Perfect 1-Factorisations of Cubic Graphs

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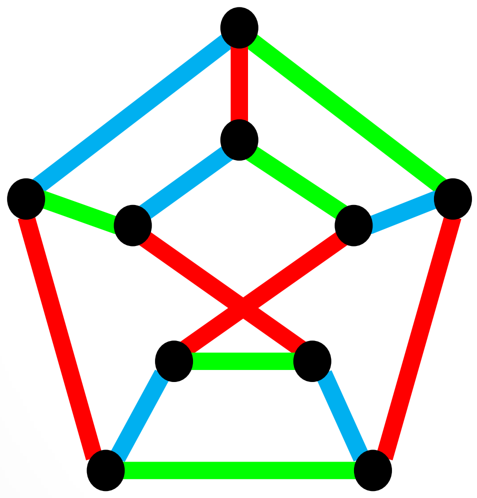
Outline

- Definitions and background
- The complete graph
- Cubic graphs
    - General results
    - Small examples

Definitions

- A 1-factor of a graph G is a 1-regular spanning subgraph of G.
- A 1-factorisation of a graph is a partition of the edges in the graph into 1-factors.

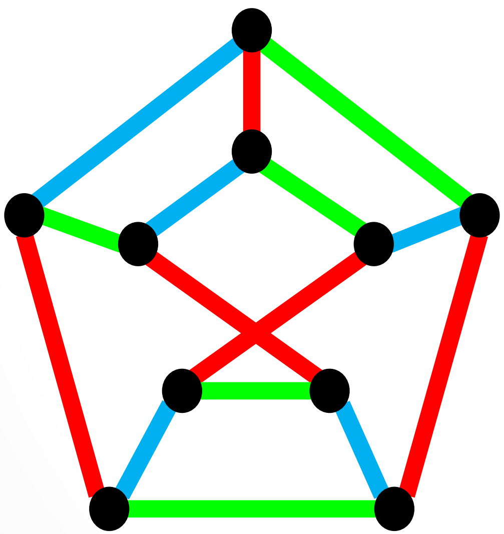
Example:



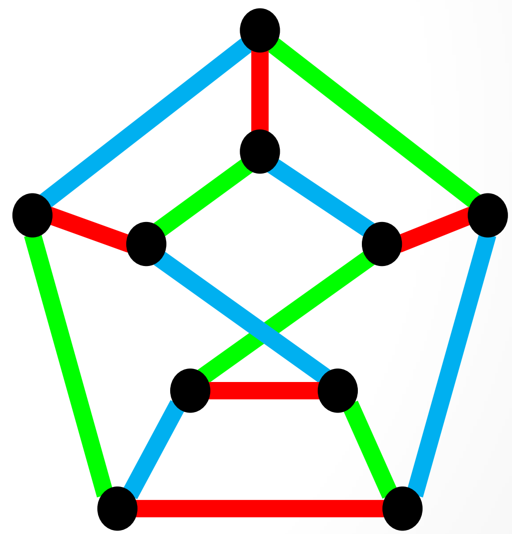
Definitions

- A 1-factorisation of a graph is perfect (P1F) if the union of any two 1-factors is a Hamilton cycle of the graph.
- A graph is cubic if every vertex has degree 3.

Example:



not a P1F



P1F

The Complete Graph

Conjecture (Kotzig 1960s):
The complete graph K_{2n} has a P1F for all $n \geq 2$.

- $2n \leq 52$.
- $2n = p + 1$ and $2n = 2p$ for odd prime p.
- Lots of sporadic examples.

