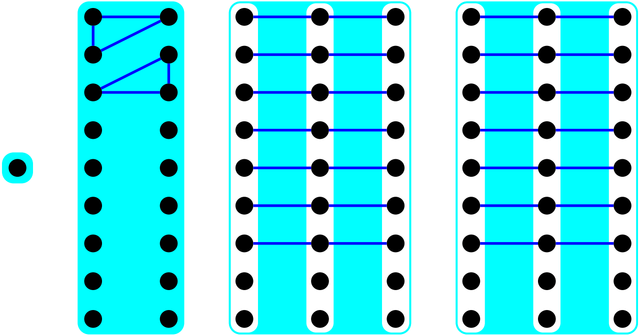

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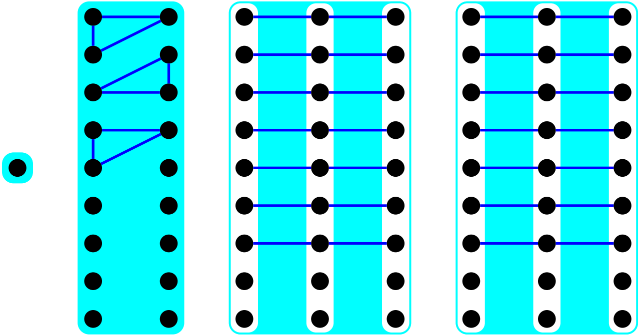

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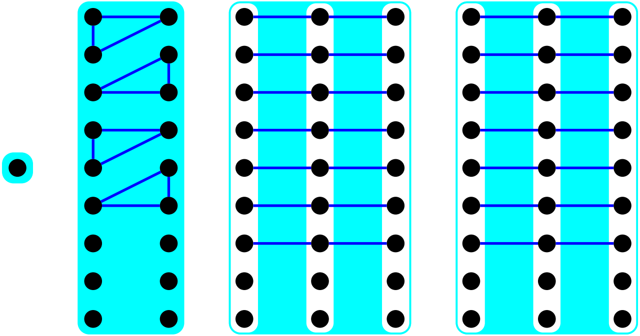

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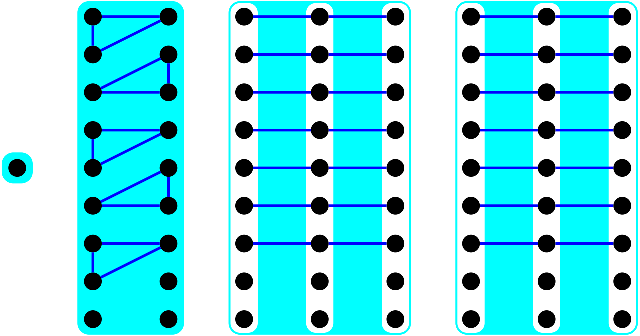

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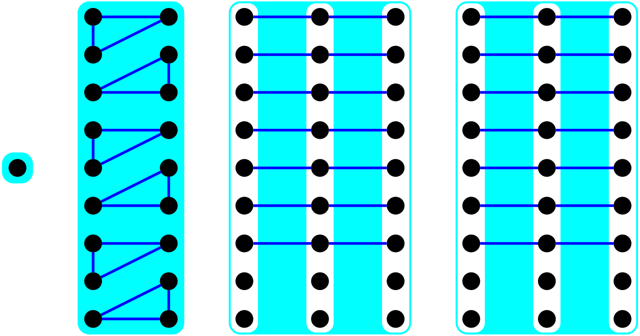

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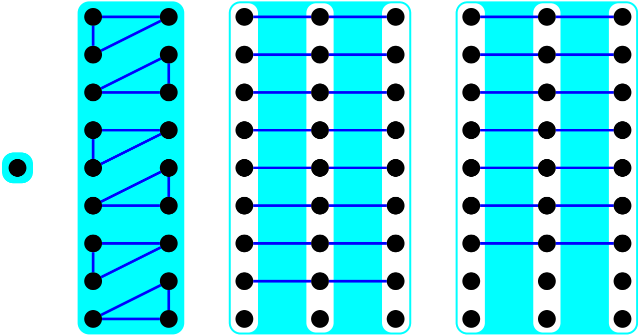

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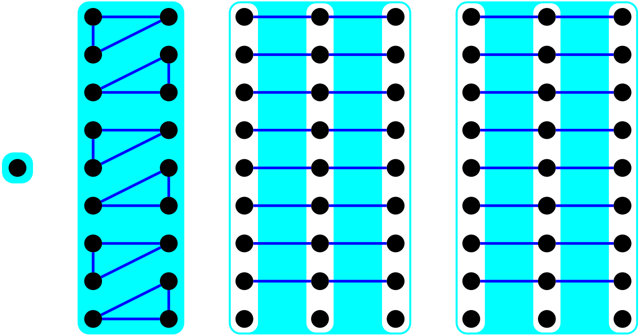

