Enumeration. Martingales. Random graphs.

Mikhail Isaev

School of Mathematical Sciences, Monash University

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Introduction

Expectation of the exponential function

We are interested in estimates for $\mathbb{E}e^Z$, where Z is a complex random variable.

$$\mathbb{E}e^Z \approx e^{\mathbb{E}Z} \quad \text{and} \quad \mathbb{E}e^Z \approx e^{\mathbb{E}Z + \frac{1}{2}\mathbb{E}(Z-\mathbb{E}Z)^2}.$$

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- when Z is small;
- when $Z = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are independent and small.

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For our purposes we needed:

- when $Z = f(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent;
- when Z is a complex martingale.

Random vectors with independent components

Theorem (I., McKay, 2017)

Let $X = (X_1, \dots, X_n)$ be a random vector with independent components taking values in $\boldsymbol{\Omega} = \boldsymbol{\Omega}_1 \times \cdots \times \boldsymbol{\Omega}_n$. Let $f : \boldsymbol{\Omega} \to \mathbb{C}$. Suppose, for any $1 \le j \ne k \le n$,

$$\sup_{x, x^j} |f(x) - f(x^j)| \le \boldsymbol{\alpha},$$

$$\sup_{x, x^j, x^k, x^{jk}} |f(x) - f(x^j) - f(x^k) + f(x^{jk})| \le \boldsymbol{\Delta},$$

where the suprema is over all $x, x^j, x^k, x^{jk} \in \boldsymbol{\Omega}$ such that

- x, x^j and x^k, x^{jk} differ only in the j-th coordinate,
- x, x^k and x^j, x^{jk} differ only in the k-th coordinate.

(A) If $\boldsymbol{\alpha} = o(n^{-1/2})$, then $\mathbb{E} e^{f(X)} = e^{\mathbb{E} f(X)} \left(1 + O(n\boldsymbol{\alpha}^2)\right)$.

(B) If $\boldsymbol{\alpha} = o(n^{-1/3})$ and $\boldsymbol{\Delta} = o(n^{-4/3})$, then

$$\mathbb{E} e^{f(X)} = e^{\mathbb{E} f + \frac{1}{2}\mathbb{E}(f - \mathbb{E} f)^2} \left(1 + O(n\boldsymbol{\alpha}^3 + n^2\boldsymbol{\alpha}^2\boldsymbol{\Delta})\, e^{\frac{1}{2}\,\mathbf{Var}\,\Im f(X)}\right).$$

Random permutations

Theorem (Greenhill, I., McKay, 2017+)

Let $X = (X_1, \ldots, X_n)$ be a uniform random element of S_n and $f : S_n \to \mathbb{C}$. Suppose, for any distinct $j, a \in \{1, \ldots, n\}$,

$$\sup_{\omega \in S_n} |f(\omega) - f(\omega \circ (ja))| \leq \boldsymbol{\alpha},$$

and, for any distinct $j, k, a, b \in \{1, \ldots, n\}$,

$$\sup_{\omega \in S_n} |f(\omega) - f(\omega \circ (ja)) - f(\omega \circ (kb)) + f(\omega \circ (ja)(kb))| \leq \boldsymbol{\Delta},$$

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Random subsets

Let $B_{n,m}$ denote the set of subsets of $\{1,\dots,n\}$ of size m.

Theorem (Greenhill, I., McKay, 2017+)

Let X be a uniform random element of $B_{n,m}$, $m \leq n/2$, and $f : B_{n,m} \to \mathbb{C}$. Suppose, for any $A \in B_{n,m}$ and $a \in A$, $j \notin A$,

$$|f(A) - f(A \oplus \{j,a\})| \leq \boldsymbol{\alpha},$$

and, for any distinct $a, b \in A$, $j, k \notin A$,

$$|f(A) - f(A \oplus \{j,a\}) - f(A \oplus \{k,b\}) + f(A \oplus \{j,k,a,b\})| \leq \boldsymbol{\Delta},$$

