Alspach's cycle decomposition problem for multigraphs

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Joint work with Darryn Bryant, Barbara Maenhaut and Ben Smith
(University of Queensland)

Part 1:

Alspach's problem

Cycle decompositions of complete graphs

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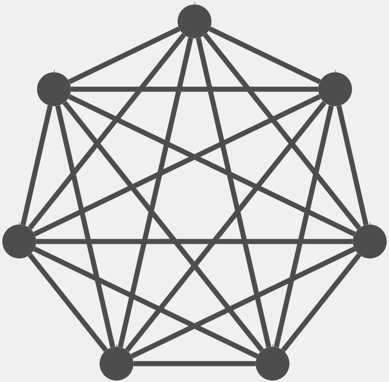

K_7

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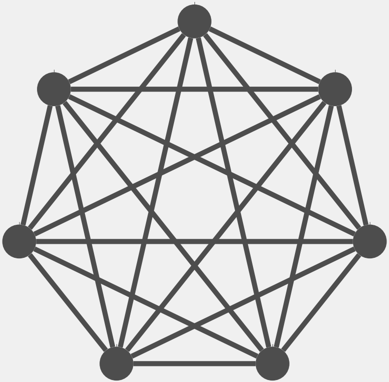

A $(7, 6, 4, 4)$-decomposition of K_7

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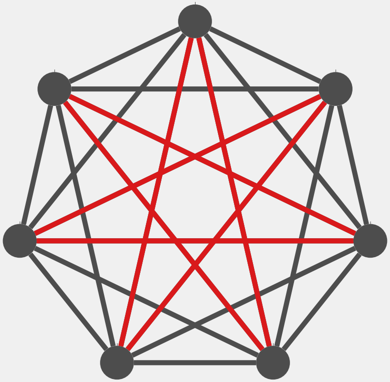

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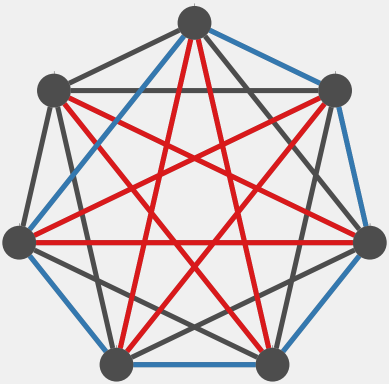

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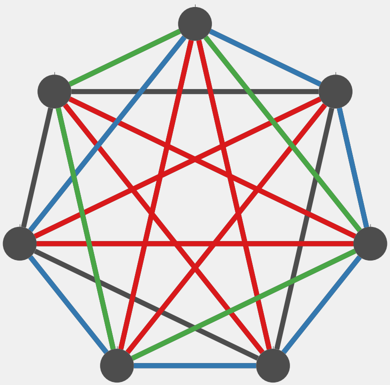

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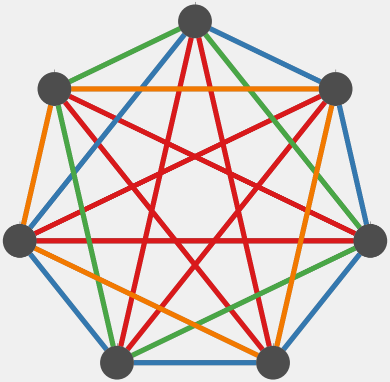

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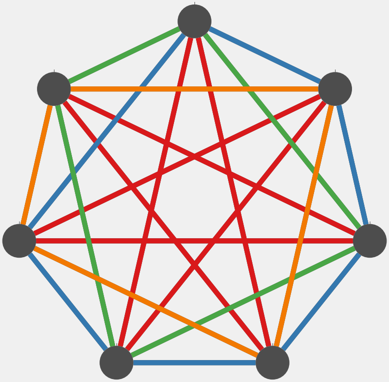

A $(7, 6, 4, 4)$-decomposition of K_7

My lists of cycle lengths will always be non-increasing.

If there exists an $(m_1, m_2, \ldots, m_t)$-decomposition of K_n then

(1) n is odd;

(2) $n \geqslant m_1, m_2, \ldots, m_t \geqslant 3$; and

(3) $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$.

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Alspach's cycle decomposition problem (1981): Prove (1), (2) and (3) are also sufficient for an $(m_1, m_2, \ldots, m_t)$-decomposition of K_n.

