A study of ℓ-link graphs

PhD Candidate: Bin Jia   Supervisor: David R. Wood

Discrete Maths Research Group Seminar
Monash University, 30 May 2014

Graphs

All graphs are loopless. Parallel edges are considered to be different.

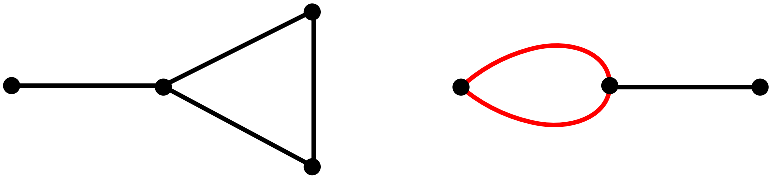


Figure: A simple graph and a graph with parallel edges

What is an ℓ-link?

An ℓ-link is a walk of length $\ell \geqslant 0$ such that consecutive edges are different.

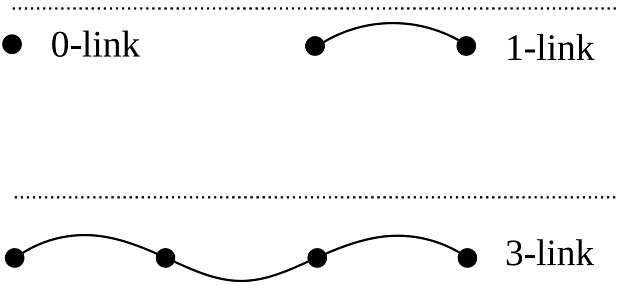


Figure: vertices, edges, paths are all ℓ-links

Some complicated ℓ-links

An ℓ-link is a walk of length $\ell \geqslant 0$ such that consecutive edges are different.

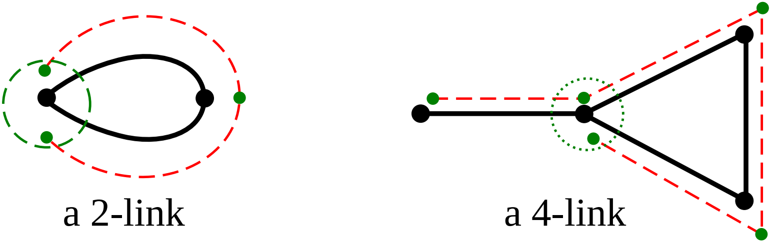


Figure: ℓ-links with repeated vertices

More complicated ℓ-links

An ℓ-link is a walk of length $\ell \geqslant 0$ such that consecutive edges are different.

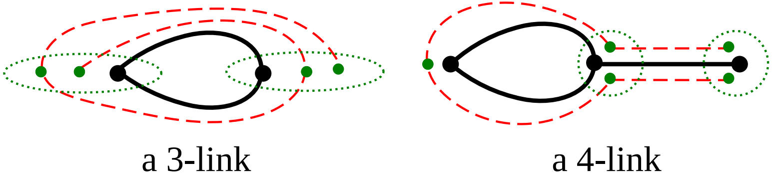


Figure: ℓ-links with repeated edges

Counterexamples

An ℓ-link is a walk of length $\ell \geqslant 0$ such that consecutive edges are different.

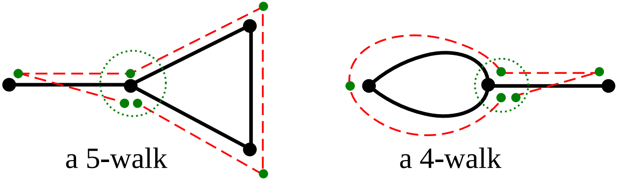
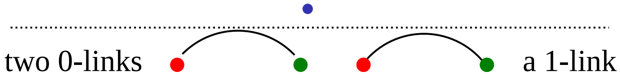


Figure: Walks that are not ℓ-links

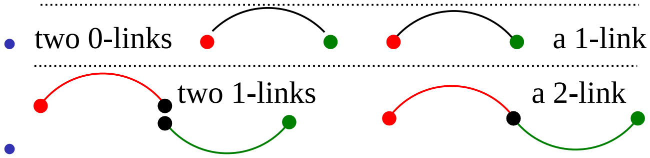
Shunting ℓ-links

An ℓ-link can be shunted to another ℓ-link in one step through an $(\ell+1)$-link.



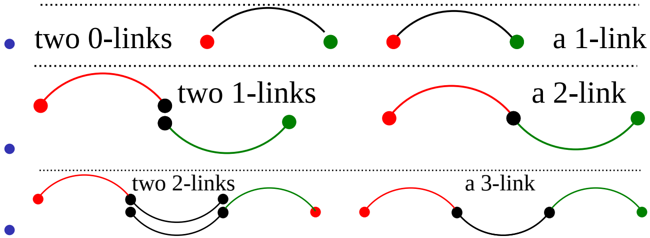
Shunting ℓ-links

An ℓ-link can be shunted to another ℓ-link in one step through an $(\ell + 1)$-link.



Shunting ℓ-links

An ℓ-link can be shunted to another ℓ-link in one step through an $(\ell + 1)$-link.



ℓ-link simple graphs

The ℓ-link simple graph of a graph G is the graph with

- vertices the ℓ-links of G; two ℓ-links are adjacent if

ℓ-link simple graphs

The ℓ-link simple graph of a graph G is the graph with

- vertices the ℓ-links of G; two ℓ-links are adjacent if
- one can be shunted onto the other through an $(\ell + 1)$-link of G.

ℓ-link simple graphs

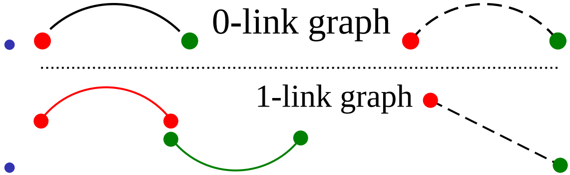
The ℓ-link simple graph of a graph G is the graph with

- vertices the ℓ-links of G; two ℓ-links are adjacent if
- one can be shunted onto the other through an $(\ell + 1)$-link of G.
- 

ℓ-link simple graphs

The ℓ-link simple graph of a graph G is the graph with

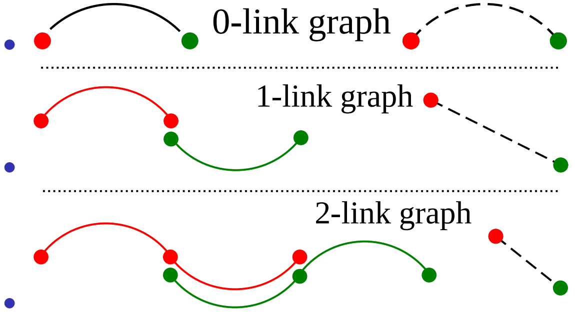
- vertices the ℓ-links of G; two ℓ-links are adjacent if
- one can be shunted onto the other through an $(\ell + 1)$-link of G.