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2. **Asymptotic normality.** Let $Z = it\frac{X - \mathbb{E}X}{\sqrt{\mathbf{Var}\, X}}$, $\phi(t) = \mathbb{E}\,\mathbf{exp}\left(it\frac{X-\mathbb{E}X}{\sqrt{\mathbf{Var}\, X}}\right)$. Then,

$$\mathbb{E}e^{Z} = \phi(t) \quad \text{and} \quad e^{\mathbb{E}Z + \frac{1}{2}\mathbb{E}(Z-\mathbb{E}Z)^2} = e^{-t^2/2}.$$

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3. Subgraph counts in random graphs with given degrees.

4. Asymptotic enumeration by complex-analytic methods.

Subgraphs counts

Random variables of our interest

1. Fix a graphical degree sequence $\vec{d} = (d_1, \dots, d_n)$.
2. Take a uniform random labelled graph $\mathcal{G}_{\vec{d}}$ with degrees d.
3. Count the number $N_P(\mathcal{G}_{\vec{d}})$ of occurrences of a given pattern P.

We want an asymptotic formula for $\mathbb{E}N_P(\mathcal{G}_{\vec{d}})$ as $n \to \infty$, where $P = P(n)$ and $\vec{d} = \vec{d}(n)$.

Let d denote the average degree. Define

$$\lambda = \frac{d}{n-1} \quad \text{and} \quad R = \frac{1}{n}\sum_{j=1}^{n}(d_j - d)^2.$$

From probabilities to subgraph counts

We employ formulae from [McKay, 1985], [McKay, 2011] for the probability of a certain pattern P (subgraph or induced subgraph) to occur at a particular place in our random graph $\mathcal{G}_{\vec{d}}$:

$$\mathbf{Pr}(\mathrm{P} \text{ occurs in } \mathcal{G}_{\vec{d}} \text{ at location L}) = \mathrm{factor}(\mathrm{P}, \boldsymbol{\lambda}) \times e^{\mathrm{stuff}(\mathrm{P},\mathrm{L},\vec{d})+o(1)}.$$

The only thing to do is to sum up over all possible locations L.

For example, for copies of a given subgraph it is reduced to estimating

$$\sum_{\boldsymbol{\sigma}\in S_n} e^{f(\boldsymbol{\sigma})} = n!\, \mathbb{E} e^{f(X)},$$

where X is a uniform random permutation.

Perfect matchings and cycles

Suppose $\min\{d, n-1-d\} \geq cn/\log n$ and $|d_j - d| \leq n^{1/2+\varepsilon}$. Then, we have

- the expected number of perfect matchings in $\mathcal{G}_{\vec{d}}$ (for even n) is

$$\frac{n!}{(n/2)!2^{n/2}} \lambda^{n/2} \exp\left(\frac{1-\lambda}{4\lambda} - \frac{R}{2\lambda^2 n} + o(1)\right);$$

- the expected number of q-cycles in $\mathcal{G}_{\vec{d}}$ (for $3 \leq q \leq n$) is

$$\frac{n!}{2q(n-q)!} \lambda^q \exp\left(-\frac{(1-\lambda)q(n-q)}{\lambda n^2} + \frac{Rq(n-2q)}{\lambda^2 n^3} + o(1)\right).$$

These expressions for the regular case ($R = 0$) were also given in [McKay, 2011].

Subgraphs isomorphic to a given one

For a given graph H with degree sequence $(h_1, \dots, h_n)$ denote $\boldsymbol{\mu}_t = \frac{1}{n}\sum_{j=1}^n h_j^t$.

Theorem 1 (Greenhill, I., McKay, 2017+).