Cubic Graphs

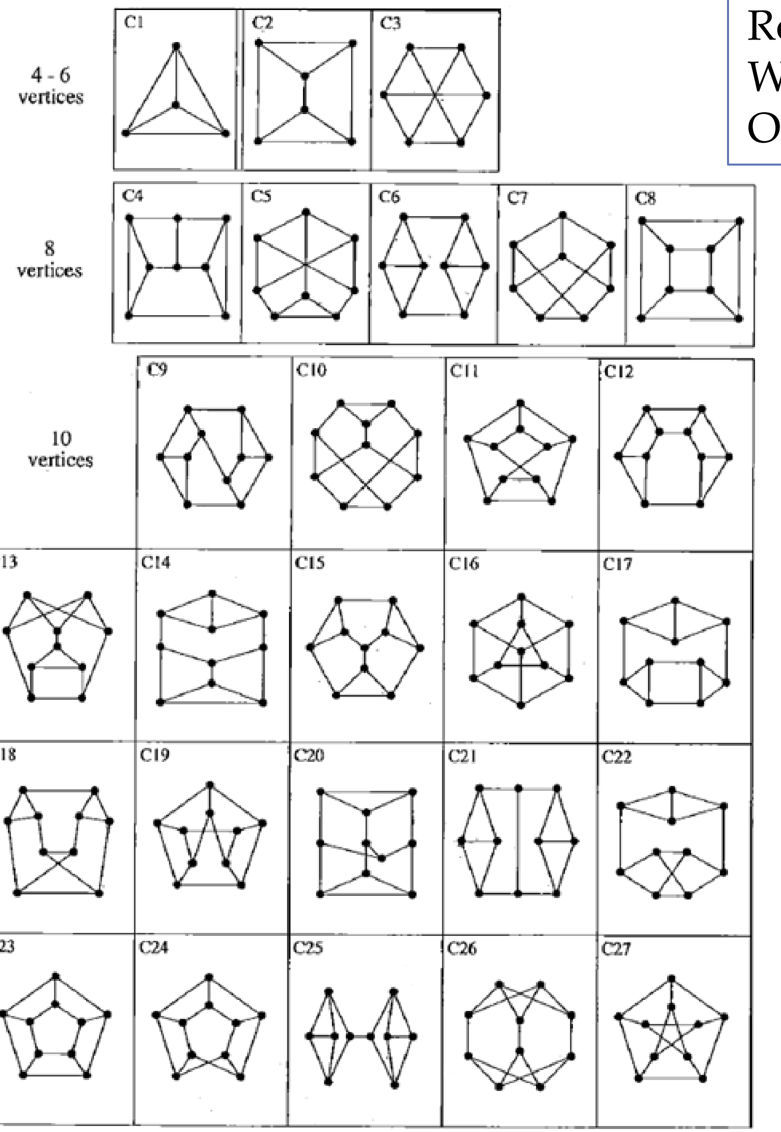
Open Problem (Kotzig 1960s):
Given a cubic graph, determine whether it has a P1F.

- There exists a cubic graph with a P1F on $2n \geq 4$ vertices.
- Results for some classes of graphs:
  - Generalised Peterson graph, $GP(n, k)$.
  - Cubic circulant graphs, $Circ(2n, \{a, n\})$.
- Other partial results and simplifications.

Cubic Graphs - Small Examples

- Connected cubic graphs on ≤ 10 vertices:

Ronald C. Read and Robin J. Wilson, *An Atlas of Graphs*, Oxford University Press, 1998.



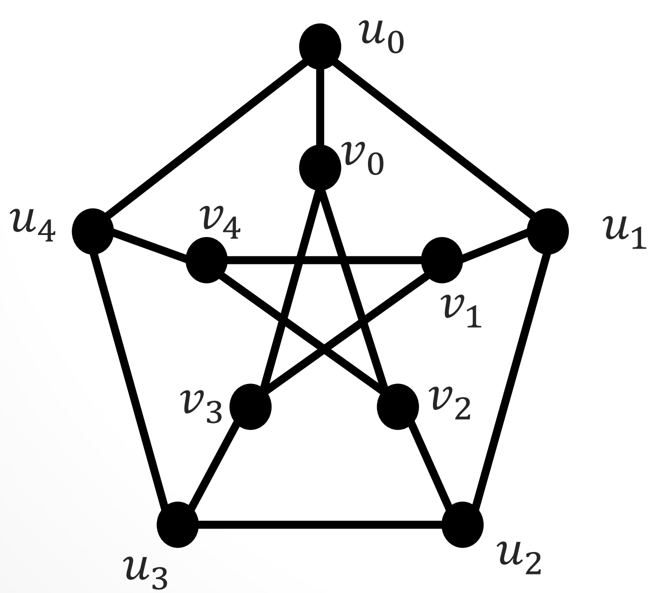
Cubic Graphs

Theorem (Kotzig, Labelle 1978):
For $r > 2$, if G is a bipartite r-regular graph that has a P1F then $|V(G)| \equiv 2 \ (mod\ 4)$.

- A bipartite cubic graph on $0\ (mod\ 4)$ vertices does not have a P1F.

Cubic Graphs

- Generalised Petersen graph $GP(n,k)$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$
  - $V = \{u_0, u_1, \dots, u_{n-1}\} \cup \{v_0, v_1, \dots, v_{n-1}\}$
  - $E = \{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+k}\ : 0 \leq i \leq n-1\}$
  - $GP(5,2)$



Cubic Graphs

Theorem (Bonvicini, Mazzuocolo 2011):