ℓ-link simple graphs

The ℓ-link simple graph of a graph G is the graph with

- vertices the ℓ-links of G; two ℓ-links are adjacent if
- one can be shunted onto the other through an $(\ell + 1)$-link of G.



ℓ-link graphs

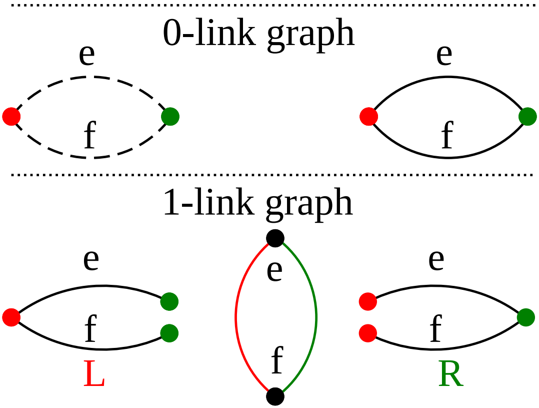
- Two ℓ-links L and R of a graph G might be shunted to each other through $\mu_G(L, R)$ different $(\ell + 1)$-links.

ℓ-link graphs

- Two ℓ-links L and R of a graph G might be shunted to each other through $\mu_G(L, R)$ different $(\ell + 1)$-links.
-

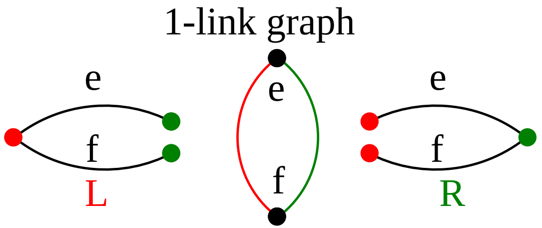
ℓ-link graphs

- Two ℓ-links L and R of a graph G might be shunted to each other through $\mu_G(L, R)$ different $(\ell + 1)$-links.
-
-



ℓ-link graphs

- Two ℓ-links L and R of a graph G might be shunted to each other through $\mu_G(L, R)$ different $(\ell + 1)$-links.
- 

- 

- For any two ℓ-links L and R of G, there are $\mu_G(L, R)$ parallel edges between them in the ℓ-link graph $\mathbb{L}_\ell(G)$ of G.

Path graphs

In the definition of ℓ-link simple graph, if the vertices are ℓ-paths of G, then the constructed graph is the ℓ-path graph $\mathbb{P}_\ell(G)$ introduced by Broersma and Hoede in 1989.

- By definition $\mathbb{L}_0(G) = G$ and $\mathbb{P}_1(G) = \mathbb{L}(G)$ (Broersma and Hoede, 1989).

Path graphs

In the definition of ℓ-link simple graph, if the vertices are ℓ-paths of G, then the constructed graph is the ℓ-path graph $\mathbb{P}_\ell(G)$ introduced by Broersma and Hoede in 1989.

- By definition $\mathbb{L}_0(G) = G$ and $\mathbb{P}_1(G) = \mathbb{L}(G)$ (Broersma and Hoede, 1989).
- If $\ell \in \{0, 1\}$, then $\mathbb{P}_\ell(G)$ is isomorphic to the underlying simple graph of $\mathbb{L}_\ell(G)$.

Path graphs

In the definition of ℓ-link simple graph, if the vertices are ℓ-paths of G, then the constructed graph is the ℓ-path graph $\mathbb{P}_\ell(G)$ introduced by Broersma and Hoede in 1989.

- By definition $\mathbb{L}_0(G) = G$ and $\mathbb{P}_1(G) = \mathbb{L}(G)$ (Broersma and Hoede, 1989).
- If $\ell \in \{0, 1\}$, then $\mathbb{P}_\ell(G)$ is isomorphic to the underlying simple graph of $\mathbb{L}_\ell(G)$.
- If $\ell \geqslant 2$, then $\mathbb{P}_\ell(G)$ is an induced subgraph of $\mathbb{L}_\ell(G)$.

Path graphs

In the definition of ℓ-link simple graph, if the vertices are ℓ-paths of G, then the constructed graph is the ℓ-path graph $\mathbb{P}_\ell(G)$ introduced by Broersma and Hoede in 1989.

- By definition $\mathbb{L}_0(G) = G$ and $\mathbb{P}_1(G) = \mathbb{L}(G)$ (Broersma and Hoede, 1989).
- If $\ell \in \{0, 1\}$, then $\mathbb{P}_\ell(G)$ is isomorphic to the underlying simple graph of $\mathbb{L}_\ell(G)$.
- If $\ell \geqslant 2$, then $\mathbb{P}_\ell(G)$ is an induced subgraph of $\mathbb{L}_\ell(G)$.
- If G is simple and $\ell \in \{0, 1, 2\}$, then $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$.

Path graphs

In the definition of ℓ-link simple graph, if the vertices are ℓ-paths of G, then the constructed graph is the ℓ-path graph $\mathbb{P}_\ell(G)$ introduced by Broersma and Hoede in 1989.

- By definition $\mathbb{L}_0(G) = G$ and $\mathbb{P}_1(G) = \mathbb{L}(G)$ (Broersma and Hoede, 1989).
- If $\ell \in \{0, 1\}$, then $\mathbb{P}_\ell(G)$ is isomorphic to the underlying simple graph of $\mathbb{L}_\ell(G)$.
- If $\ell \geqslant 2$, then $\mathbb{P}_\ell(G)$ is an induced subgraph of $\mathbb{L}_\ell(G)$.
- If G is simple and $\ell \in \{0, 1, 2\}$, then $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$.
- If $girth(G) > \ell \geqslant 2$ then $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$.

ℓ-arc graphs

Godsil and Royle [Algebraic graph theory] defined the ℓ-arc graph $\mathbb{A}_\ell(G)$ of G as the digraph with vertices the ℓ-arcs of G. If an ℓ-arc $\vec{L}$ can be shunted to another $\vec{R}$ in one step, then there is an arc from $\vec{L}$ to $\vec{R}$ in $\mathbb{A}_\ell(G)$.

