Extending (part of) the Bruck-Ryser-Chowla Theorem to Coverings

Daniel Horsley (Monash University, Australia)

Joint work with Darryn Bryant, Melinda Buchanan, Barbara Maenhaut, and Victor Scharaschkin (University of Queensland)

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I acknowledge the four institutions at which I have been employed during the refereeing process.

Balanced Incomplete Block Designs

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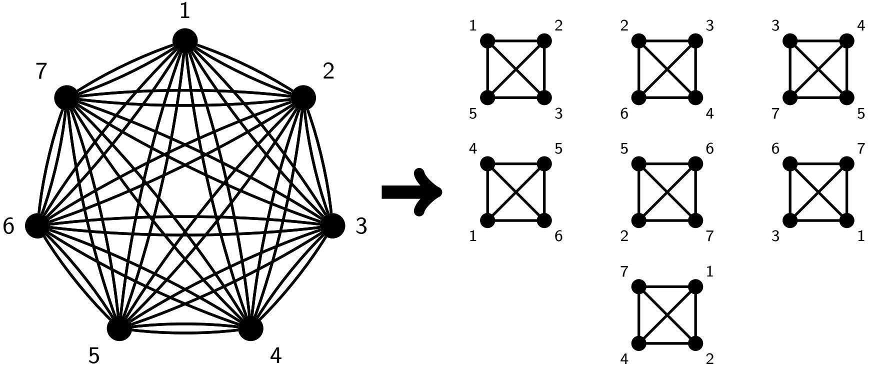

A (v, k, λ)-BIBD with $v = 7$, $k = 4$, $\lambda = 2$, having $b = 7$ blocks.

When do BIBDs exist?

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Obvious Necessary Conditions If there exists an (v, k, λ)-BIBD then

(1) $\lambda(v-1) \equiv 0 \pmod{k-1}$;

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- if v is even, then $k - \lambda$ is a perfect square; and
- if v is odd, then $z^2 = (k-\lambda)x^2 + (-1)^{(v-1)/2}\lambda y^2 = 0$ has a solution for integers x, y, z, not all zero.

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There are very few examples of (v, k, λ)-BIBDs which are known not to exist, but which are not ruled out by the above results.

Pair covering designs

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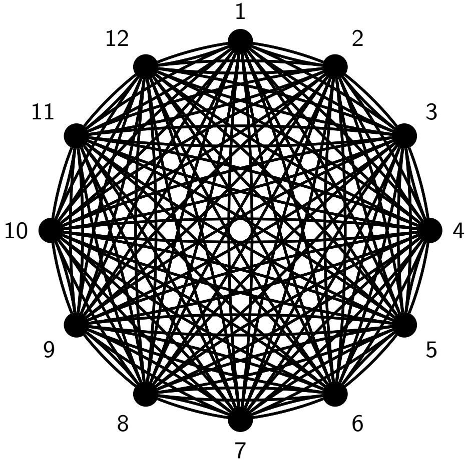
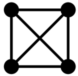
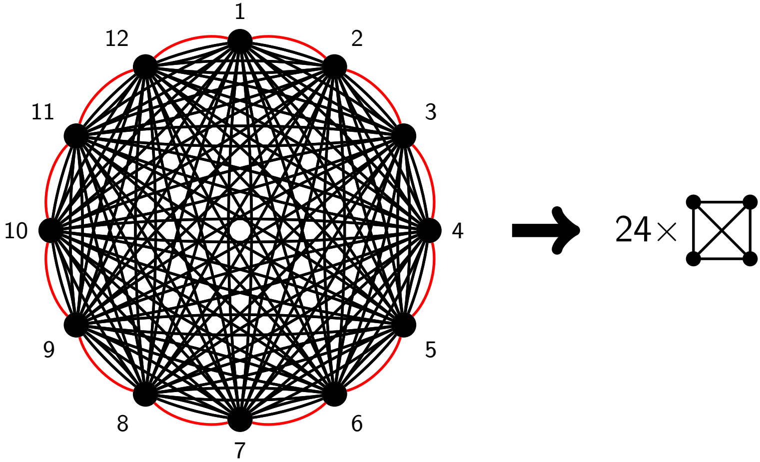




$v = 12,\ k = 4,\ \lambda = 2.$

Pair covering designs



A $(12, 4, 2)$-covering.

