The sandwich conjecture of random regular graphs and more

Mikhail Isaev

(joint work with P. Gao and B.D. McKay)

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Introduction

Random graphs

The parameter n is the number of vertices. All graphs are labelled.

- $\mathcal{G}(n,p)$ **model**: every pair of vertices is connected in the graph with probability p independently from every other edge.
- $\mathcal{G}(n,m)$ **model**: we take a uniform random element of the set of graphs on n vertices with m edges.
- $\mathcal{R}(n,d)$ **model**: we take a uniform random element of the set of d-regular graphs on n vertices (we always assume dn is even).

The sandwich conjecture

Conjecture (by Kim and Vu in [Advances in Math., 2004])

For $d \gg \log n$, there is a random triple (G_1, R, G_2) of graphs on n vertices which marginal distributions are

$$G_1 \sim \mathcal{G}(n, p_1), \qquad R \sim \mathcal{R}(n, d), \qquad G_2 \sim \mathcal{G}(n, p_2),$$

for some $p_1 = \frac{d}{n}(1 - o(1))$ and $p_2 = \frac{d}{n}(1 + o(1))$, and

$$\Pr(G_1 \subseteq R \subseteq G_2) = 1 - o(1).$$

Kim and Vu managed to prove the sandwich conjecture for the range $\log n \ll d \leq n^{1/3 - o(1)}$ with a defect in one side: $\mathcal{R}(n, d)$ is not completely contained in $\mathcal{G}(n, p_2)$.

Recent progress towards the sandwich conjecture

Dudek, Frieze, Ruciński, abd Šileikis [J. Comb. Theory B, 2017] showed that, for all $d = o(n)$, $\mathcal{G}(n, (1 - o(1))\frac{d}{n}) \subseteq \mathcal{R}(n, d)$ a.a.s.

Theorem (Gao, I., McKay)

Let ε be any positive constant. Then the following holds a.a.s.

(i) For $d \geq n^{2/3+\varepsilon}$ the sandwich conjecture holds.

(ii) For $d \geq n^{1/2}$ we have $\mathcal{R}(n, d) \subseteq \mathcal{G}(n, \varepsilon\frac{d}{n}\log n)$.

(iii) For $d \leq n^{1/2}$ we have $\mathcal{R}(n, d) \subseteq \mathcal{G}(\varepsilon n^{-1/2}\log n)$.

Coupling procedure

Another way to generate $\mathcal{G}(n,p)$

Procedure $M(n,m)$.

1. Take $M := \emptyset$.
2. Repeat m times: take jk uniformly at random from K_n and add it to M (in case the edge jk was not in M yet).
3. Return M.

If $D \sim \mathrm{Po}(\lambda)$ then $M(n,D) \sim \mathcal{G}(n,p)$ with $p = 1 - e^{-\lambda/\binom{n}{2}}$.

Let $M_\xi(n,m)$ be the random graph defined similarly to $M(n,m)$ but with some rejection probability ξ at Step 2. Then,

$$M_\xi(n,D) \sim \mathcal{G}(n,p_\xi) \text{ with } p_\xi = 1 - e^{-\lambda(1-\xi)/\binom{n}{2}}.$$

Kim and Vu relied on the algorithm of [Steger and Wormald, Combin.Probab. Comput., 1999] and the asymptotic formula for the number of d-regular graphs.

Coupling $\mathcal{G}(n,p) \subseteq \mathcal{R}(n,d)$.

Procedure $R(n,d)$.

1. Take $R := \emptyset$.
2. Repeat until R is d-regular: take jk uniformly at random from K_n and add it to R with probability

$$\frac{\Pr(jk \in \mathcal{R}(n,d) \mid R \subset \mathcal{R}(n,d))}{\max_{jk \notin R} \Pr(jk \in \mathcal{R}(n,d) \mid R \subset \mathcal{R}(n,d))} \tag{1}$$

   (in case the edge jk was not in R yet).
3. Return R.