Suppose $\min\{d, n-1-d\} \geq cn/\log n$ and $|d_j - d| \leq n^{1/2+\varepsilon}$. Let H be a graph with degrees $(h_1, \dots, h_n)$ and $m = O(n^{1+2\varepsilon})$ edges such that, for all $1 \leq j \leq n$,

$$h_j = O(n^{1/2+\varepsilon}), \qquad \frac{(d_j - d)^3 \boldsymbol{\mu}_3}{\boldsymbol{\lambda}^3} = o(n^2).$$

Then, the expected number of subgraphs in $\mathcal{G}_{\vec{d}}$ isomorphic to H is

$$\frac{n!}{|\mathrm{Aut}(H)|}\boldsymbol{\lambda}^m \exp\Bigg(-\frac{1-\boldsymbol{\lambda}}{4\boldsymbol{\lambda}}(2\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1^2 - 2\boldsymbol{\mu}_1) + \frac{R}{2\boldsymbol{\lambda}^2 n}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1^2 - \boldsymbol{\mu}_1) - \frac{1-\boldsymbol{\lambda}^2}{6\boldsymbol{\lambda}^2 n}\boldsymbol{\mu}_3 - \frac{1-\boldsymbol{\lambda}}{\boldsymbol{\lambda} n^2}\sum_{jk\in E(H)} h_j h_k + o(1)\Bigg),$$

where Aut(H) is the automorphism group of H.

A similar result holds for the number of induced copies of H.

Spanning trees

Let $\tau_{\vec{d}}$ denote the number of spanning trees in $\mathcal{G}_{\vec{d}}$.

Theorem (Greenhill, I., McKay, 2017+)

Suppose $\min\{d, n-1-d\} \geq cn/\log n$ and $|d_j - d| \leq n^{1/2+\varepsilon}$, then

$$\mathbb{E}\tau_{\vec{d}} = n^{n-2}\lambda^{n-1} \exp\left(-\frac{1-\lambda}{2\lambda} - \frac{R}{2\lambda^2 n} + o(1)\right).$$

Theorem (Greenhill, I., McKay, Kwan, 2017)

Suppose that number of edges is at least $n + \frac{1}{2}d_{\max}^4$ (i.e. $d_{\max}^4 \leq (d-2)n$), then

$$\mathbb{E}\tau_{\vec{d}} = \frac{(d-1)^{1/2}}{(d-2)^{3/2}n}\left(\prod_{j=1}^{n} d_j\right)\left(\frac{(d-1)^{d-1}}{d^{d/2}(d-2)^{d/2-1}}\right)^n \times$$
$$\exp\left(\frac{6d^2-14d+7}{4(d-1)^2} + \frac{R}{2(d-1)^3} + \frac{(2d^2-4d+1)R^2}{4(d-1)^3 d^2} + o(1)\right).$$

Asymptotic enumeration

Complex-analytic approach

1. We write combinatorial counts in terms of multivariate generation functions.

Example: the number of d-regular graphs on n vertices

$$RG(n, d) = [x_1^d \cdots x_n^d] \prod_{1 \le j < k \le n} (1 + x_j x_k).$$

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$$\mathrm{RG}(n, d) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{1 \le j < k \le n} (1 + z_j z_k)}{z_1^{d+1} \cdots z_n^{d+1}} \, dz_1 \cdots , dz_n.$$

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3. By choosing appropriate contours, we approximate the value of the integral which is mostly given by small neighbourhoods of concentration points.

Integrals we work with

Typically the problem is reduced to the estimation of integrals of the following form:

$$\int_B \mathbf{exp}(-x^T Ax + f(x))dx = c\,\mathbb{E}e^{f(X_B)},$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, f is a multi-variable polynomial of low degree with complex coefficients and X_B is a gaussian random variable truncated to B.

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Example: if nd is even and $\mathbf{min}\{d, n-1-d\}$ grows sufficiently fast as $n \to \infty$

$$RG(n,d) \approx \frac{2\,(2\pi)^{-n}}{(\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n}{2}}} \int_{\|x\|_\infty \leq \frac{n^\varepsilon}{(\lambda(1-\lambda)n)^{1/2}}} \mathbf{exp}\left(\sum_{\ell=2}^{\ell_{\mathbf{max}}} \sum_{j<k} c_\ell (x_j + x_k)^\ell\right) dx,$$

where $\lambda = d/(n-1)$ is the density of such a graph and $c_2 = -\frac{1}{2}\lambda(1-\lambda)$.