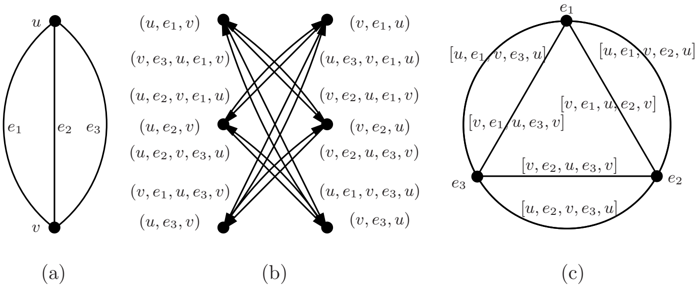


Figure: A multigraph, its 1-arc graph, and 1-link graph

Incidence patterns

Introduced by Grünbaum (1969), an incidence pattern is a function that maps given graphs or similar objects to graphs. So the constructions of line graphs, ℓ-path graphs, ℓ-arc graphs and ℓ-link graphs are all incidence patterns. Two general problems have been proposed by Grunbaum as the characterization of constructed graphs and the determination of original graphs.

R&D problems

Grünbaum's general problems for incidence pattern can be stated more precisely for ℓ-link graphs. For each integer $\ell \geqslant 0$ and every finite graph H:

Recognition problem Decide whether H is an ℓ-link graph.

Determination problem Find the set of ℓ-roots of H.

0-link graph

- Each simple graph is a 0-path graph of itself.

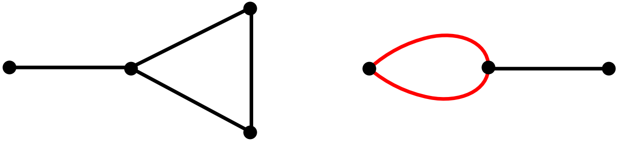


Figure: It is worthy to define the ℓ-link graphs to be multigraphs

0-link graph

- Each simple graph is a 0-path graph of itself.
- The 0-path graph of a graph G is the underlying simple graph of G.

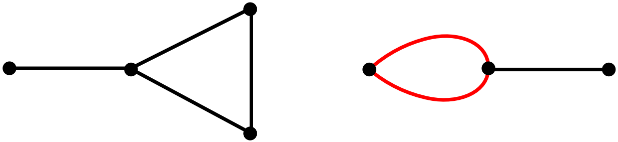


Figure: It is worthy to define the ℓ-link graphs to be multigraphs

0-link graph

- Each simple graph is a 0-path graph of itself.
- The 0-path graph of a graph G is the underlying simple graph of G.
- Each graph is a 0-link graph of itself.

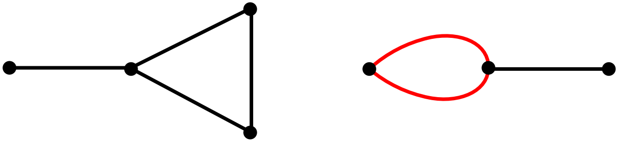


Figure: It is worthy to define the ℓ-link graphs to be multigraphs

Line graph

[J. Krausz, 1943] A graph is a line graph of some simple graph if and only if it is simple and admits a partition of edges in which each part induces a complete subgraph so that every vertex lies in at most two of these subgraphs.

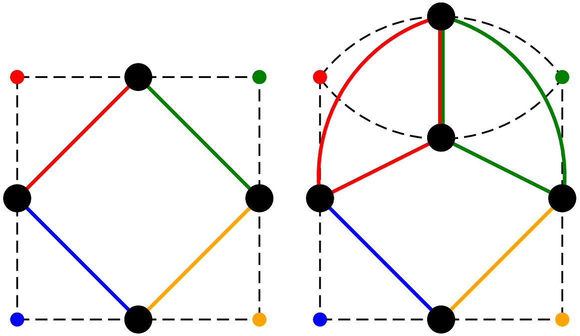


Figure: Characterization of line graphs

Line graph

[J. Krausz, 1943] A graph is a line graph of some simple graph if and only if it is simple and admits a partition of edges in which each part induces a complete subgraph so that every vertex lies in at most two of these subgraphs.

- The statement may be false for multigraphs.

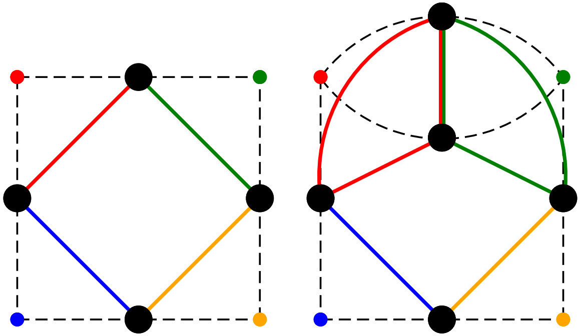


Figure: Characterization of line graphs

1-link graph

A graph H is a 1-link graph if and only if

- $E(H)$ can be partitioned such that each part induces a complete simple subgraph of H;

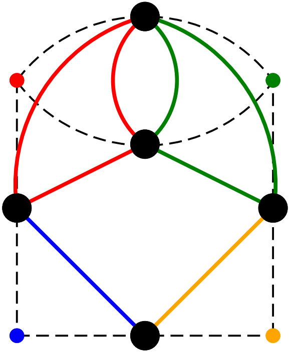


Figure: A line graph is a underlying simple graph of a 1-link graph

1-link graph

A graph H is a 1-link graph if and only if

- $E(H)$ can be partitioned such that each part induces a complete simple subgraph of H;
- and each vertex of H is in at most two such subgraphs.

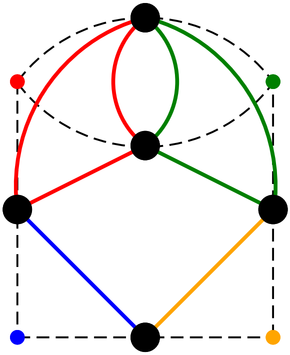


Figure: A line graph is a underlying simple graph of a 1-link graph

line graphs VS 1-link graphs

Let H be a simple graph. The following are equivalent:

- H is a line graph (introduced by Whitney, 1932).

line graphs VS 1-link graphs

Let H be a simple graph. The following are equivalent:

- H is a line graph (introduced by Whitney, 1932).
- H is a 1-path graph (Defined by Broersma and Hoede, 1989).

line graphs VS 1-link graphs

Let H be a simple graph. The following are equivalent:

- H is a line graph (introduced by Whitney, 1932).
- H is a 1-path graph (Defined by Broersma and Hoede, 1989).
- By duplicating some edges of H we can obtain a 1-link graph.

line graphs VS 1-link graphs

Let H be a simple graph. The following are equivalent:

- H is a line graph (introduced by Whitney, 1932).
- H is a 1-path graph (Defined by Broersma and Hoede, 1989).
- By duplicating some edges of H we can obtain a 1-link graph.
- By duplicating some edges we can obtain a graph H' which admits an edge partition such that each part induces a complete simple subgraph of H', and that each vertex of H' is in at most two such subgraphs.

