Alspach’s cycle decomposition problem for multigraphs

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Joint work with Darryn Bryant, Barbara Maenhaut and Ben Smith
(University of Queensland)
Part 1: 

Alspach’s problem
Cycle decompositions of complete graphs
Cycle decompositions of complete graphs

cycle decomposition: set of cycles in a graph such that each edge of the graph appears in exactly one cycle.
Cycle decompositions of complete graphs

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\[ K_7\]
Cycle decompositions of complete graphs

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A \((7, 6, 4, 4)\)-decomposition of \(K_7\)
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A \((7, 6, 4, 4)\)-decomposition of \(K_7\)

My lists of cycle lengths will always be non-increasing.
If there exists an \((m_1, m_2, \ldots, m_t)\)-decomposition of \(K_n\) then
(1) \(n\) is odd;
(2) \(n \geq m_1, m_2, \ldots, m_t \geq 3\); and
(3) \(m_1 + m_2 + \cdots + m_t = \binom{n}{2}\).
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Alspach’s cycle decomposition problem (1981): Prove (1), (2) and (3) are also sufficient for an \((m_1, m_2, \ldots, m_t)\)-decomposition of \(K_n\).
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Alspach’s cycle decomposition problem (1981): Prove (1), (2) and (3) are also sufficient for an \((m_1, m_2, \ldots, m_t)\)-decomposition of \(K_n\).

Alspach also posed the equivalent problem for \(K_{n-l}\) when \(n\) is even.
History (fixed cycle length)

When does there exist an $(m, m, \ldots, m)$-decomposition of $K_n$?

Kirkman (1846): solution for $m = 3$

Walecki (1890): solution for $m = n$

Kotzig (1965): solution for $n \equiv 1 \pmod{2}, m \equiv 0 \pmod{4}$

Rosa (1966): solution for $n \equiv 1 \pmod{2}, m \equiv 2 \pmod{4}$

Rosa (1966): solution for $m = 5$ and $m = 7$

Rosa, Huang (1975): solution for $m = 6$

Bermond, Huang, Sotteau (1978): reduction of the problem for even $m$

Hoffman, Lindner, Rodger (1989): reduction of the problem for odd $m$

Alspach, Gavlas, ˇSajna (2001–2002): solution for each $m$
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History (varied cycle lengths)

When does there exist an \((m_1, \ldots, m_t)\)-decomposition of \(K_n\)?

(1969+): results on Oberwolfach problem etc.

Heinrich, Horák, Rosa (1989): solution for \(\{m_1, \ldots, m_t\} \subseteq \{2^k, 2^k+1\}, \{3, 4, 6\}\), \(\{n-2, n-1, n\}\)

Adams, Bryant, Khodkar (1998): solution for \(m_1 \leq 10\) and \(|\{m_1, \ldots, m_t\}| \leq 2\)

Balister (2001): solution for \(\{m_1, \ldots, m_t\} \subseteq \{3, 4, 5\}\)

Balister (2001): solution for \(n\) large and \(m_1 \leq \lfloor \frac{n-11}{20} \rfloor\)

Bryant, Maenhaut (2004): solution for \(\{m_1, \ldots, m_t\} \subseteq \{3, n\}\)

Bryant, Horsley (2009): solution for \(m_t \geq n+5\)

Bryant, Horsley (2010): solution for \(m_1 \leq n-1\) and \(m_1 \leq 2m_2\)

Bryant, Horsley (2010): solution for large \(n\)

Remember \(m_1 \geq m_2 \geq \cdots \geq m_t\).
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Remember \(m_1 \geq m_2 \geq \cdots \geq m_t\).
The solution to Alspach’s problem

Theorem. There is an $(m_1, m_2, \ldots, m_t)$-decomposition of $K_n$ if and only if

1. $n$ is odd;
2. $n \geq m_1, m_2, \ldots, m_t \geq 3$; and
3. $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$.

The analogous result for $K_n - I$ when $n$ is even also holds.

– Bryant, Horsley, Pettersson (2014)
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Part 2:

Generalisation to multigraphs
Cycle decompositions of complete multigraphs
Cycle decompositions of complete multigraphs

$2K_8$
A $(8^3, 3^{10}, 2)$-decomposition of $2K_8$
Cycle decompositions of complete multigraphs

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A \((8^3, 3^{10}, 2)\)-decomposition of \(2K_8\)
When does there exist an \((m, m, \ldots, m)\)-decomposition of \(\lambda K_n\)?

Hanani (1961): solution for \(m = 3\).
Huang, Rosa (1973): solution for \(m = 4\).
Huang, Rosa (1975): solution for \(m = 5\) and \(m = 6\).
Bermond, Sotteau (1977): solution for \(m = 7\).
Bermond, Huang, Sotteau (1978): solution for \(m \in \{8, 10, 12, 14\}\).
Smith (2010): solution for \(m = \lambda\).
Bryant, Horsley, Maenhaut, Smith (2011): solution for each \(m\).