The $\boldsymbol{\beta}$-model of random graph

A random graph model with independent adjacencies

$$p_{jk} = \frac{e^{\boldsymbol{\beta}_j + \boldsymbol{\beta}_k}}{1 + e^{\boldsymbol{\beta}_j + \boldsymbol{\beta}_k}}.$$

is known as $\boldsymbol{\beta}$-model, where $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n) \in \mathbb{R}^n$.

Let $\mathcal{G}_{\boldsymbol{\beta}}$ denote the $\boldsymbol{\beta}$-model where $\boldsymbol{\beta}$ is the solution of the system

$$\sum_{k: j \neq k} p_{jk} = d_j, \qquad (1 \leq j \leq n).$$

for a given degree sequence $\vec{d} = (d_1, \ldots, d_n)$.

The model $\mathcal{G}_{\boldsymbol{\beta}}$ behaves similar to $\mathcal{G}_{\vec{d}}$ in many ways.

Some corollaries

Using the formula $\mathbb{E}e^{f(X)} \approx e^{\mathbb{E}f(X)+\frac{1}{2}\mathbb{E}(f(X)-\mathbb{E}f(X))^2}$ it was obtained in [I., McKay, 2017] that (for the dense case):

- Models $\boldsymbol{\mathcal{G}}_{\boldsymbol{\beta}}$ and $\boldsymbol{\mathcal{G}}_{\vec{d}}$ agree for small subgraph probabilities.

$$P_{\vec{d}}(H^+, H^-) \approx \prod_{jk\in H^+} p_{jk} \prod_{jk\in H^-} (1 - p_{jk}).$$

- Let

$$X_{\vec{d}} = |\boldsymbol{\mathcal{G}}_{\vec{d}} \cap Y| \quad \text{and} \quad X_{\boldsymbol{\beta}} = |\boldsymbol{\mathcal{G}}_{\boldsymbol{\beta}} \cap Y|,$$

where Y is a set of edges. Then, we have

$$\mathbf{Pr}(|X_{\vec{d}} - \mathbb{E}X_{\boldsymbol{\beta}}| > t|Y|^{1/2}) \leq ce^{-2t\,\mathbf{min}\{t, n^{1/6}(\log n)^{-3}\}}.$$

Cumulant expansion

Moments and cumulants

One could use series $\mathbb{E}e^{tZ} = \sum_{s=0}^{\infty} \frac{t^r}{r!}\mathbb{E}Z^r$. However, for $Z = f(X_B)$, we would need to estimate a big number of moments, before they become negligible.

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Instead, we employ the cumulant expansion

$$\mathbb{E}e^{tZ} = \mathbf{exp}\left(\sum_{r=1}^{\infty} \frac{t^r}{r!}\boldsymbol{\kappa}_r(Z)\right).$$

Recall that cumulants $\boldsymbol{\kappa}_r(Z)$ could be defined by

$$\boldsymbol{\kappa}_r(Z) = \sum_{\boldsymbol{\pi}} (|\boldsymbol{\pi}| - 1)!(-1)^{|\boldsymbol{\pi}|-1} \prod_{P\in\boldsymbol{\pi}} \mathbb{E}Z^{|P|},$$

where the sum is over all partitions $\boldsymbol{\pi}$ of $\{1, \ldots, n\}$. For example,

$$\boldsymbol{\kappa}_1(Z) = \mathbb{E}Z, \qquad \boldsymbol{\kappa}_2(Z) = \mathbb{E}Z^2 - (\mathbb{E}Z)^2 = \mathbb{E}(Z - \mathbb{E}Z)^2,$$
$$\boldsymbol{\kappa}_3(Z) = \mathbb{E}Z^3 - 3(\mathbb{E}Z^2)(\mathbb{E}Z) + 2(\mathbb{E}Z)^3 = \mathbb{E}(Z - \mathbb{E}Z)^3,$$
$$\boldsymbol{\kappa}_4(Z) = \mathbb{E}Z^4 - 4(\mathbb{E}Z^3)(\mathbb{E}Z) - 3(\mathbb{E}Z^2)^2 + 12(\mathbb{E}Z^2)(\mathbb{E}Z)^2 - 6(\mathbb{E}Z)^4.$$