Whitney's isomorphism theorem

The line graphs of K_3 and $K_{1,3}$ are isomorphic to K_3. The line graphs of K_0 and K_1 are isomorphic to K_0. These are the only pairs of nonisomorphic connected graphs with isomorphic line graphs.

Whitney's isomorphism theorem

The line graphs of K_3 and $K_{1,3}$ are isomorphic to K_3. The line graphs of K_0 and K_1 are isomorphic to K_0. These are the only pairs of nonisomorphic connected graphs with isomorphic line graphs.

- We characterised all nonisomorphic pairs with isomorphic 1-link graphs.

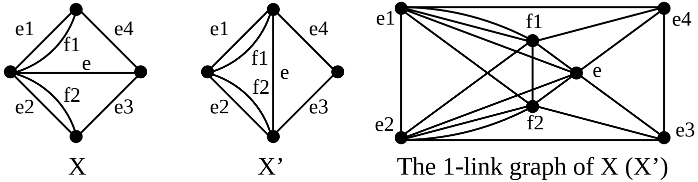


Figure: A pair of connected graphs with isomorphic 1-link graphs

From multigraphs to simple graphs

For $s \geqslant 1$ and each graph G, $(\mathbb{L}_\ell(G))^{\langle s\rangle} \cong \mathbb{L}_{\ell s}(G^{\langle s\rangle})$.

From multigraphs to simple graphs

For $s \geqslant 1$ and each graph G, $(\mathbb{L}_\ell(G))^{\langle s \rangle} \cong \mathbb{L}_{\ell s}(G^{\langle s \rangle})$.

- $(\mathbb{L}_1(G))^{\langle s \rangle} \cong \mathbb{L}_s(G^{\langle s \rangle})$.

From multigraphs to simple graphs

For $s \geqslant 1$ and each graph G, $(\mathbb{L}_\ell(G))^{\langle s\rangle} \cong \mathbb{L}_{\ell s}(G^{\langle s\rangle})$.

- $(\mathbb{L}_1(G))^{\langle s\rangle} \cong \mathbb{L}_s(G^{\langle s\rangle})$.
- This projects multigraphs to simple graphs.

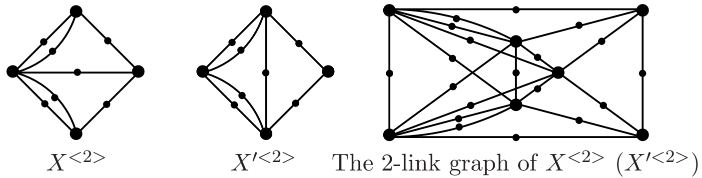


Figure: A pair of connected graphs with isomorphic 2-link (2-path) graphs

Applications of Whitney's theorem

Xueliang Li and Yan Liu proved that there exists no triple of nonnull connected simple graphs with isomorphic connected 2-path graphs.

Applications of Whitney's theorem

Xueliang Li and Yan Liu proved that there exists no triple of nonnull connected simple graphs with isomorphic connected 2-path graphs.

- $(\mathbb{L}_1(G))^{\langle 2\rangle} \cong \mathbb{L}_2(G^{\langle 2\rangle})$.

Applications of Whitney's theorem

Xueliang Li and Yan Liu proved that there exists no triple of nonnull connected simple graphs with isomorphic connected 2-path graphs.

- $(\mathbb{L}_1(G))^{\langle 2 \rangle} \cong \mathbb{L}_2(G^{\langle 2 \rangle})$.
- There exists no triple of connected graphs with isomorphic 1-link graphs.

Uniqueness of original graphs

Xueliang Li proved that simple graphs of minimum degree $\geqslant 3$ are isomorphic if and only if their 2-path graphs are isomorphic. Li also conjectured that the assertion is true when minimum degree is 2, which was shown to be false by Aldred et al.

Uniqueness of original graphs

Xueliang Li proved that simple graphs of minimum degree $\geqslant 3$ are isomorphic if and only if their 2-path graphs are isomorphic. Li also conjectured that the assertion is true when minimum degree is 2, which was shown to be false by Aldred et al.

- $(\mathbb{L}_1(G))^{\langle \ell \rangle} \cong \mathbb{L}_\ell(G^{\langle \ell \rangle})$.

Uniqueness of original graphs

Xueliang Li proved that simple graphs of minimum degree $\geqslant 3$ are isomorphic if and only if their 2-path graphs are isomorphic. Li also conjectured that the assertion is true when minimum degree is 2, which was shown to be false by Aldred et al.

- $(\mathbb{L}_1(G))^{\langle \ell \rangle} \cong \mathbb{L}_\ell(G^{\langle \ell \rangle})$.
- For each $\ell \geqslant 2$, there are infinitely many pairs of simple graphs of minimum degree 2 with isomorphic ℓ-link (and ℓ-path) graphs.

Uniqueness of original graphs

Let $\ell, s \geqslant 2$, and G and X be connected graphs of $\delta(G) \geqslant 3$ such that G is simple and $\mathbb{L}_\ell(G) \cong \mathbb{L}_s(X)$. In each of the following cases, $\ell = s$, $G \cong X$, $Aut(G) \cong Aut(\mathbb{L}_\ell(G))$, and every isomorphism from $\mathbb{L}_\ell(G)$ to $\mathbb{L}_s(X)$ is induced by an isomorphism from G to X:

Uniqueness of original graphs

Let $\ell, s \geqslant 2$, and G and X be connected graphs of $\delta(G) \geqslant 3$ such that G is simple and $\mathbb{L}_\ell(G) \cong \mathbb{L}_s(X)$. In each of the following cases, $\ell = s$, $G \cong X$, $Aut(G) \cong Aut(\mathbb{L}_\ell(G))$, and every isomorphism from $\mathbb{L}_\ell(G)$ to $\mathbb{L}_s(X)$ is induced by an isomorphism from G to X:

- $\Delta(G) \geqslant 4$.