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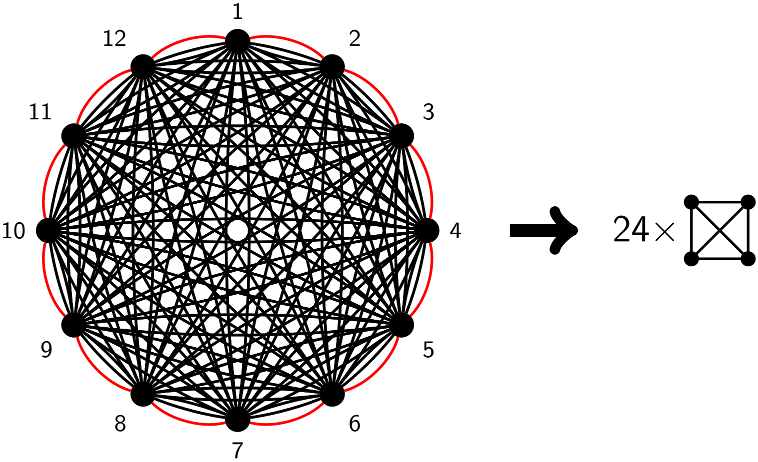
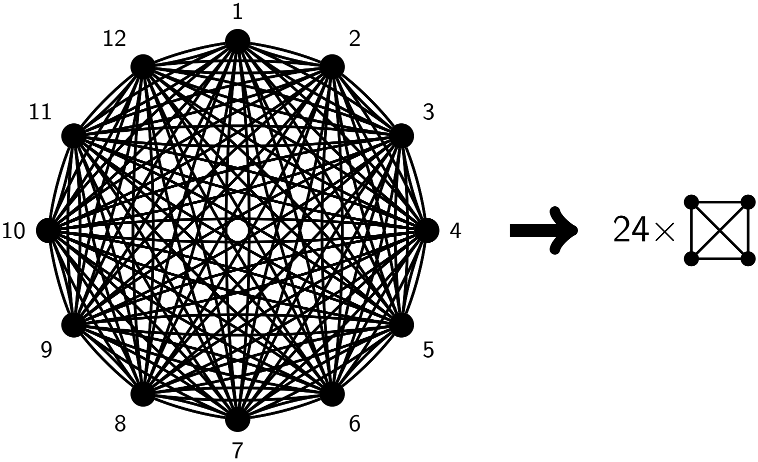


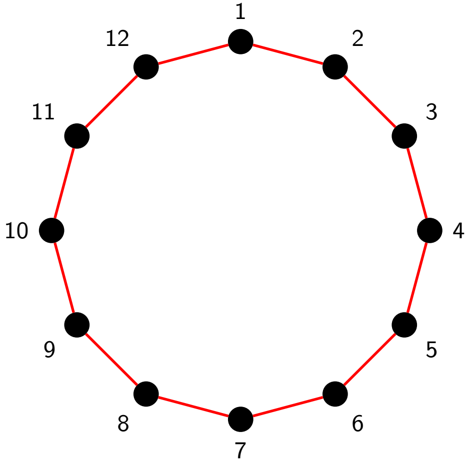
A $(12, 4, 2)$-covering with a C_{12} excess.

Pair covering designs



Any $(12, 4, 2)$-covering with 24 blocks will have a 2-regular excess.

Pair covering designs



A C_{12} excess.

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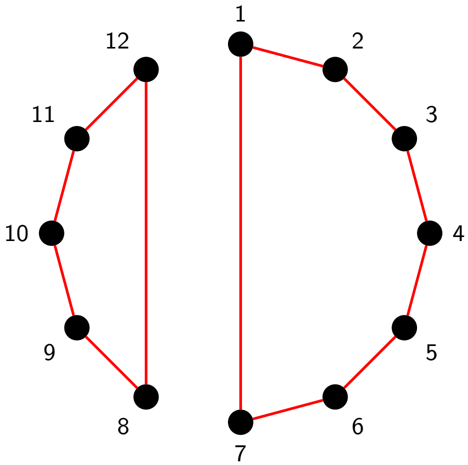

A $C_7 \cup C_5$ excess.

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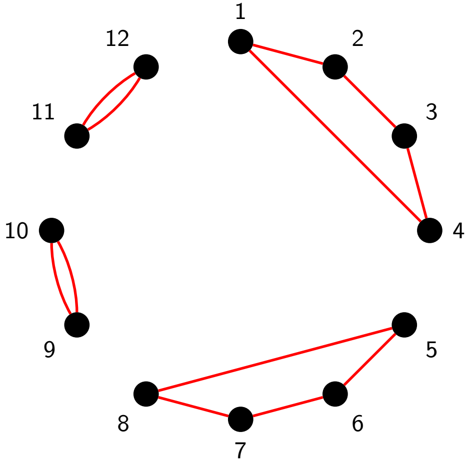

A $C_4 \cup C_4 \cup C_2 \cup C_2$ excess.