Overview of specifics

For enumeration of regular graphs we have $f(x) = \sum_{\ell=3}^{\ell_{max}} \sum_{j<k} c_\ell (x_j + x_k)^\ell$.

Main difficulties:

- $Z = f(X_B)$ is not a sum of independent random variables.
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  In particular, the following bound of error terms is not good enough:

  $$|\mathbb{E}(We^Z)| \leq \|W\|_\infty \mathbb{E}|e^Z| = \|W\|_\infty \mathbb{E}e^{\Re Z},$$

  because $\mathbb{E}e^{\Re Z}$ could be much bigger than $\mathbb{E}e^Z$.

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In all previous works the method is limited to the "dense" range (where $\mathbb{E}e^{\Re Z}$ is of the same order as $\mathbb{E}e^Z$) and the formulae $\mathbb{E}e^Z \approx e^{\mathbb{E}Z}$ and $\mathbb{E}e^Z \approx e^{\mathbb{E}Z + \frac{1}{2}\mathbb{V}Z}$ are used.

The gap between sparse and dense

Conjecture [McKay, Wormald, 1991]

Suppose dn is even and $0 < d < n - 1$, then the number of d-regular graphs on n vertices is

$$\mathrm{RG}(n, d) = \left(\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right)^{\binom{n}{2}} \binom{n-1}{d}^{n} \sqrt{2}\, e^{1/4+o(1)}$$

as $n \to \infty$, where $\lambda = \frac{d}{n-1}$.

We know it is true for the following cases:

- $\min\{d, n-1-d\} = o(n^{1/2})$ (McKay and Wormald, 1991).
- $\min\{d, n-1-d\} \geq c\frac{n}{\log n}$ (McKay and Wormald, 1990).

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Our results for d-regular graphs

Let $L = \boldsymbol{\lambda}(1-\boldsymbol{\lambda}) = \frac{d(n-1-d)}{(n-1)^2}$ and define $\boldsymbol{\delta}(n,d)$ by

$$RG(n,d) = \left(\boldsymbol{\lambda}^{\boldsymbol{\lambda}}(1-\boldsymbol{\lambda})^{1-\boldsymbol{\lambda}}\right)^{\binom{n}{2}} \binom{n-1}{d}^{n} \sqrt{2}\, e^{1/4+\boldsymbol{\delta}(n,d)}.$$

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Theorem (I., McKay)

Suppose dn is even and $n^{1/7+\varepsilon} \leq d \leq \frac{n-1}{2}$, then (up to the error term $O(n/d^7)$)

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Suppose dn is even and $n^{1/7+\varepsilon} \leq d \leq \frac{n-1}{2}$, then (up to the error term $O(n/d^7)$)

$$\begin{aligned}
\delta(n,d) = O(n^{-1}) &= -\frac{1}{4n} + O\left(\frac{1}{dn}\right) \\
&= -\frac{1}{4n} + \frac{2-23L}{24Ln^2} + O\left(\frac{1}{dn^2}\right) \\
&= -\frac{1}{4n} + \frac{2-23L}{24Ln^2} + \frac{22-129L}{24Ln^3} + O\left(\frac{1}{d^2n^2}\right) \\
&= -\frac{1}{4n} + \frac{2-23L}{24Ln^2} + \frac{22-129L}{24Ln^3} - \frac{3-115L+483L^2}{12L^2n^4} + O\left(\frac{1}{d^2n^3}\right).
\end{aligned}$$

Regular graphs on 4, 5, 6 vertices

RG(4,1) = RG(4,2) = 3.