Uniqueness of original graphs

Let $\ell, s \geqslant 2$, and G and X be connected graphs of $\delta(G) \geqslant 3$ such that G is simple and $\mathbb{L}_\ell(G) \cong \mathbb{L}_s(X)$. In each of the following cases, $\ell = s$, $G \cong X$, $Aut(G) \cong Aut(\mathbb{L}_\ell(G))$, and every isomorphism from $\mathbb{L}_\ell(G)$ to $\mathbb{L}_s(X)$ is induced by an isomorphism from G to X:

- $\Delta(G) \geqslant 4$.
- G contains a triangle.

Uniqueness of original graphs

Let $\ell, s \geqslant 2$, and G and X be connected graphs of $\delta(G) \geqslant 3$ such that G is simple and $\mathbb{L}_\ell(G) \cong \mathbb{L}_s(X)$. In each of the following cases, $\ell = s$, $G \cong X$, $Aut(G) \cong Aut(\mathbb{L}_\ell(G))$, and every isomorphism from $\mathbb{L}_\ell(G)$ to $\mathbb{L}_s(X)$ is induced by an isomorphism from G to X:

- $\Delta(G) \geqslant 4$.
- G contains a triangle.
- $girth(G) \geqslant 5$ and X contains an $(s-1)$-link whose end vertices are of degree at least 3.

Recognition and determination algorithms

Roussopoulos (1973) gave an $\max\{m, n\}$-time algorithm for determining the simple graph G from its line graph H.

Recognition and determination algorithms

Roussopoulos (1973) gave an $\max\{m, n\}$-time algorithm for determining the simple graph G from its line graph H.

- A graph is a 1-link graph if and only if it is the induced subgraph of the strong product of the line graph of a simple graph and some K_t^2.

Recognition and determination algorithms

Roussopoulos (1973) gave an $\max\{m, n\}$-time algorithm for determining the simple graph G from its line graph H.

- A graph is a 1-link graph if and only if it is the induced subgraph of the strong product of the line graph of a simple graph and some K_t^2.
- It costs linear time $O(m)$ to decide if H is a 1-link graph, and to find all its 1-roots.

Recognition and determination algorithms

Roussopoulos (1973) gave an $\max\{m, n\}$-time algorithm for determining the simple graph G from its line graph H.

- A graph is a 1-link graph if and only if it is the induced subgraph of the strong product of the line graph of a simple graph and some K_t^2.
- It costs linear time $O(m)$ to decide if H is a 1-link graph, and to find all its 1-roots.
- For each finite graph H, it costs linear time $O(m)$ to decide whether there exists an integer $\ell \geqslant 1$ and a graph G with $\delta(G) \geqslant 3$, such that $H \cong \mathbb{L}_\ell(G)$.

Recognition and determination algorithms

Roussopoulos (1973) gave an $\max\{m, n\}$-time algorithm for determining the simple graph G from its line graph H.

- A graph is a 1-link graph if and only if it is the induced subgraph of the strong product of the line graph of a simple graph and some K_t^2.
- It costs linear time $O(m)$ to decide if H is a 1-link graph, and to find all its 1-roots.
- For each finite graph H, it costs linear time $O(m)$ to decide whether there exists an integer $\ell \geqslant 1$ and a graph G with $\delta(G) \geqslant 3$, such that $H \cong \mathbb{L}_\ell(G)$.
- All such pairs (G, ℓ) can be obtained in linear time $O(m)$.

2-link graph

Let H be a 2-link graph. Then it has a vertex partition $\mathcal{V}$ and an edge partition $\mathcal{E}$ such that

- Each part of $\mathcal{V}$ is an independent set of H;

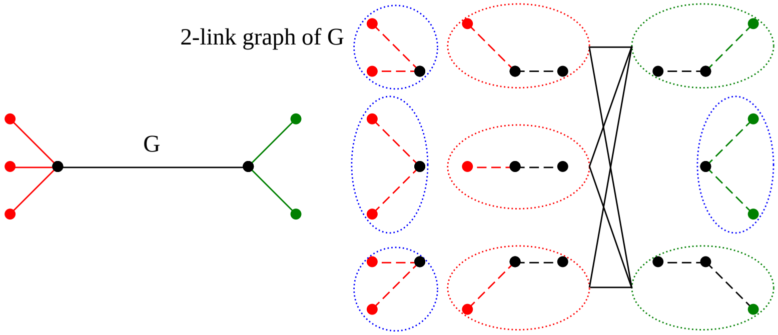


Figure: A 2-link graph of a simple graph is a 2-path graph

2-link graph

Let H be a 2-link graph. Then it has a vertex partition $\mathcal{V}$ and an edge partition $\mathcal{E}$ such that

- Each part of $\mathcal{V}$ is an independent set of H;
- Each part of $\mathcal{E}$ induces a complete bipartite graph.

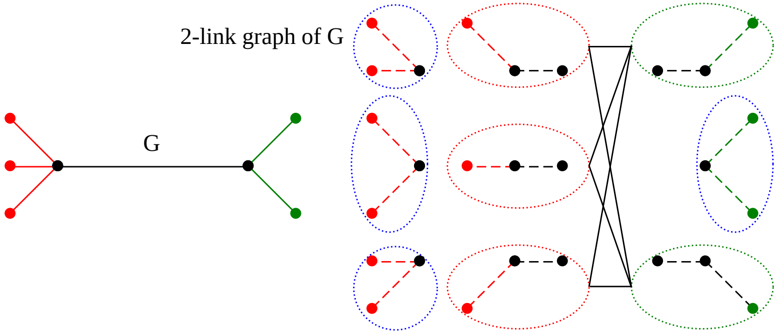


Figure: A 2-link graph of a simple graph is a 2-path graph

Standard partition

$(\mathcal{V}, \mathcal{E})$ is called a standard partition of H if further:

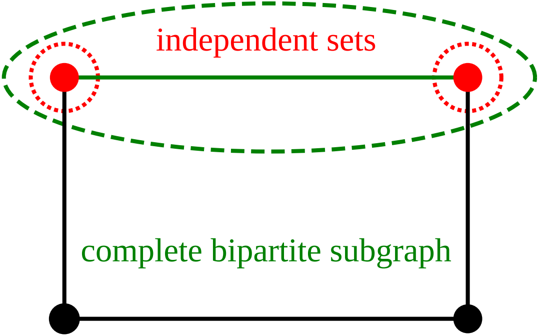


Figure: The ℓ-link graph of a cycle is itself

Standard partition

$(\mathcal{V}, \mathcal{E})$ is called a standard partition of H if further:

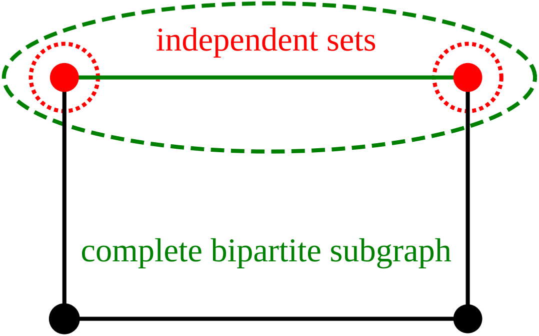


Figure: The ℓ-link graph of a cycle is itself

Standard partition

$(\mathcal{V}, \mathcal{E})$ is called a standard partition of H if further:

- Each part of $\mathcal{E}$ is incident to exactly two parts of $\mathcal{V}$.

