Extending (part of) the Bruck-Ryser-Chowla Theorem to Coverings

Daniel Horsley (Monash University, Australia)

Joint work with Darryn Bryant, Melinda Buchanan, Barbara Maenhaut, and Victor Scharaschkin (University of Queensland)
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I acknowledge the four institutions at which I have been employed during the refereeing process.
Balanced Incomplete Block Designs

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
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</table>

An $(v, k, \lambda)$-BIBD with $v = 7$, $k = 4$, $\lambda = 2$, having $b = 7$ blocks.
A $(v, k, \lambda)$-BIBD with $v = 7$, $k = 4$, $\lambda = 2$, having $b = 7$ blocks.
When do BIBDs exist?

Obvious Necessary Conditions

If there exists an \((v, k, \lambda)\)-BIBD then

\[(1) \lambda (v - 1) \equiv 0 \pmod{k - 1};\]
\[(2) \lambda v (v - 1) \equiv 0 \pmod{k (k - 1)}\].

Fischer's Inequality (1940)

Any \((v, k, \lambda)\)-BIBD has at least \(v\) blocks.

Bruck-Ryser-Chowla Theorem (1950)

If a \((v, k, \lambda)\)-BIBD with exactly \(v\) blocks exists then

\[\text{if } v \text{ is even, then } k - \lambda \text{ is a perfect square; and}\]
\[\text{if } v \text{ is odd, then } z^2 = (k - \lambda) x^2 + (-1)^{(v - 1)/2} \lambda y^2 = 0 \text{ has a solution for integers } x, y, z, \text{ not all zero.}\]

There are very few examples of \((v, k, \lambda)\)-BIBDs which are known not to exist, but which are not ruled out by the above results.
When do BIBDs exist?

**Obvious Necessary Conditions** If there exists an \((v, k, \lambda)\)-BIBD then

1. \(\lambda(v - 1) \equiv 0 \pmod{k - 1}\);
2. \(\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}\).
When do BIBDs exist?

**Obvious Necessary Conditions**  If there exists an \((\nu, k, \lambda)\)-BIBD then

1. \(\lambda(\nu - 1) \equiv 0 \pmod{k - 1}\);
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**Fischer’s Inequality (1940)**  Any \((\nu, k, \lambda)\)-BIBD has at least \(\nu\) blocks.

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There are very few examples of \((v, k, \lambda)\)-BIBDs which are known not to exist, but which are not ruled out by the above results.
Pair covering designs

$v = 12$, $k = 4$, $\lambda = 2$. 
Pair covering designs

\[ v = 12, \quad k = 4, \quad \lambda = 2. \]
Pair covering designs

A (12, 4, 2)-covering.
A $(12, 4, 2)$-covering with a $C_{12}$ excess.
Pair covering designs

Any \((12, 4, 2)\)-covering with 24 blocks will have a 2-regular excess.
Pair covering designs

A $C_{12}$ excess.
Pair covering designs

A $C_7 \cup C_5$ excess.
Pair covering designs

A $C_4 \cup C_4 \cup C_2 \cup C_2$ excess.
Bounds on coverings

Let $C_\lambda(v, k)$ be the minimum number of blocks required for a $(v, k, \lambda)$-covering. Schönhheim Bound

$$C_\lambda(v, k) \geq L_\lambda(v, k)$$

where

$$L_\lambda(v, k) = \left\lceil \frac{v}{k} \left\lceil \frac{\lambda(v - 1)}{k - 1} \right\rceil \right\rceil.$$

Hanani

$$C_\lambda(v, k) \geq L_\lambda(v, k) + 1 \text{ when } \lambda(v - 1) \equiv 0 \pmod{k - 1} \text{ and } \lambda v(v - 1) \equiv 1 \pmod{k}.$$

There are few general results which increase this lower bound (most are for specific $(v, k, \lambda)$ and involve computer search).
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There are few general results which increase this lower bound (most are for specific $(v, k, \lambda)$ and involve computer search).
General improvements to the Schönhheim Bound

Fischer's Inequality and the Bruck-Ryser-Chowla Theorem establish the non-existence of certain coverings whose excess would necessarily be empty.

Bose and Connor (1952) used similar methods to establish the non-existence of certain coverings whose excess would necessarily be 1-regular.

Our results focus on non-existence of certain coverings whose excess would necessarily be 2-regular.

Todorov (1989) established the non-existence of certain coverings with $b < v$ and $\lambda = 1$. 
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Todorov (1989) established the non-existence of certain coverings with $b < v$ and $\lambda = 1$. 
Our results

**Fischer-type result**  Any \((v, k, \lambda)\)-covering with a 2-regular excess has at least \(v\) blocks, unless \((v, k, \lambda) = (3\lambda + 6, 3\lambda + 3, \lambda)\) for \(\lambda \geq 1\) or \((v, k, \lambda) \in \{(8, 4, 1), (14, 6, 1), (14, 8, 2)\}\).
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**BRC-type result**  If a \((v, k, \lambda)\)-covering with \(v\) blocks with a 2-regular excess exists for \(v\) even, then one of \(k - \lambda - 2\) or \(k - \lambda + 2\) is a perfect square, unless \((v, k, \lambda) = (\lambda + 4, \lambda + 2, \lambda)\) for even \(\lambda \geq 1\).
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**Theorem** \(C_\lambda(v, k) \geq L_\lambda(v, k) + 1\) when