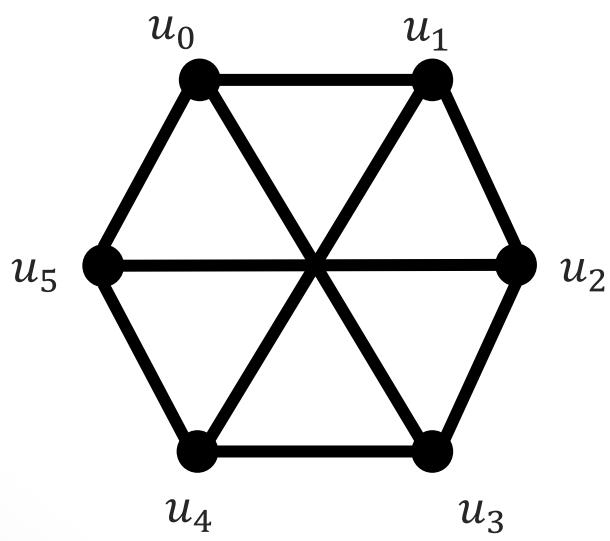
1. $GP(n,1)$ has a P1F iff $n = 3$;
2. $GP(n,2)$ has a P1F iff $n \equiv 3,4 \ (mod\ 6)$; and
3. $GP(n,3)$ has a P1F iff $n = 9$.

- Completely solved for $1 \leq k \leq 3$ (given here).
- Partial results for other values of n and k.

Cubic Graphs

- Cubic circulant graphs: $Circ(2n, \{a, n\})$, where $a \in \{1,2\}$
  - $V = \{u_0, u_1, \ldots, u_{2n-1}\}$
  - $E = \{u_i u_{i+a},\ u_i u_{i+n} : 0 \le i \le 2n-1\}$

Example: $Circ(6, \{1, 3\})$:



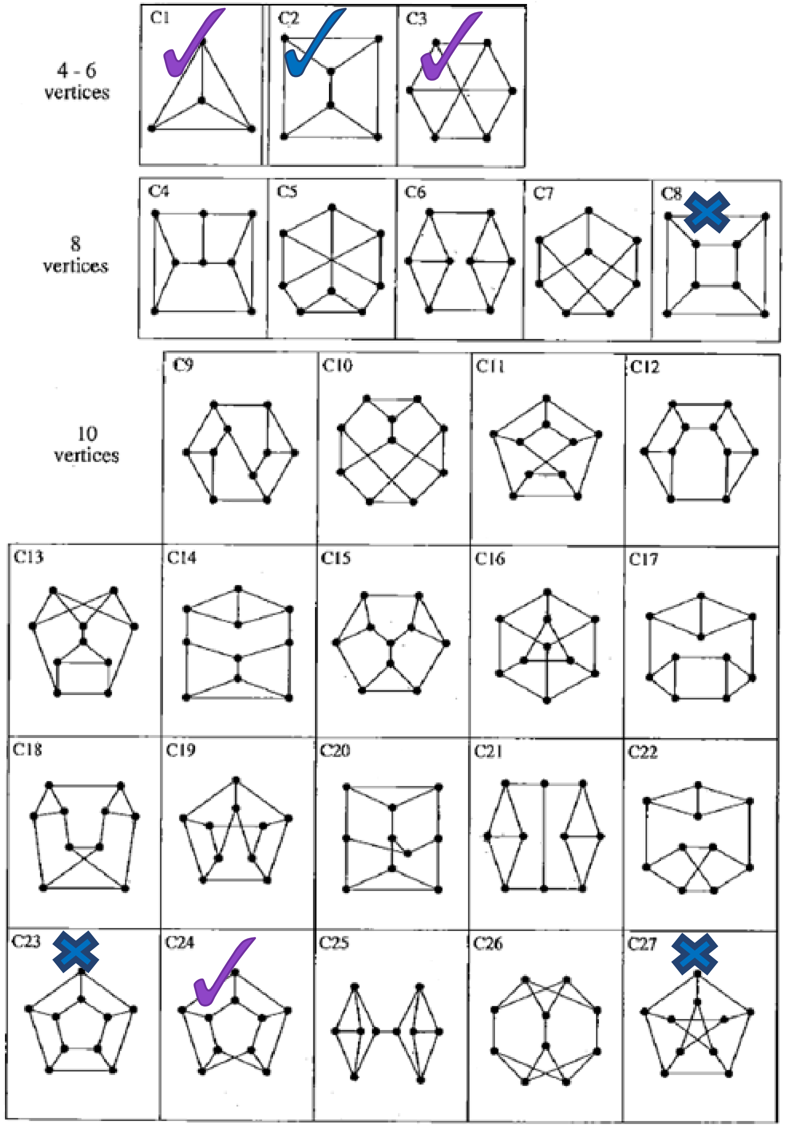
Cubic Graphs

Theorem (Herke, Maenhaut 2013): For an integer $n \geq 2$ and $a \in \{1,2\}$, $Circ(2n, \{a, n\})$ has a P1F iff it is isomorphic to one of following:

1. $Circ(4, \{1, 2\})$;
2. $Circ(6, \{a, 3\})$, $a \in \{1,2\}$;
3. $Circ(2n, \{1, n\})$ for $2n > 6$ and n odd.