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Alspach also posed the equivalent problem for $K_n - I$ when n is even.

History (fixed cycle length)

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When does there exist an $(m, m, \ldots, m)$-decomposition of K_n?

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When does there exist an $(m, m, \dots, m)$-decomposition of K_n?

Kirkman (1846): solution for $m = 3$

Walecki (1890): solution for $m = n$

Kotzig (1965): solution for $n \equiv 1 \pmod{2m}$, $m \equiv 0 \pmod 4$

Rosa (1966): solution for $n \equiv 1 \pmod{2m}$, $m \equiv 2 \pmod 4$

Rosa (1966): solution for $m = 5$ and $m = 7$

Rosa, Huang (1975): solution for $m = 6$

Bermond, Huang, Sotteau (1978): reduction of the problem for even m

Hoffman, Lindner, Rodger (1989): reduction of the problem for odd m

Alspach, Gavlas, Šajna (2001–2002): solution for each m

History (varied cycle lengths)

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When does there exist an $(m_1, \ldots, m_t)$-decomposition of K_n?

Remember $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_t$.

History (varied cycle lengths)

When does there exist an $(m_1, \ldots, m_t)$-decomposition of K_n?

(1969+): results on Oberwolfach problem etc.

Heinrich, Horák, Rosa (1989): solution for $\{m_1, \ldots, m_t\} \subseteq \{2^k, 2^{k+1}\}$, $\{3, 4, 6\}$, $\{n-2, n-1, n\}$

Adams, Bryant, Khodkar (1998): solution for $m_1 \leqslant 10$ and $|\{m_1, \ldots, m_t\}| \leqslant 2$

Balister (2001): solution for $\{m_1, \ldots, m_t\} \subseteq \{3, 4, 5\}$

Balister (2001): solution for n large and $m_1 \leqslant \lfloor \frac{n-112}{20} \rfloor$

Bryant, Maenhaut (2004): solution for $\{m_1, \ldots, m_t\} \subseteq \{3, n\}$

Bryant, Horsley (2009): solution for $m_t \geqslant \frac{n+5}{2}$

Bryant, Horsley (2010): solution for $m_1 \leqslant \frac{n-1}{2}$ and $m_1 \leqslant 2m_2$

Bryant, Horsley (2010): solution for large n

Remember $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_t$.

The solution to Alspach's problem

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Theorem. There is an $(m_1, m_2, \ldots, m_t)$-decomposition of K_n if and only if

(1) n is odd;

(2) $n \geqslant m_1, m_2, \ldots, m_t \geqslant 3$; and

(3) $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$.

The analogous result for $K_n - I$ when n is even also holds.

– Bryant, Horsley, Pettersson (2014)

Part 2:

Generalisation to multigraphs

Cycle decompositions of complete multigraphs

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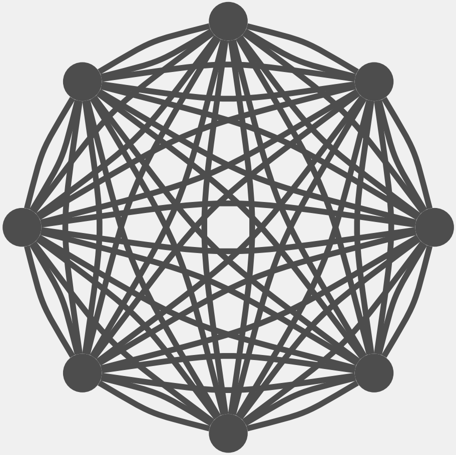
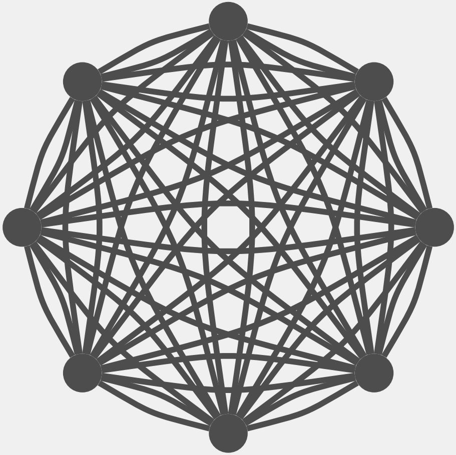


