The existence of *k*-chromatic cycle systems

Daniel Horsley and David Pike

m-cycle systems

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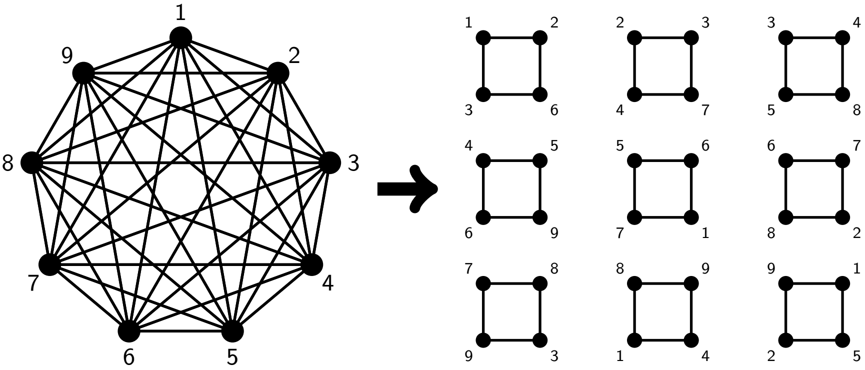

A 4-cycle system of order 9

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For each $m \geq 3$, we call values of v which satisfy (1), (2) and (3) *m-admissible*.

Partial m-cycle systems

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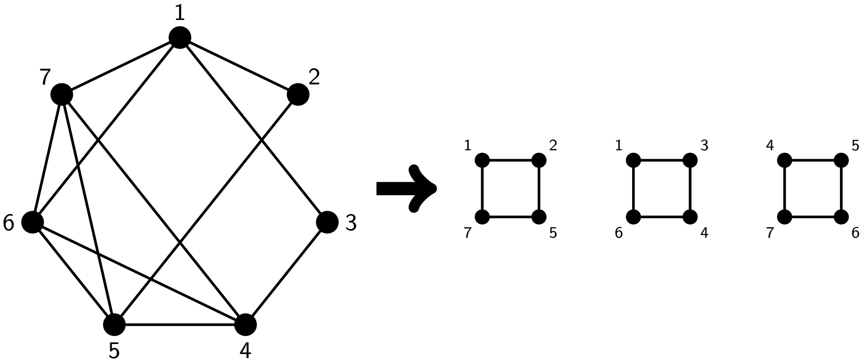
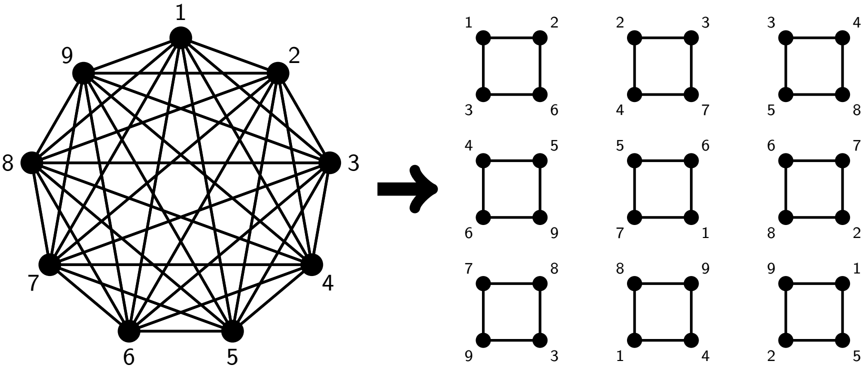


A partial 4-cycle system of order 7

Weak colourings

A *weak k-colouring* of a cycle system is a colouring of the vertices of the system with k colours which does not result in any monochromatic cycle.



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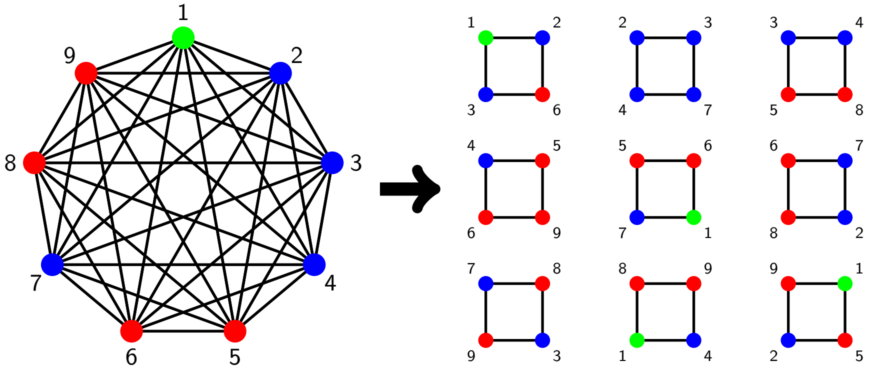