Small Examples

- Apply results for generalised Petersen and cubic circulant graphs:



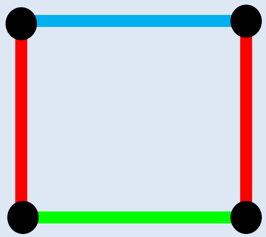
$GP(n, k)$

$Circ\ (2n, \{a, n\})$

Small Examples

Lemma 1a:

If a cubic graph on more than 4 vertices has a P1F then any 4-cycles must be factorised as:

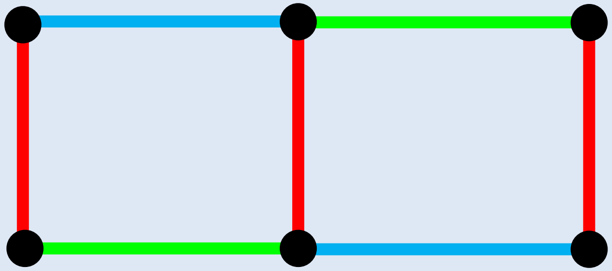


- Must have all three 1-factors in the 4-cycle (otherwise not Hamilton cycle).

Small Examples

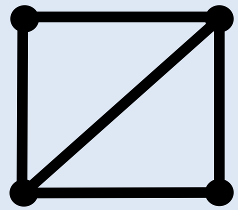
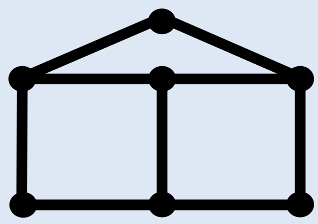
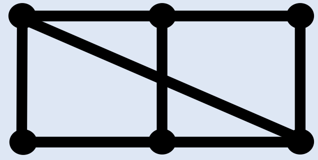
Lemma 1b:

If a cubic graph on more than 6 vertices has a P1F then any two 4-cycles that share an edge must be factorised as:



Small Examples

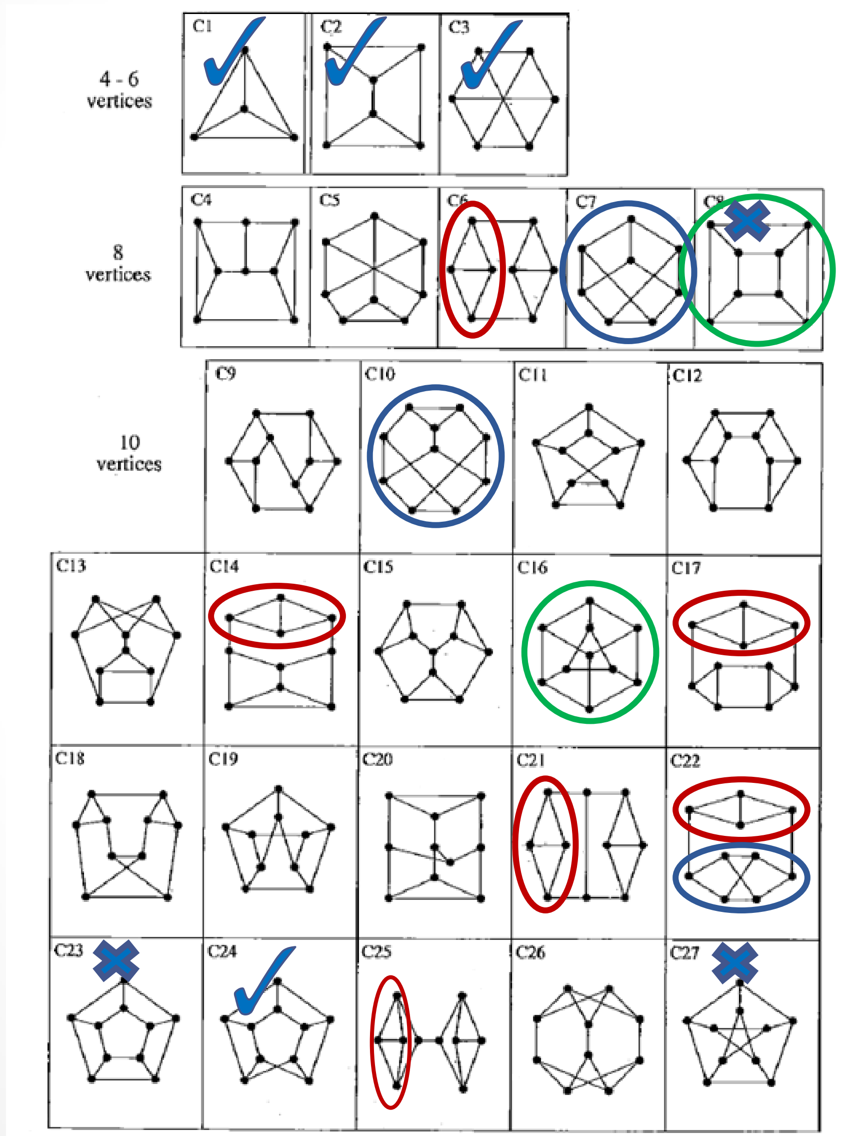
Lemma 2: Let G be a cubic graph that has a P1F.