$2K_8$

Cycle decompositions of complete multigraphs



A $(8^3, 3^{10}, 2)$-decomposition of $2K_8$

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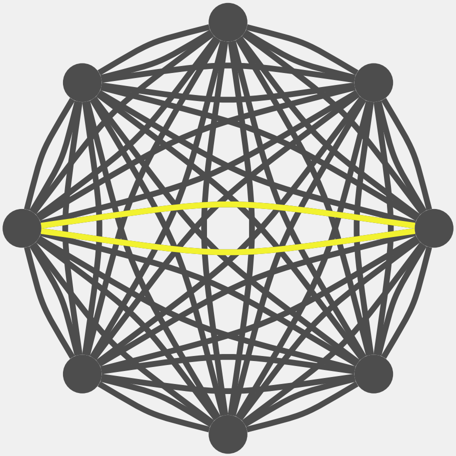

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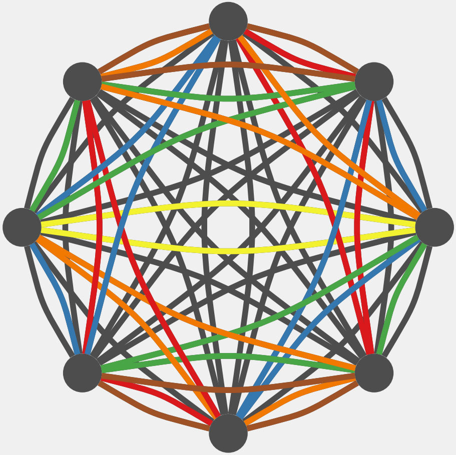

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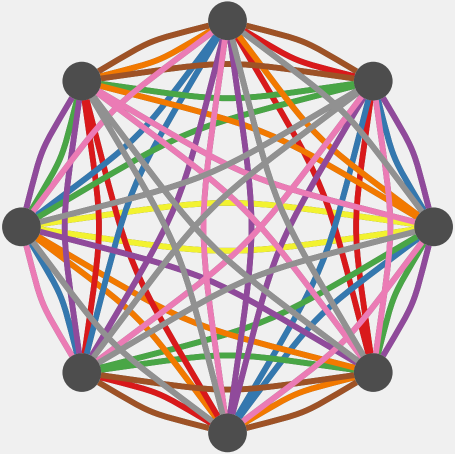

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Hanani (1961): solution for $m = 3$

Huang, Rosa (1973): solution for $m = 4$

Huang, Rosa (1975): solution for $m = 5$ and $m = 6$

Bermond, Sotteau (1977): solution for $m = 7$.

Bermond, Huang, Sotteau (1978): solution for $m \in \{8, 10, 12, 14\}$

Smith (2010): solution for $m = \lambda$

Bryant, Horsley, Maenhaut, Smith (2011): solution for each m

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Very little work on the case of varied cycle lengths.

The solution to Alspach's problem for multigraphs

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Theorem. There is an $(m_1, m_2, \ldots, m_t)$-decomposition of λK_n if and only if

(1) $\lambda(n-1)$ is even;

(2) $n \geqslant m_1, m_2, \ldots, m_t \geqslant 2$;

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(4) $|\{i : m_i = 2\}| \leqslant \frac{\lambda-1}{2}\binom{n}{2}$ if λ is odd; and

(5) $m_1 \leqslant 2 + \sum_{i=2}^{t}(m_i - 2)$ if λ is even.

The analogous result for $\lambda K_n - I$ when $\lambda(n-1)$ is odd also holds.

– Bryant, Horsley, Maenhaut, Smith (2015+)

Remember $m_1 \geqslant m_2, \ldots, m_t$.