Idea: to achieve $M_\xi(n,D) \subseteq R(n,d)$ we only need to show that a.a.s. (1) is bounded below by $1-\xi$ for the first D iterations of Step 2.

What is left to show?

Let $S \sim \mathcal{G}(n,p)$. Take a t-factor $T \subseteq S$ uniformly at random.

Toy problem

For which values of p and t we can show a.a.s.

$$\Pr_S(uv \in T) = (1 + o(1))\frac{t}{pn}$$

simultaneusly for all edges $uv \in S$?

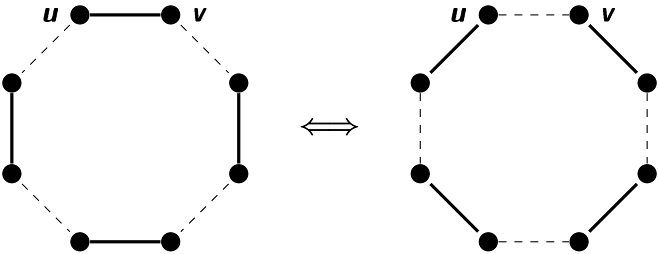
During the coupling procedure, p ranges from 1 to $1 - \frac{d}{n}$ and t ranges from d to 0.

It is fairly easy to resolve the toy problem for $d = o(n)$ which gives us $\mathcal{G}(n, \frac{d}{n}(1 - o(1))) \subseteq \mathcal{R}(n,d)$, see [Dudek et al., 2017].

The containment $\mathcal{R}(n,d) \subseteq \mathcal{G}(n, \frac{d}{n}(1 - o(1)))$ is equivalent to $\mathcal{G}(n, 1 - \frac{d}{n} - o(\frac{d}{n})) \subseteq \mathcal{R}(n, n-d)$. So we need $p = o(1)$ for that.

Two key ideas

Switchings



The number of ways to switch $\Longrightarrow$ is $p^4 t^3 n^3 (1 + o(1))$.

The number of ways to switch $\Longleftarrow$ is $p^3 t^4 n^2 (1 + o(1))$.

This works for $p \geq \varepsilon n^{-1/2} \log n$ and $t = o(pn)$.

Complex-analytic approach

The probability can be expressed as a ratio of two integrals:

$$\mathbf{Pr}_S(uv \in T) = \frac{t}{pn}(1+o(1)) \frac{\oint \cdots \oint \frac{\prod_{jk \in S-uv}(1+z_j z_k)}{z_1^{d+1} \cdots z_n^{d+1}/z_u z_v} dz_1 \ldots dz_n}{\oint \cdots \oint \frac{\prod_{jk \in S}(1+z_j z_k)}{z_1^{d+1} \cdots z_n^{d+1}} dz_1 \ldots dz_n}.$$

Then, we estimate these multidimensional complex integrals using the machinery of [I., McKay, Random Struct. Algor., 2017] and get that

$$\frac{\frac{1}{(2\pi)^{n/2}|Q_S|} e^{\mathbb{E}g(X) - \frac{1}{2}\mathbb{E}h(X)^2 + o(1)}}{\frac{1}{(2\pi)^{n/2}|Q_{S-uv}|} e^{\mathbb{E}\tilde{g}(\tilde{X}) - \frac{1}{2}\mathbb{E}\tilde{h}(\tilde{X})^2 + o(1)}} = 1 + o(1).$$

This works for $p \geq n^{-1/3+\varepsilon}$ and $\min\{t, pn - t\} \gg pn/\log n$.

...AND MORE

More sandwiches

1) Our result actually covers random graphs with given degree sequence $(d_1, \ldots, d_n)$ that $d_j = d(1 + o(1))$.

2) Similar sandwiching results holds for the model G_p and random subgraph of G with given degrees (chosen uniformly).

3) There are immediate corollaries of the form $\mathcal{R}(n, d_1) \subseteq \mathcal{R}(n, d_2)$.

Thank you for your attention!