- If $|V(G)| > 4$, then  is not a subgraph.
- If $|V(G)| > 7$, then  is not a subgraph.
- If $|V(G)| > 6$, then  is not a subgraph.

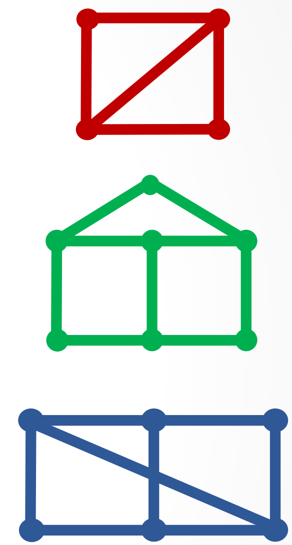
- "Forbidden subgraphs"

Small Examples

- Graphs with ‘forbidden subgraphs do not have P1Fs:

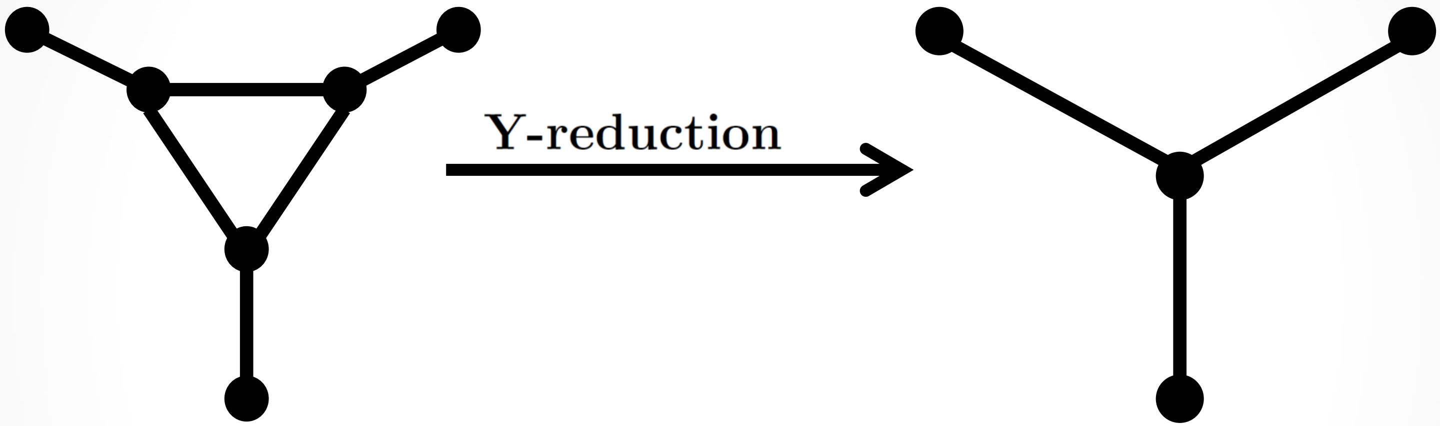


Forbidden subgraphs:



Cubic Graphs

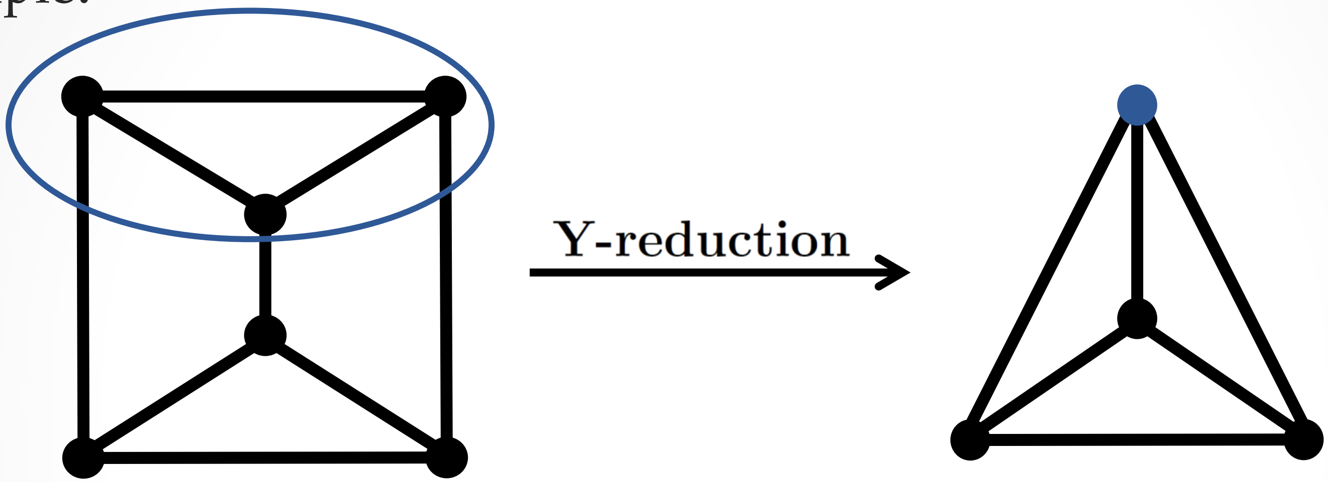
- Y-reduction operation.