RG(5,2) =12.

RG(6,1) = RG(6,4) =15.

RG(6,2) = RG(6,3) =70.

Regular graphs on 4, 5, 6 vertices

RG(4,1) = RG(4,2) = 3.

F0 = 3.23, F1 = 3.03, F2 = 2.92, F3 = 2.87, F4 = 2.84.

RG(5,2) =12.

F0 = 13.79, F1 = 13.11, F2 = 12.79, F3 = 12.61, F4 = 12.50.

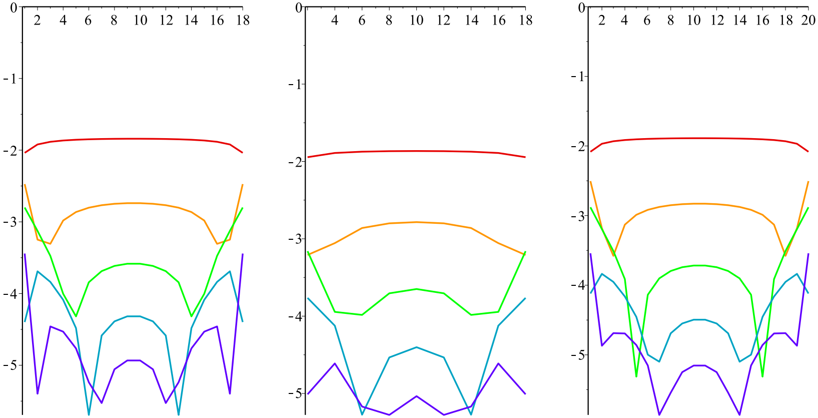
RG(6,1) = RG(6,4) =15.

F0 = 15.60, F1 = 14.96, F2 = 14.78, F3 = 14.80, F4 = 14.91.

RG(6,2) = RG(6,3) =70.

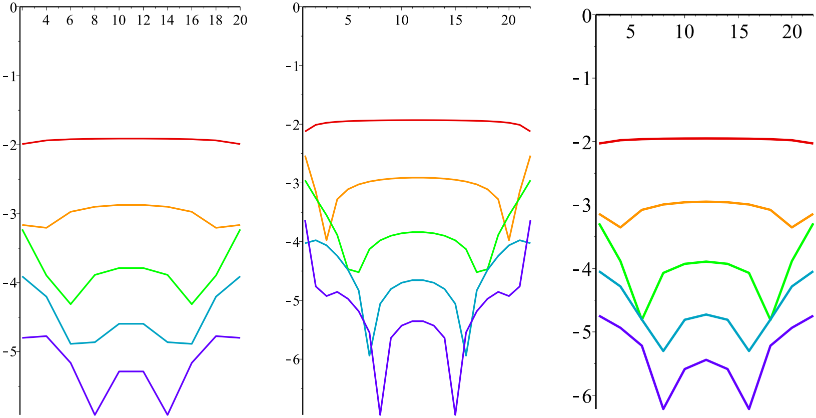
F0 = 74.96, F1 = 71.90, F2 = 70.68, F3 = 70.18, F4 = 69.92.

Plots of $\log_{10}\left(\frac{|Fx-RG|}{RG}\right)$ for $n = 20, 21, 22$



$n = 20.$ $n = 21.$ $n = 22.$

Plots of $\log_{10}\left(\frac{|Fx - RG|}{RG}\right)$ for $n = 23, 24, 25$



$n = 23.$ $n = 24.$ $n = 25.$

Perfect matchings

Note that $\mathrm{RG}(2k, 1) = (2k-1)(2k-3)\cdots 1 = \frac{(2k)!}{2^k k!}$.

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$$F0 = 2.73002 \cdot 10^{78}.$$
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Conclusion: the cumulant expansion not only helps to extend the range of complex analytic-methods, but also give more accurate approximations.

Thank you for your attention!