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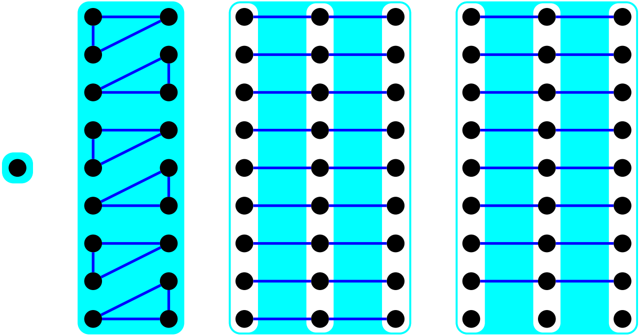

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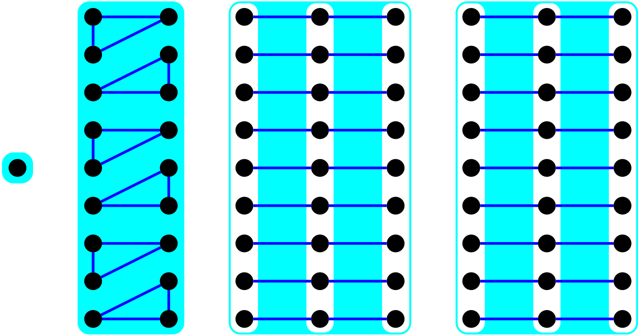

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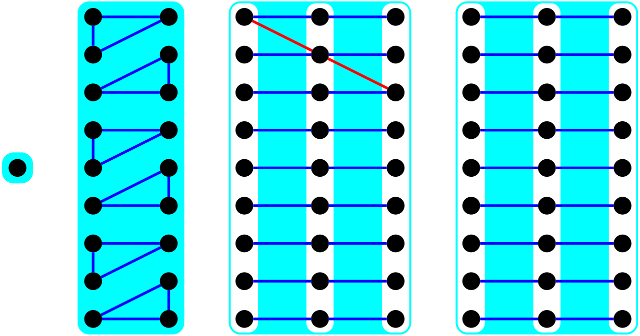

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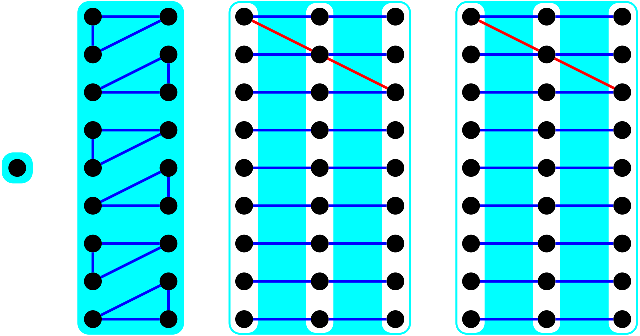

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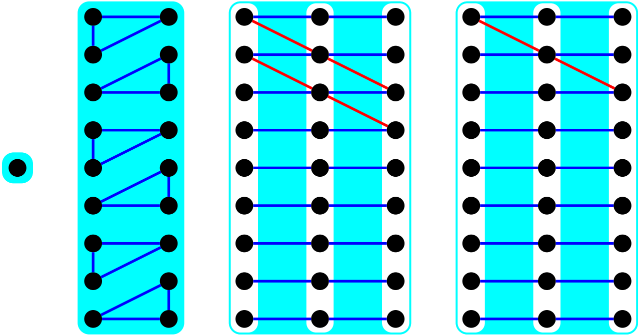

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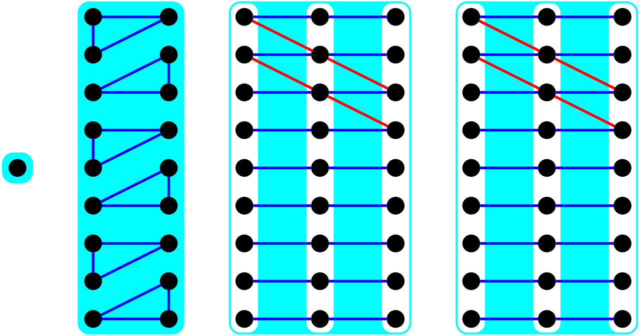