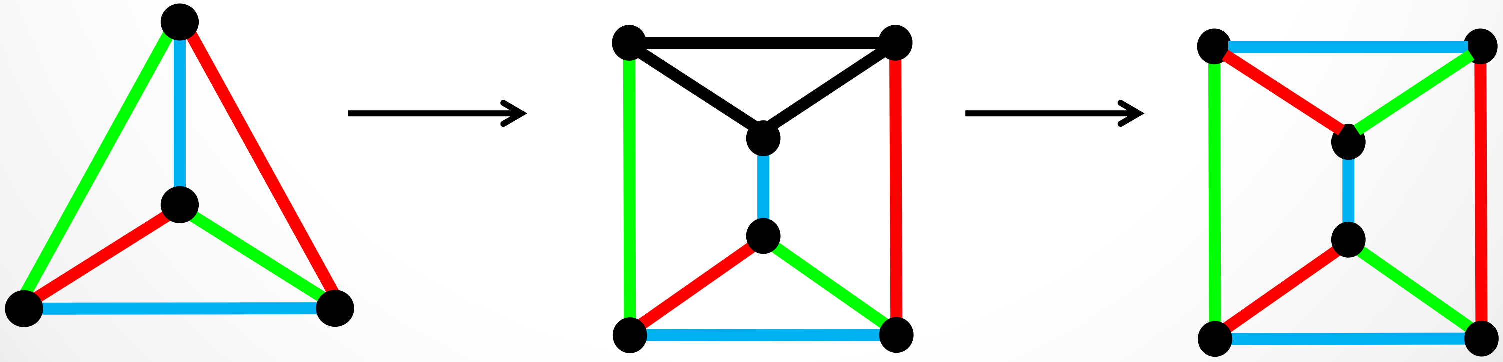
Lemma: A cubic graph has a P1F if and only if its Y-reduction has a P1F.

Cubic Graphs

Example:

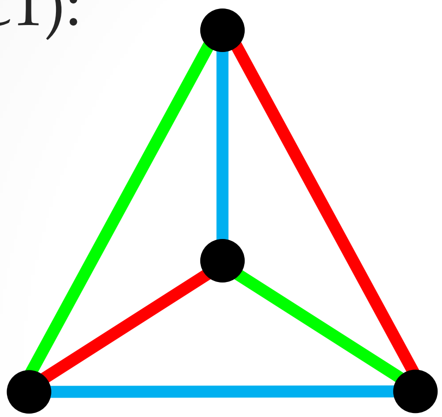


- Construct a P1F from P1F of Y-reduced graph

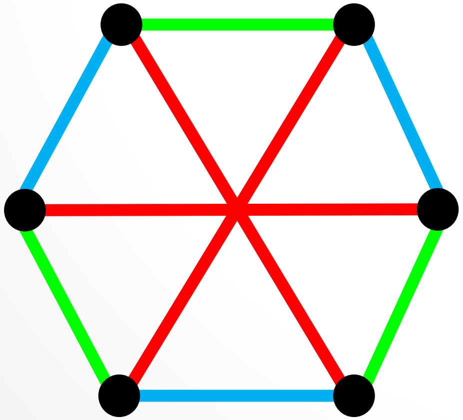


Small Examples

- Some cubic graphs that have P1Fs
- K_4 (C1):