1. \(\lambda(v - 1) + 2 \equiv 0 \pmod{k - 1}\);  
2. \(\lambda v(v - 1) + 2v \equiv 0 \pmod{k(k - 1)}\);  
3. \(v \leq \frac{k^2-k-2}{\lambda} + 1\); and  
4. if \(v = \frac{k^2-k-2}{\lambda} + 1\) then \(v\) is even and neither \(k - \lambda - 2\) nor \(k - \lambda + 2\) is a perfect square;

unless \((v, k, \lambda)\) is in the exceptions listed above.
Incidence matrices

The incidence matrix $M$ of a $(v, k, \lambda)$-covering is a $v \times b$ matrix whose $(i, j)$ entry is 1 if point $i$ is in block $j$ and 0 otherwise.

$$
\begin{bmatrix}
\begin{array}{cccccc}
b_1 & b_2 & x_1 & x_2 & x_1 & x_2 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0
\end{array}
\end{bmatrix}
$$

We will be interested in the matrix $MM^T$. 
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$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
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$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

point $x_1$
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$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
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1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0
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\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0
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\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
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\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

We will be interested in the matrix $MM^T$. 
If $M$ is the incidence matrix of a $(10, k, \lambda)$-covering with excess $C_{10}$, then $MM^T$ is the $10 \times 10$ matrix
\[
\begin{pmatrix}
    r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 \\
    \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda \\
    \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 \\
\end{pmatrix}.
\]
What does $MM^T$ look like?

If $M$ is the incidence matrix of a $(10, k, \lambda)$-covering with excess $C_6 \cup C_4$, $MM^T$ is the $10 \times 10$ matrix

$$
\begin{pmatrix}
  r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda + 1 & \lambda & \lambda & \lambda & \lambda \\
  \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
  \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
  \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda \\
  \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda & \lambda & \lambda \\
  \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & \lambda & \lambda + 1 \\
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  \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 \\
  \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 \\
  \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r
\end{pmatrix}.
$$
What does $MM^T$ look like?

If $M$ is the incidence matrix of a $(10, k, \lambda)$-covering with excess $C_5 \cup C_3 \cup C_2$, $MM^T$ is the $10 \times 10$ matrix

$$
\begin{pmatrix}
    r & \lambda + 1 & \lambda & \lambda & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda + 1 & \lambda & \lambda & \lambda + 1 & r & \lambda & \lambda & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda & \lambda & r & \lambda + 1 & \lambda + 1 & \lambda & \lambda & \lambda \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & r & \lambda + 1 & \lambda \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 1 & \lambda + 1 & r & \lambda \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & r & \lambda + 2 \\
    \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda + 2
\end{pmatrix}.
$$
Proof of our results

**Lemma** If $M$ is the incidence matrix of a $(v, k, \lambda)$-covering with a 2-regular excess then

$$\det(MM^T) = rk(r - \lambda + 2)^{t-1}(r - \lambda - 2)^e z^2$$

for some non-zero integer $z$, where $r$ is the number of blocks on each point, $t$ is the number of cycles in the excess, and $e$ is the number of cycles of even length.
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**Fischer-type result**  Any $(v, k, \lambda)$-covering with a 2-regular excess has at least $v$ blocks, unless $(v, k, \lambda) = (3\lambda + 6, 3\lambda + 3, \lambda)$ for $\lambda \geq 1$ or $(v, k, \lambda) \in \{(8, 4, 1), (14, 6, 1), (14, 8, 2)\}$. 
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**Proof sketch** If $r - \lambda > 2$ then $\det( MM^T ) \neq 0$ and it follows that $\text{rank}(M) = v$. 
Proof of our results

**Lemma**  If $M$ is the incidence matrix of a $(v, k, \lambda)$-covering with a 2-regular excess then
\[ \det(MM^T) = rk(r - \lambda + 2)^{t-1}(r - \lambda - 2)^ez^2 \]
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Proof of our results

**Lemma** If $M$ is the incidence matrix of a $(v, k, \lambda)$-covering with a 2-regular excess then

\[ \det(MM^T) = rk(r - \lambda + 2)^{t-1}(r - \lambda - 2)^e z^2 \]

for some non-zero integer $z$, where $r$ is the number of blocks on each point, $t$ is the number of cycles in the excess, and $e$ is the number of cycles of even length.

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**Proof sketch** Note $\det(MM^T) = (\det(M))^2$ and $r = k$, so if $r - \lambda > 2$ and $k - \lambda - 2$ and $k - \lambda + 2$ are not perfect squares then $t$ is odd and $e$ is even.
Notes and future plans

Notes

- We also considered the case of $K_{1,k} \cup K_2 \cup \cdots \cup K_2$ excesses.
- Very similar results can be obtained for packings.
- Our results establish the non-existence of certain $(K_k - e)$-decompositions of $\lambda K_v$.

Future plans

- Considering other kinds of excesses.
- Adapting the “hard” half of the Bruck-Ryser-Chowla Theorem.