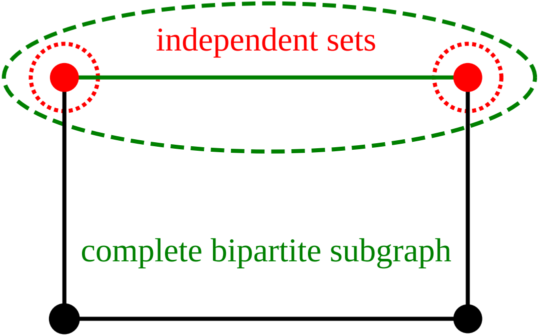


Figure: The ℓ-link graph of a cycle is itself

Standard partition

$(\mathcal{V}, \mathcal{E})$ is called a standard partition of H if further:

- Each part of $\mathcal{E}$ is incident to exactly two parts of $\mathcal{V}$.
- Each vertex of H is incident to exactly two parts of $\mathcal{E}$.

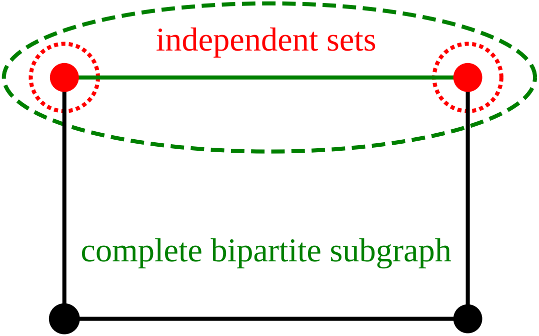


Figure: The ℓ-link graph of a cycle is itself

Characterization of 2-link graphs

A graph is a 2-link graph of some graph of minimum degree at least 2 if and only if it admits a standard partition $(\mathcal{V}, \mathcal{E})$ such that:

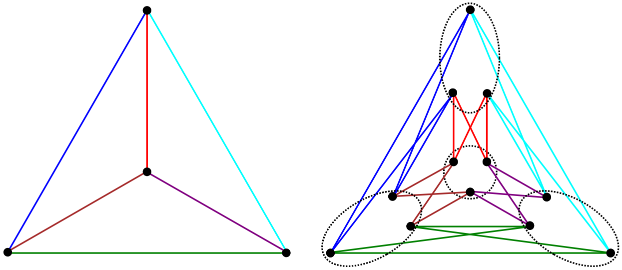


Figure: The 2-link graph of K_4

Characterization of 2-link graphs

A graph is a 2-link graph of some graph of minimum degree at least 2 if and only if it admits a standard partition $(\mathcal{V}, \mathcal{E})$ such that:

- If $E, F \in \mathcal{E}$ are incident to some $V \in \mathcal{V}$, then they are incident to exactly one vertex of V.

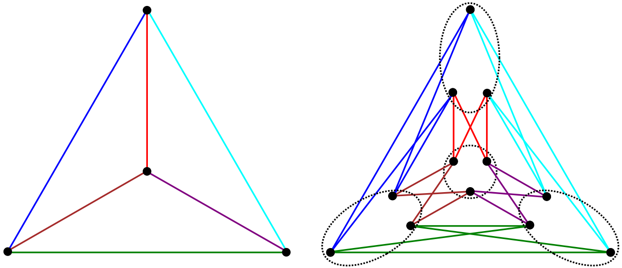


Figure: The 2-link graph of K_4

Characterization of 2-link graphs

A graph is a 2-link graph of some graph of minimum degree at least 2 if and only if it admits a standard partition $(\mathcal{V}, \mathcal{E})$ such that:

- If $E, F \in \mathcal{E}$ are incident to some $V \in \mathcal{V}$, then they are incident to exactly one vertex of V.

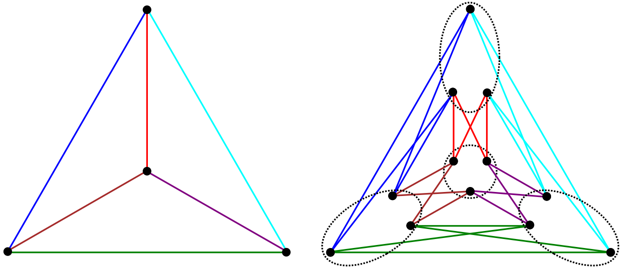


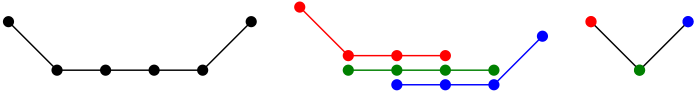
Figure: The 2-link graph of K_4

- $H \cong \mathbb{L}_2(G)$, where $G := H_{(\mathcal{V},\mathcal{E})}$.

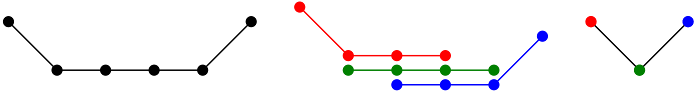
Reduction of ℓ-links

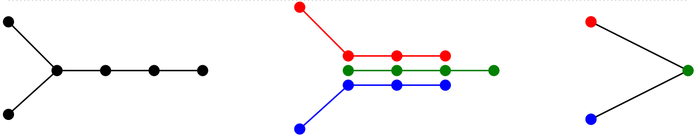
- Let $\ell \geq 2$. Then an ℓ-link of G corresponds to a 2-link of the $(\ell - 2)$-link graph of G.