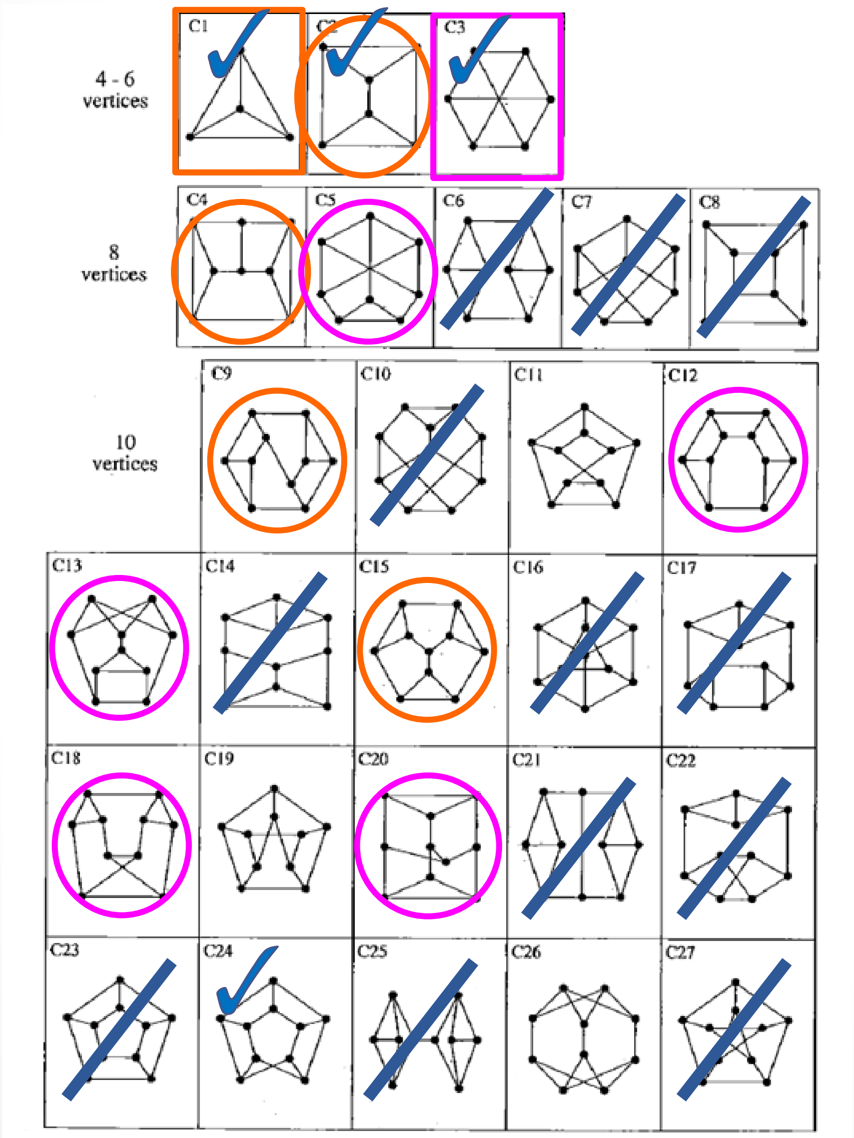


- $Circ(6,\ \{1,3\})$ (C3):



Small Examples

- Apply Y-reductions where possible:

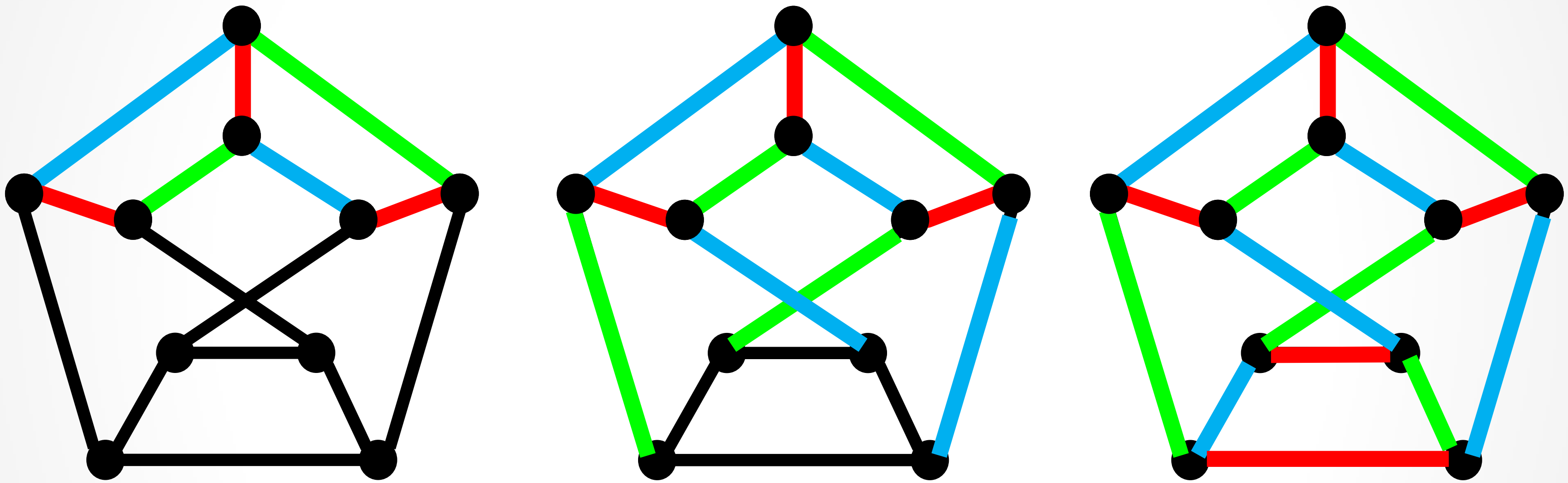


K_4

$Circ(6, \{1, 3\})$

Small Examples

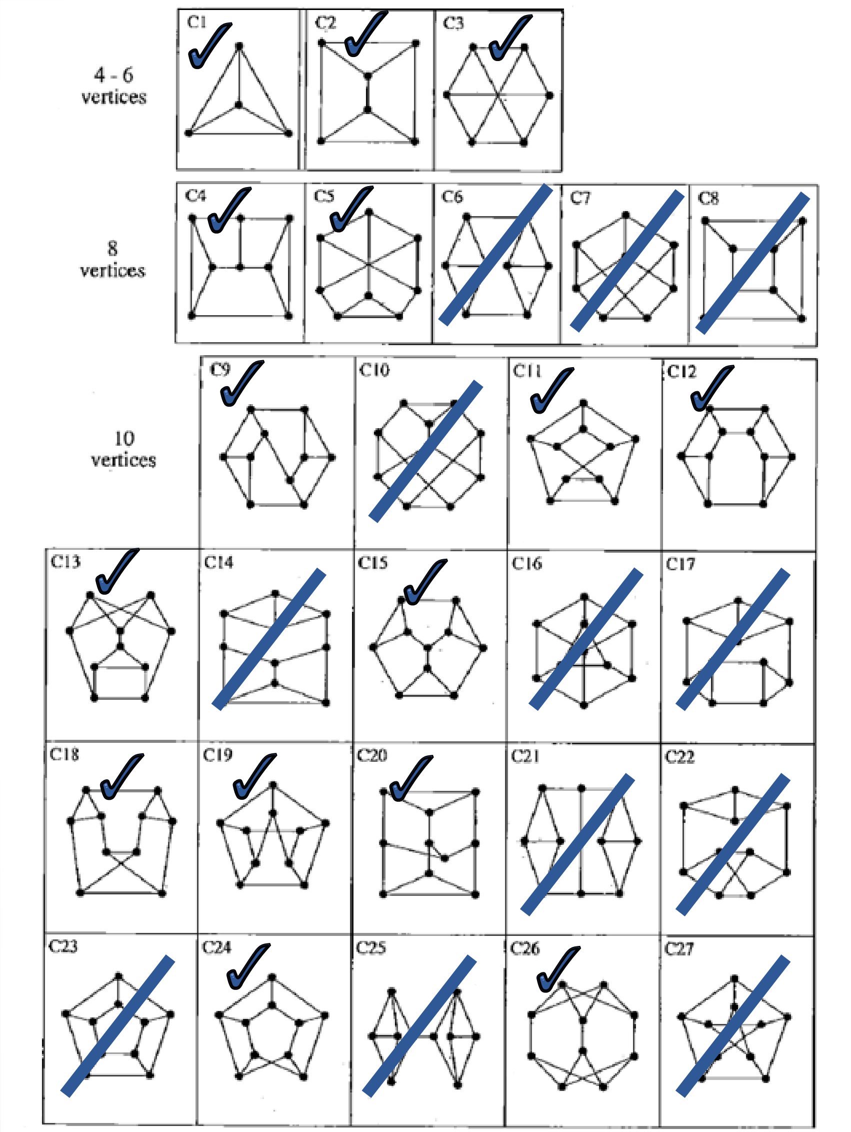
- C11: P1F constructed using 4-cycle lemmas



- C19, C26 also have P1Fs

Small Examples

- Connected cubic graphs on ≤ 10 vertices:



Summary

Open Problem (Kotzig 1960s):
Given a cubic graph, determine whether it has a P1F.

- $GP(n,k)$ solved for $k \leq 3$ and some values of n and k
- Cubic circulant graphs $Circ(2n, \{a, n\})$ solved
- Y-reduction operation
- 4-cycles and 'forbidden subgraphs'
- There were still a few graphs that needed examples

References

1. S. Bonvicini and G. Mazzuoccolo, Perfect one-factorizations in generalized Petersen graphs, Ars Combinatoria, **99** (2011), 33-43.

2. S. Herke, B. Maenhaut, Perfect 1-factorisations of circulants with small degree, Electronic Journal of Combinatorics, **20** (2013), P58.

3. G. Mazzuoccolo, Perfect one-factorizations in line-graphs and planar graphs, Australasian Journal of Combinatorics, **41** (2008), 227-233.

4. E. Seah, Perfect one-factorizations of the complete graph – a survey, Bulletin of the Institute of Combinatorics and its Applications, **1** (1991), 59-70.