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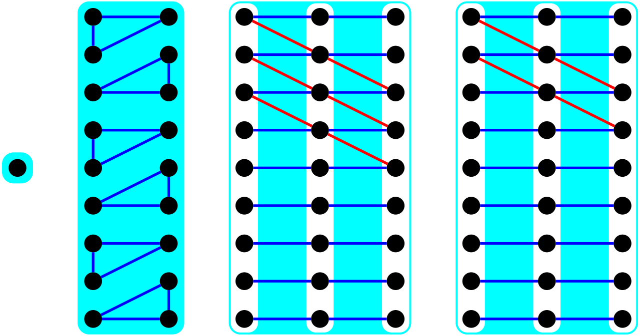

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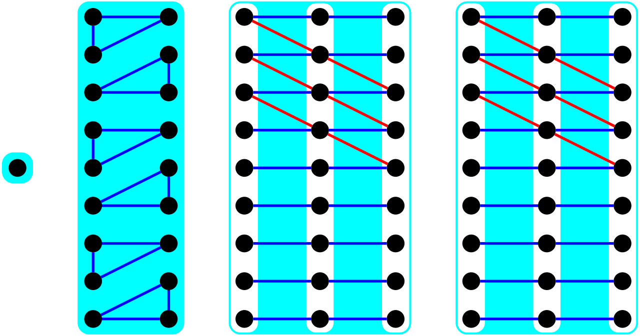

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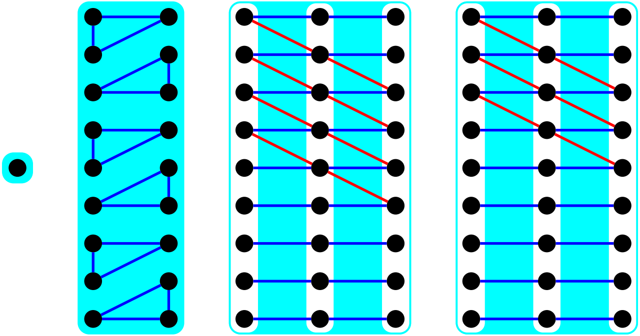

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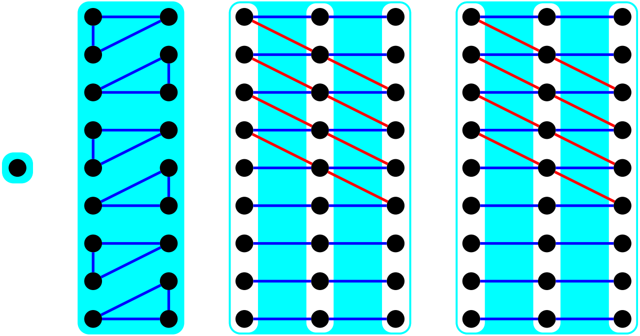

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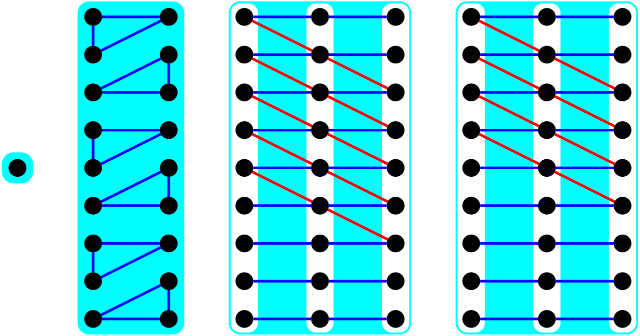

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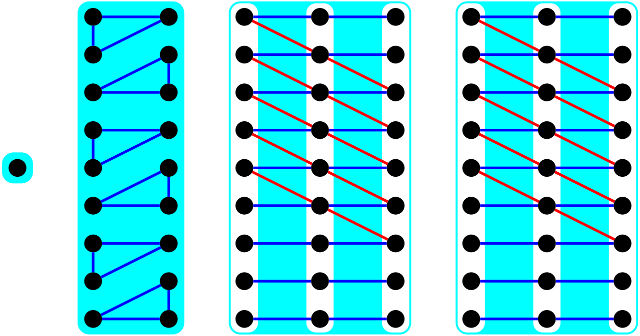

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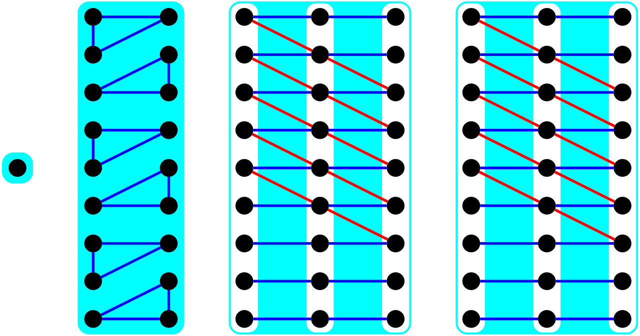