Very little work on the case of varied cycle lengths.
History (complete multigraphs)

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The solution to Alspach’s problem for multigraphs

Theorem. There is an 

\((m_1, m_2, \ldots, m_t)\)-decomposition of \(\lambda K_n\) if and only if

1. \(\lambda(n-1)\) is even;
2. \(n \geq m_1, m_2, \ldots, m_t \geq 2;\)
3. \(m_1 + m_2 + \cdots + m_t = \lambda \left(\binom{n}{2}\right)\);
4. \(|\{i : m_i = 2\}| \leq \lambda - 1\) if \(\lambda\) is odd; and
5. \(m_1 \leq 2 + \sum_{i=2}^{t} (m_i - 2)\) if \(\lambda\) is even.

The analogous result for \(\lambda K_n - I\) when \(\lambda(n-1)\) is odd also holds.

– Bryant, Horsley, Maenhaut, Smith (2015+)

Remember \(m_1 \geq m_2, \ldots, m_t\).
The solution to Alspach’s problem for multigraphs

**Theorem.** There is an \((m_1, m_2, \ldots, m_t)\)-decomposition of \(\lambda K_n\) if and only if

1. \(\lambda(n - 1)\) is even;
2. \(n \geq m_1, m_2, \ldots, m_t \geq 2\);
3. \(m_1 + m_2 + \cdots + m_t = \lambda(n)\);
4. \(|\{i : m_i = 2\}| \leq \frac{\lambda - 1}{2} \binom{n}{2}\) if \(\lambda\) is odd; and
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Remember \(m_1 \geq m_2, \ldots, m_t\).
Why is $|\{i : m_i = 2\}| \leq \frac{\lambda - 1}{2} \binom{n}{2}$ necessary when $\lambda$ is odd?
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There is no \((6, 4, 2^{23})\)-decomposition of \(2K_8\).

For this to exist there would have to be a graph \( G \) with 5 edges such that \( 2G \) has a \((6, 4)\)-decomposition.
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In general, the cycles of length greater than 2 must decompose \( 2G \) for some (multi)graph \( G \).
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In general, the cycles of length greater than 2 must decompose $2G$ for some (multi)graph $G$.

**Lemma.** If there is a $(m_1, \ldots, m_t)$-decomposition of $2G$ for some (multi)graph $G$, then $m_1 \leq 2 + \sum_{i=2}^{t} (m_i - 2)$. 
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An $(18, 8, 6, 5, 4, 3)$-decomposition of $2G$
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An $(18, 8, 6, 5, 4, 3)$-decomposition of $2G$
Proof of sufficiency

Reduction lemma.
If there is a decomposition of $\lambda K^n$ for each $(\lambda, n)$-ancestor list, then our main theorem holds for $\lambda K^n$. 

$(\lambda, n)$-ancestor lists are of the form $(n^\alpha, k^\beta, 3^\gamma)$. 

$\lambda$-induction lemma.
If our main theorem holds for $K^n$ and $2K^n$, then there is a decomposition of $\lambda K^n$ for each $(\lambda, n)$-ancestor list.

Many $n$'s lemma.
There is a decomposition of $2K^n$ for each $(\lambda, n)$-ancestor list containing at least $n - 3/2$ occurrences of $n$.

Few $n$'s lemma.
If our main theorem holds for $2K^n - 1$, then there is a decomposition of $2K^n$ for each $(\lambda, n)$-ancestor list containing less than $n - 3/2$ occurrences of $n$. 
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**Reduction lemma.** If there is a decomposition of $\lambda K_n$ for each $(\lambda, n)$-ancestor list, then our main theorem holds for $\lambda K_n$.

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**Few $n$’s lemma.** If our main theorem holds for $2K_{n-1}$, then there is a decomposition of $2K_n$ for each $(\lambda, n)$-ancestor list containing less than $\frac{n-3}{2}$ occurrences of $n$. 
The solution to Alspach’s problem for multigraphs

**Theorem.** There is an \((m_1, m_2, \ldots, m_t)\)-decomposition of \(\lambda K_n\) if and only if

1. \(\lambda(n - 1)\) is even;
2. \(n \geq m_1, m_2, \ldots, m_t \geq 2\);
3. \(m_1 + m_2 + \cdots + m_t = \lambda\left(\begin{array}{c}n \\end{array}\right)\);
4. \(|\{i : m_i = 2\}| \leq \frac{\lambda - 1}{2} \left(\begin{array}{c}n \\end{array}\right)\) if \(\lambda\) is odd; and
5. \(m_1 \leq 2 + \sum_{i=2}^{t}(m_i - 2)\) if \(\lambda\) is even.

The analogous result for \(\lambda K_n - I\) when \(\lambda(n - 1)\) is odd also holds.

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3. \(m_1 + m_2 + \cdots + m_t = \lambda \binom{n}{2}\);
4. \(|\{i : m_i = 2\}| \leq \frac{\lambda - 1}{2} \binom{n}{2}\) if \(\lambda\) is odd; and
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Remember \(m_1 \geq m_2, \ldots, m_t\).
That’s all.