Not a weak 3-colouring

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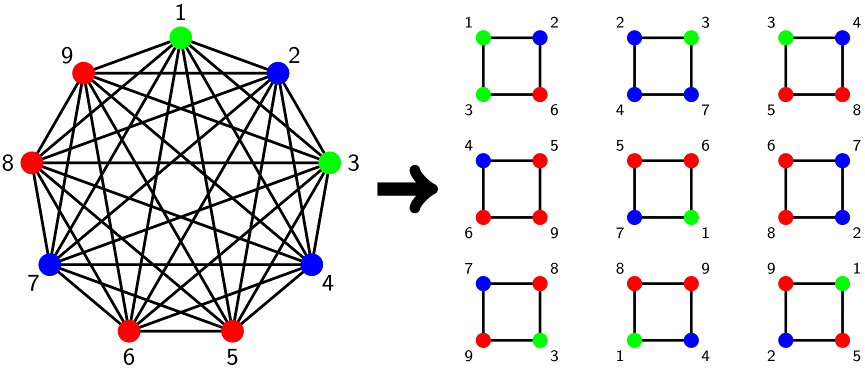

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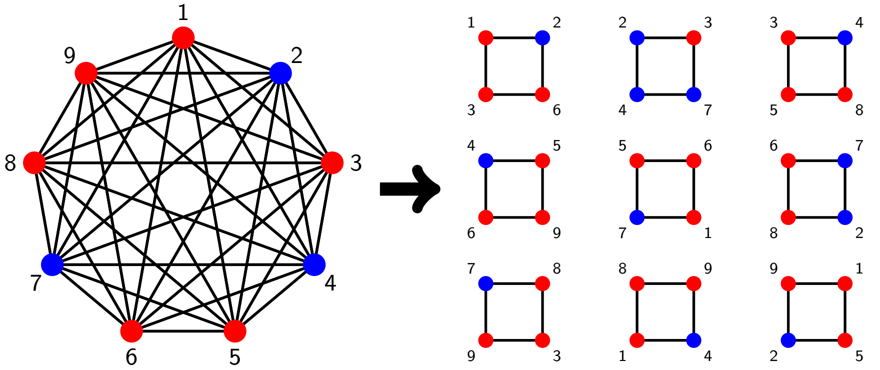

A weak 2-colouring

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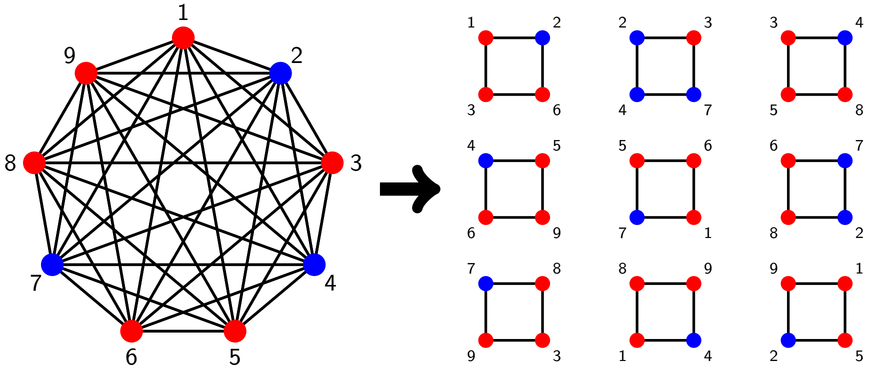

A cycle system is *weakly k-chromatic* if it has a weak k-colouring but no weak $(k-1)$-colouring.

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- For each $k \geq 3$ there is an integer $v_{k,3}$ such that for all 3-admissible $v \geq v_{k,3}$ there is a k-chromatic Steiner triple system of order v.

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- For $m \geq 4$, $k = 2$: an infinite family of v
  (Milici, Tuza)

Our result

For $k \geq 2$, $m \geq 3$ with $(k, m) \neq (2, 3)$, there is an integer $v_{k,m}$ such that there exists a k-chromatic m-cycle system of order v for all m-admissible integers $v \geq v_{k,m}$.

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Proof Outline:

First find a k-chromatic partial cycle system.

Then embed this in a (complete) cycle system without changing its chromatic number.

Finding a k-chromatic partial cycle system

Theorem (Erdős, Hajnal, Lovász): For all $m \geq 3$ and $k \geq 2$ there is a k-chromatic partial K_m system.

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Lemma: For all $m \geq 3$ and $k \geq 2$ there is a k-chromatic partial m-cycle system.

Embedding

Lemma: Let $k \geq 2$ and $m \geq 4$ and $(k, m) \notin \{(2,4),(2,5)\}$. A k-chromatic partial m-cycle system of order u can be embedded in a k-chromatic (complete) m-cycle system of order v for all m-admissible $v \geq 2m(u+1)$.

Our result (again)

Let k and m be integers such that $k \geq 2$, $m \geq 3$ and $(k, m) \neq (2, 3)$. Then there is an integer $v_{k,m}$ such that there exists a weakly k-chromatic m-cycle system of order v for all m-admissible integers $v \geq v_{k,m}$.

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Furthermore, for fixed m there are constants c_1 and c_2 such that

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