Bounds on coverings

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There are few general results which increase this lower bound (most are for specific (v, k, λ) and involve computer search).

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- Our results focus on non-existence of certain coverings whose excess would necessarily be 2-regular.
- Todorov (1989) established the non-existence of certain coverings with $b < v$ and $\lambda = 1$.

Our results

Fischer-type result Any (v, k, λ)-covering with a 2-regular excess has at least v blocks, unless $(v, k, \lambda) = (3\lambda + 6, 3\lambda + 3, \lambda)$ for $\lambda \geq 1$ or $(v, k, \lambda) \in \{(8, 4, 1), (14, 6, 1), (14, 8, 2)\}$.

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BRC-type result If a (v, k, λ)-covering with v blocks with a 2-regular excess exists for v even, then one of $k - \lambda - 2$ or $k - \lambda + 2$ is a perfect square, unless $(v, k, \lambda) = (\lambda + 4, \lambda + 2, \lambda)$ for even $\lambda \geq 1$.

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Theorem $C_\lambda(v, k) \geq L_\lambda(v, k) + 1$ when

- $\lambda(v - 1) + 2 \equiv 0 \pmod{k - 1}$;
- $\lambda v(v - 1) + 2v \equiv 0 \pmod{k(k - 1)}$;
- $v \leq \frac{k^2-k-2}{\lambda} + 1$; and
- if $v = \frac{k^2-k-2}{\lambda} + 1$ then v is even and neither $k - \lambda - 2$ nor $k - \lambda + 2$ is a perfect square;

unless (v, k, λ) is in the exceptions listed above.

Incidence matrices

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$$\begin{array}{r} \\ \text{point } x_1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccccc} & & & & & & & & & b_2 & & \end{array} \\ \left(\begin{array}{cccccccccccc} & & & & & & & & & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & \color{red}{0} & 0 & 0 \\ & & & & & & & & & & & \end{array}\right) \end{array}$$

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What does MM^T look like?

If M is the incidence matrix of a $(10, k, \lambda)$-covering with excess C_{10}, then MM^T is the 10×10 matrix

$$\begin{pmatrix}
r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 \\
\lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\
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\lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 \\
\lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & r
\end{pmatrix}.$$

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If M is the incidence matrix of a $(10, k, \lambda)$-covering with excess $C_6 \cup C_4$, MM^T is the 10×10 matrix

$$\begin{pmatrix} r & \lambda+1 & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda \\ \lambda+1 & \lambda & \lambda & \lambda & \lambda+1 & r & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & r & \lambda+1 & \lambda & \lambda+1 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & r & \lambda+1 \\ \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda+1 & \lambda & \lambda+1 & r \end{pmatrix}.$$

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$$\begin{pmatrix}
r & \lambda+1 & \lambda & \lambda & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda+1 & r & \lambda+1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
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\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & r & \lambda+2 \\
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\end{pmatrix}.$$

Proof of our results

Lemma If M is the incidence matrix of a (v, k, λ)-covering with a 2-regular excess then

$$\det(MM^T) = rk(r - \lambda + 2)^{t-1}(r - \lambda - 2)^e z^2$$

for some non-zero integer z, where r is the number of blocks on each point, t is the number of cycles in the excess, and e is the number of cycles of even length.

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Proof sketch If $r - \lambda > 2$ then $\det(MM^T) \neq 0$ and it follows that $\text{rank}(M) = v$.

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Proof sketch Note $\det(MM^T) = (\det(M))^2$ and $r = k$, so if $r - \lambda > 2$ and $k - \lambda - 2$ and $k - \lambda + 2$ are not perfect squares then t is odd and e is even.

Notes and future plans

Notes

- We also considered the case of $K_{1,k} \cup K_2 \cup \cdots \cup K_2$ excesses.
- Very similar results can be obtained for packings.
- Our results establish the non-existence of certain $(K_k - e)$-decompositions of λK_v.

Future plans

- Considering other kinds of excesses.
- Adapting the "hard" half of the Bruck-Ryser-Chowla Theorem.