Reduction of ℓ-links

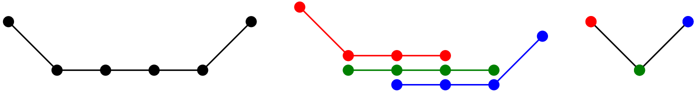
- Let $\ell \geq 2$. Then an ℓ-link of G corresponds to a 2-link of the $(\ell-2)$-link graph of G.
- 

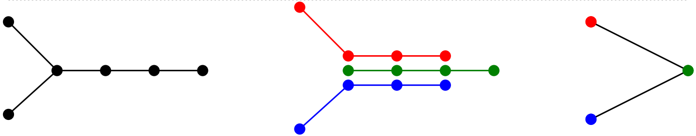
Reduction of ℓ-links

- Let $\ell \geq 2$. Then an ℓ-link of G corresponds to a 2-link of the $(\ell - 2)$-link graph of G.
- 
- But not vice versa.



Reduction of ℓ-links

- Let $\ell \geq 2$. Then an ℓ-link of G corresponds to a 2-link of the $(\ell - 2)$-link graph of G.
- 
- But not vice versa.



- $\mathbb{L}_{\ell}(G)$ is an induced subgraph of the 2-link graph of $\mathbb{L}_{\ell-2}(G)$.

Characterization of 3-links graphs

Let H be a graph. Then H is a 3-link graph of some graph of minimum degree at least 2 if and only if there is a standard partition $(\mathcal{V}, \mathcal{E})$ of H and a partition $\mathcal{K}$ of $\mathcal{E}$ such that:

- $H_{(\mathcal{V},\mathcal{E})}$ is a 1-link graph with an edge partition $\mathcal{K}$.

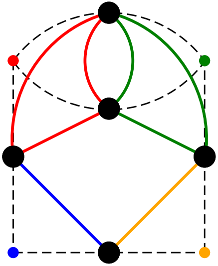


Figure: We only choose 2-links with two different colors

Characterization of 3-links graphs

Let H be a graph. Then H is a 3-link graph of some graph of minimum degree at least 2 if and only if there is a standard partition $(\mathcal{V}, \mathcal{E})$ of H and a partition $\mathcal{K}$ of $\mathcal{E}$ such that:

- $H_{(\mathcal{V},\mathcal{E})}$ is a 1-link graph with an edge partition $\mathcal{K}$.
- If $E, F \in \mathcal{E}$ are incident to $V \in \mathcal{V}$ in $H_{(\mathcal{V},\mathcal{E})}$,

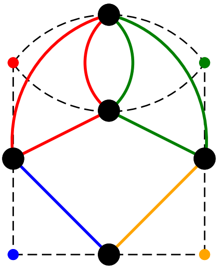


Figure: We only choose 2-links with two different colors

Characterization of 3-links graphs

Let H be a graph. Then H is a 3-link graph of some graph of minimum degree at least 2 if and only if there is a standard partition $(\mathcal{V}, \mathcal{E})$ of H and a partition $\mathcal{K}$ of $\mathcal{E}$ such that:

- $H_{(\mathcal{V},\mathcal{E})}$ is a 1-link graph with an edge partition $\mathcal{K}$.
- If $E, F \in \mathcal{E}$ are incident to $V \in \mathcal{V}$ in $H_{(\mathcal{V},\mathcal{E})}$,
- then they are incident in H if and only if E and F are in different parts of $\mathcal{K}$.

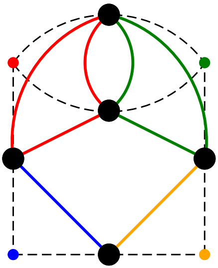


Figure: We only choose 2-links with two different colors

Characterization of ℓ-link graphs

Let $\ell \geq 4$ be an integer. Then a graph H is an ℓ-link graph of a graph of minimum degree at least 2 if and only if H admits a standard partition $(\mathcal{V}, \mathcal{E})$ such that:

- $H_{(\mathcal{V},\mathcal{E})}$ is an $(\ell - 2)$-link graph of a graph of minimum degree at least 2, with a standard partition $(\mathcal{V}_{\ell-2}, \mathcal{E}_{\ell-2})$.

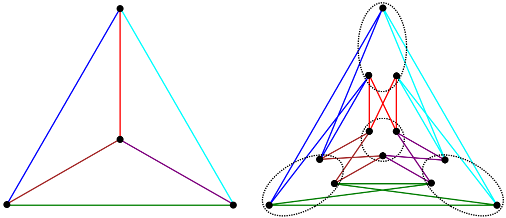


Figure: We only choose 2-links with two different colors

Characterization of ℓ-link graphs

Let $\ell \geq 4$ be an integer. Then a graph H is an ℓ-link graph of a graph of minimum degree at least 2 if and only if H admits a standard partition $(\mathcal{V}, \mathcal{E})$ such that:

- $H_{(\mathcal{V},\mathcal{E})}$ is an $(\ell - 2)$-link graph of a graph of minimum degree at least 2, with a standard partition $(\mathcal{V}_{\ell-2}, \mathcal{E}_{\ell-2})$.
- For any two different parts E, F of $\mathcal{E}$ and any part V of $\mathcal{V}$, E, F and V are incident at one vertex of H if and only if $E_{\mathcal{E}}$ and $F_{\mathcal{E}}$ are incident to $V_{\mathcal{V}}$, and correspond to two edges of $H_{(\mathcal{V},\mathcal{E})}$ that are in different parts of $\mathcal{E}_{\ell-2}$.

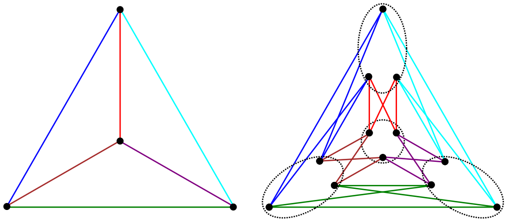


Figure: We only choose 2-links with two different colors

Chromatic number

Recall that the chromatic number $\chi(G)$ of G is the smallest integer $t \geq 0$ such that $V(G)$ can be colored by t colors and any two adjacent vertices are assigned to different colors.

Coloring 2-link graphs

Let H be a 2-link graph of G. Then

- H is homomorphic to G.

