The sandwich conjecture of random regular graphs and more

Mikhail Isaev
(joint work with P. Gao and B.D. McKay)

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Introduction
Random graphs

The parameter \( n \) is the number of vertices. All graphs are labelled.

- **\( G(n, p) \) model**: every pair of vertices is connected in the graph with probability \( p \) independently from every other edge.

- **\( G(n, m) \) model**: we take a uniform random element of the set of graphs on \( n \) vertices with \( m \) edges.

- **\( R(n, d) \) model**: we take a uniform random element of the set of \( d \)-regular graphs on \( n \) vertices (we always assume \( dn \) is even).
The sandwich conjecture

Conjecture (by Kim and Vu in [Advances in Math., 2004])

For $d \gg \log n$, there is a random triple $(G_1, R, G_2)$ of graphs on $n$ vertices which marginal distributions are

$$ G_1 \sim \mathcal{G}(n, p_1), \quad R \sim \mathcal{R}(n, d), \quad G_2 \sim \mathcal{G}(n, p_2), $$

for some $p_1 = \frac{d}{n}(1 - o(1))$ and $p_2 = \frac{d}{n}(1 + o(1))$, and

$$ \Pr(G_1 \subseteq R \subseteq G_2) = 1 - o(1). $$

Kim and Vu managed to prove the sandwich conjecture for the range $\log n \ll d \leq n^{1/3-o(1)}$ with a defect in one side: $\mathcal{R}(n, d)$ is not completely contained in $\mathcal{G}(n, p_2)$. 
Recent progress towards the sandwich conjecture

Dudek, Frieze, Ruciński, and Šileikis [J. Comb. Theory B, 2017] showed that, for all $d = o(n)$, $\mathcal{G}(n, (1 - o(1))\frac{d}{n}) \subseteq \mathcal{R}(n, d)$ a.a.s.

**Theorem (Gao, I., McKay)**

Let $\varepsilon$ be any positive constant. Then the following holds a.a.s.

(i) For $d \geq n^{2/3} + \varepsilon$ the sandwich conjecture holds.
(ii) For $d \geq n^{1/2}$ we have $\mathcal{R}(n, d) \subseteq \mathcal{G}(n, \varepsilon \frac{d}{n} \log n)$.
(iii) For $d \leq n^{1/2}$ we have $\mathcal{R}(n, d) \subseteq \mathcal{G}(\varepsilon n^{-1/2} \log n)$. 
Coupling procedure
Another way to generate $\mathcal{G}(n, p)$

Procedure $M(n, m)$.

1. Take $M := \emptyset$.
2. Repeat $m$ times: take $jk$ uniformly at random from $K_n$ and add it to $M$ (in case the edge $jk$ was not in $M$ yet).
3. Return $M$.

If $D \sim \text{Po}(\lambda)$ then $M(n, D) \sim \mathcal{G}(n, p)$ with $p = 1 - e^{-\lambda/(n^2)}$.

Let $M_\xi(n, m)$ be the random graph defined similarly to $M(n, m)$ but with some rejection probability $\xi$ at Step 2. Then,

$$M_\xi(n, D) \sim \mathcal{G}(n, p_\xi) \text{ with } p_\xi = 1 - e^{-\lambda(1-\xi)/(n^2)}.$$

Coupling $\mathcal{G}(n, p) \subseteq \mathcal{R}(n, d)$.

Procedure $\mathcal{R}(n, d)$.

1. Take $\mathcal{R} := \emptyset$.

2. Repeat until $\mathcal{R}$ is $d$-regular: take $jk$ uniformly at random from $K_n$ and add it to $\mathcal{R}$ with probability

$$\frac{\Pr(jk \in \mathcal{R}(n, d) \mid \mathcal{R} \subseteq \mathcal{R}(n, d))}{\max_{jk \notin \mathcal{R}} \Pr(jk \in \mathcal{R}(n, d) \mid \mathcal{R} \subseteq \mathcal{R}(n, d))}$$  \hspace{1cm} (1)

(in case the edge $jk$ was not in $\mathcal{R}$ yet).

3. Return $\mathcal{R}$.

Idea: to achieve $M_\xi(n, D) \subseteq \mathcal{R}(n, d)$ we only need to show that a.a.s. (1) is bounded below by $1 - \xi$ for the first $D$ iterations of Step 2.
What is left to show?

Let $S \sim \mathcal{G}(n, p)$. Take a $t$-factor $T \subseteq S$ uniformly at random.

**Toy problem**

For which values of $p$ and $t$ we can show a.a.s.

$$\Pr_S(\{uv \in T\}) = (1 + o(1)) \frac{t}{pn}$$

simultaneously for all edges $uv \in S$?

During the coupling procedure, $p$ ranges from 1 to $1 - \frac{d}{n}$ and $t$ ranges from $d$ to 0.

It is fairly easy to resolve the toy problem for $d = o(n)$ which gives us

$\mathcal{G}(n, \frac{d}{n}(1 - o(1))) \subseteq \mathcal{R}(n, d)$, see [Dudek et al., 2017].

The containment $\mathcal{R}(n, d) \subseteq \mathcal{G}(n, \frac{d}{n}(1 - o(1)))$ is equivalent to

$\mathcal{G}(n, 1 - \frac{d}{n} - o(\frac{d}{n})) \subseteq \mathcal{R}(n, n - d)$. So we need $p = o(1)$ for that.
Two key ideas
The number of ways to switch $\implies$ is $p^4 t^3 n^3 (1 + o(1))$.

The number of ways to switch $\impliedby$ is $p^3 t^4 n^2 (1 + o(1))$.

This works for $p \geq \varepsilon n^{-1/2} \log n$ and $t = o(pn)$.
Complex-analytic approach

The probability can be expressed as a ratio of two integrals:

\[
\Pr_S(uv \in T) = \frac{t}{pn} (1 + o(1)) \frac{\int \cdots \int \frac{\prod_{jk \in S-uv} (1+z_j z_k)}{z_1^{d+1} \cdots z_n^{d+1}/z_u z_v} \, dz_1 \cdots dz_n}{\int \cdots \int \frac{\prod_{jk \in S} (1+z_j z_k)}{z_1^{d+1} \cdots z_n^{d+1}} \, dz_1 \cdots dz_n}.
\]

Then, we estimate these multidimensional complex integrals using the machinery of [I., McKay, Random Struct. Algor., 2017] and get that

\[
\frac{1}{(2\pi)^{n/2} |Q_S|} e^{\mathbb{E}g(X) - \frac{1}{2} \mathbb{E}h(X)^2 + o(1)} = 1 + o(1).
\]

\[
\frac{1}{(2\pi)^{n/2} |Q_{S-uv}|} e^{\mathbb{E}\tilde{g}(\tilde{X}) - \frac{1}{2} \mathbb{E}\tilde{h}(\tilde{X})^2 + o(1)} = 1 + o(1).
\]

This works for \( p \geq n^{-1/3+\varepsilon} \) and \( \min\{t, pn - t\} \gg pn/\log n \).
...AND MORE
1) Our result actually covers random graphs with given degree sequence 
\((d_1, \ldots, d_n)\) that \(d_j = d(1 + o(1))\).

2) Similar sandwiching results holds for the model \(G_p\) and random 
subgraph of \(G\) with given degrees (chosen uniformly).

3) There are immediate corollaries of the form \(\mathcal{R}(n, d_1) \subseteq \mathcal{R}(n, d_2)\).
Thank you for your attention!