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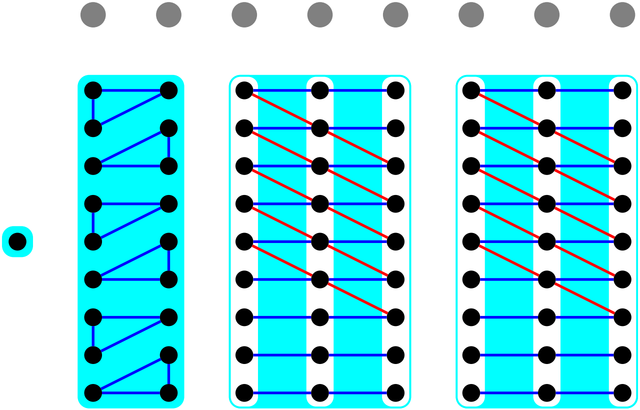

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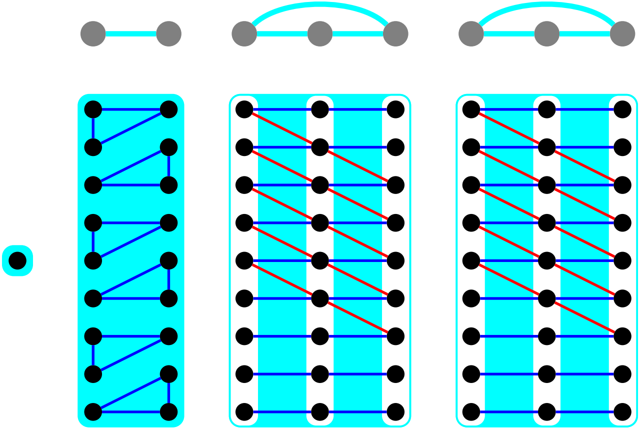

General case

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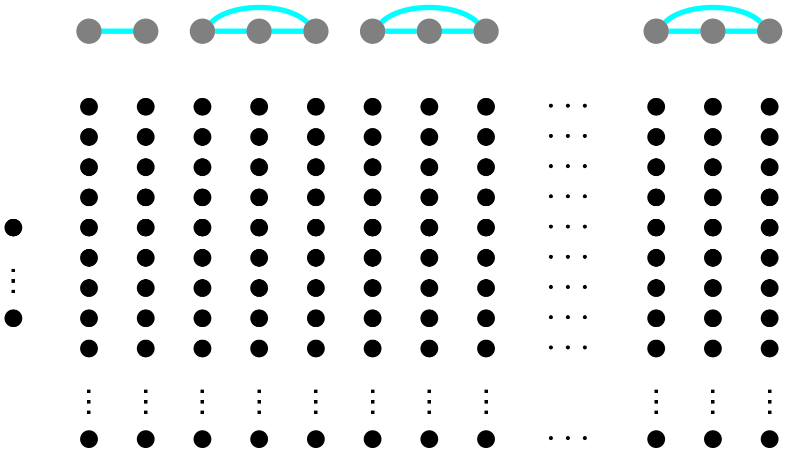

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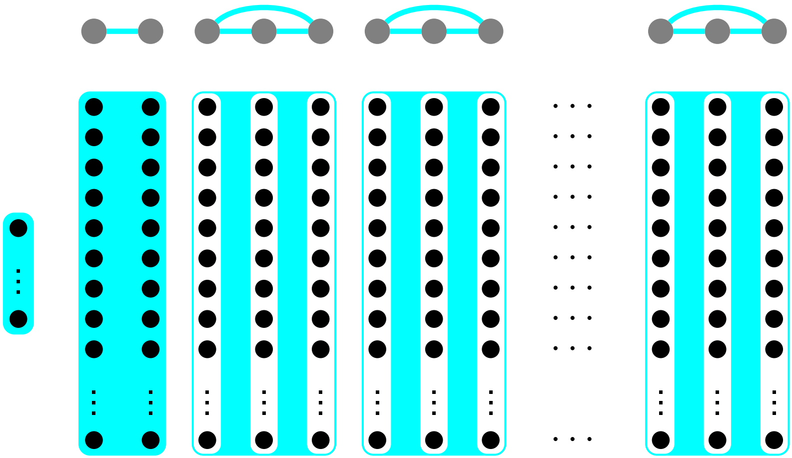

Theorem [Colbourn, Horsley, Wang (2011)] For each sufficiently large v, there is a maximum PSTS(v) admitting an m-pessimal ordering where $m = \frac{v}{3}(1 - \mathrm{o}(1))$.

Future directions

Future directions

- Can we improve our result to $m = \frac{v}{3} - \mathrm{O}(1)$?
- Can we make this construction better by making it recursive?
- Prove the colouring conjecture.
- Latin square equivalents of these problems.
- For sufficiently large v, is there a maximum PSTS(v) that admits all v-feasible colourings?

That’s all.