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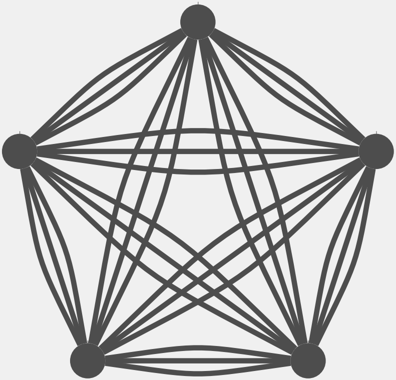

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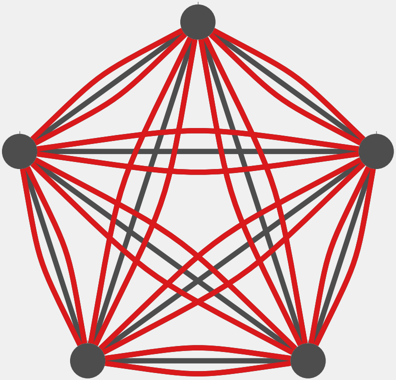

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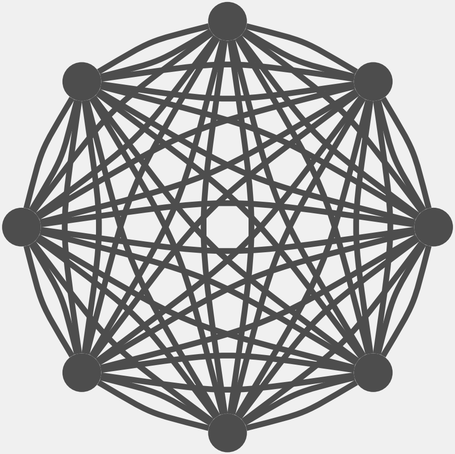

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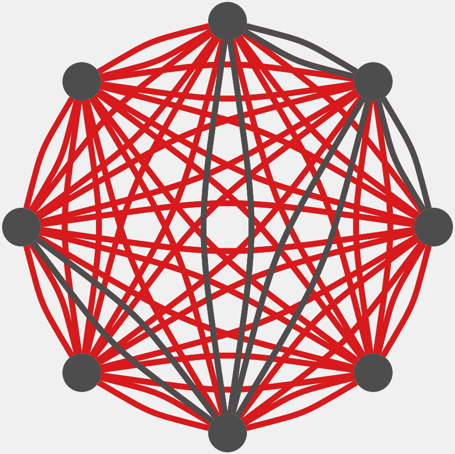

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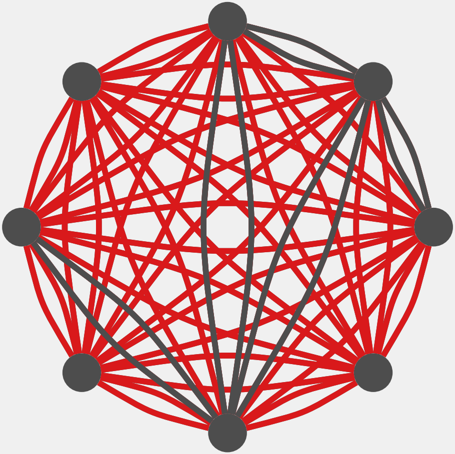

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For this to exist there would have to be a graph G with 5 edges such that $2G$ has a $(6, 4)$-decomposition.

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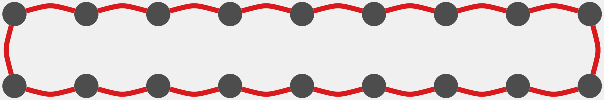

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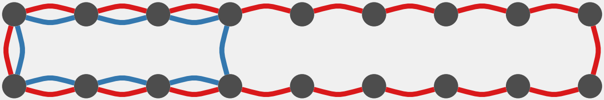

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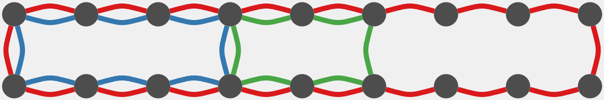

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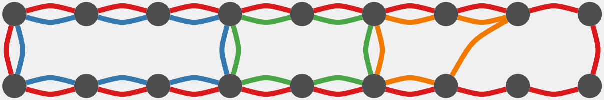

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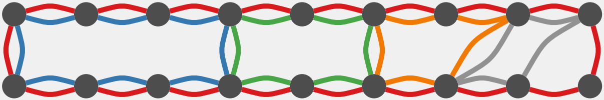

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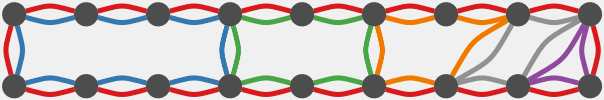

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The analogous result for $\lambda K_n - I$ when $\lambda(n-1)$ is odd also holds.

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