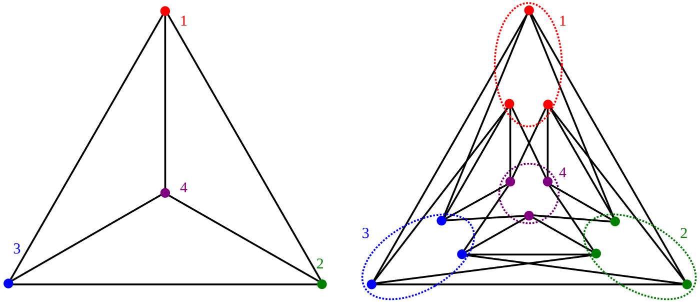


Figure: H inherits a coloring from G

Coloring 2-link graphs

Let H be a 2-link graph of G. Then

- H is homomorphic to G.
- So H can be colored by $\{1, 2, \ldots, \chi := \chi(G)\}$.

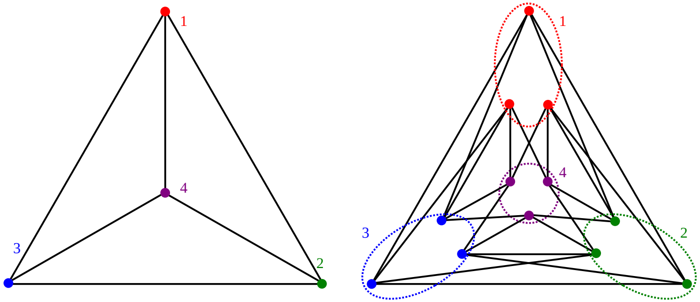


Figure: H inherits a coloring from G

Every vertex of H is adjacent to at most two parts of $\mathcal{V}$.

- For each $v \in V(H)$ colored by χ,

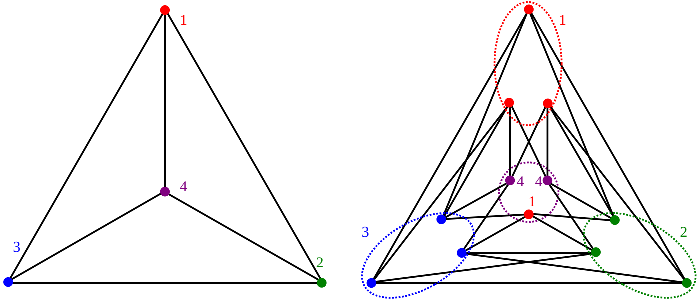


Figure: Change a color to the smallest possible integer

Every vertex of H is adjacent to at most two parts of $\mathcal{V}$.

- For each $v \in V(H)$ colored by χ,
- the color of v can be replaced by one of $\{1, 2, 3\}$.

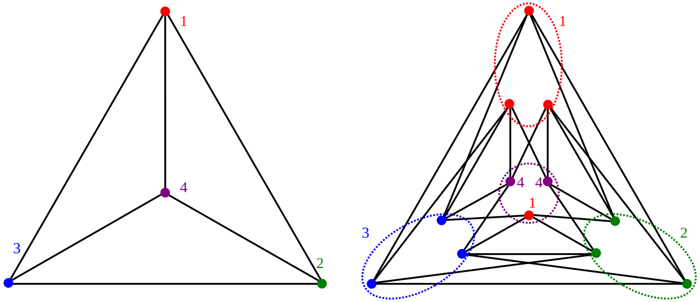


Figure: Change a color to the smallest possible integer

H is homomorphic to G, so H can be colored by the color set $\{1, 2, \ldots, \chi(G)\}$. For each $v \in V(H)$ colored by χ, the color of v can be replaced by one of $\{1, 2, 3\}$.

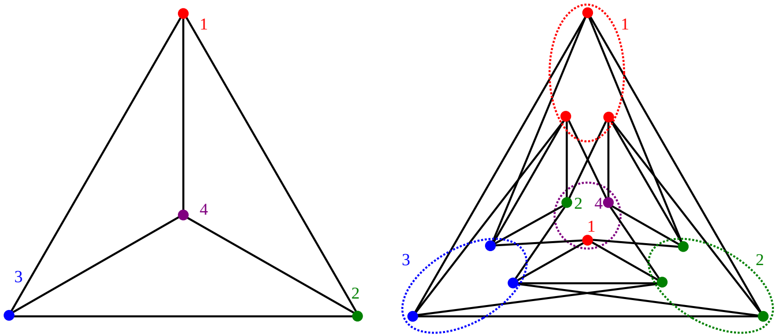


Figure: Change a color to the smallest possible integer

Let $\chi_\ell := \chi(\mathbb{L}_\ell(G))$.

- Repeat the process we obtain $\chi_2 \leq \lfloor \frac{2\chi}{3} \rfloor + 1$.

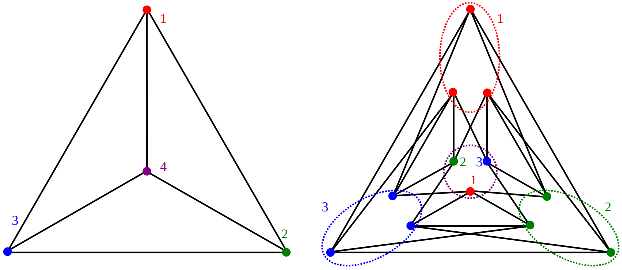


Figure: Change a color to the smallest possible integer

Let $\chi_\ell := \chi(\mathbb{L}_\ell(G))$.

- Repeat the process we obtain $\chi_2 \leq \lfloor \frac{2\chi}{3} \rfloor + 1$.
- Note $\mathbb{L}_\ell(G)$ is a subgraph of the 2-link graph of $\mathbb{L}_{\ell-2}(G)$.

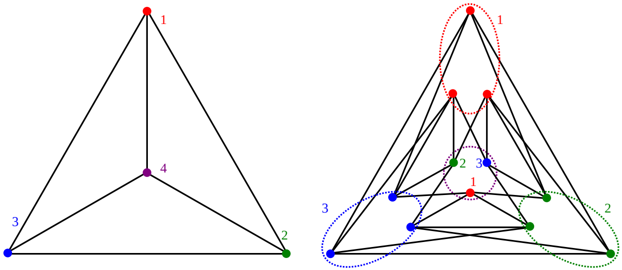


Figure: Change a color to the smallest possible integer

Let $\chi_\ell := \chi(\mathbb{L}_\ell(G))$.

- Repeat the process we obtain $\chi_2 \leq \lfloor \frac{2\chi}{3} \rfloor + 1$.
- Note $\mathbb{L}_\ell(G)$ is a subgraph of the 2-link graph of $\mathbb{L}_{\ell-2}(G)$.
- So $\chi_\ell \leq \lfloor \frac{2\chi_{\ell-2}}{3} \rfloor + 1$.

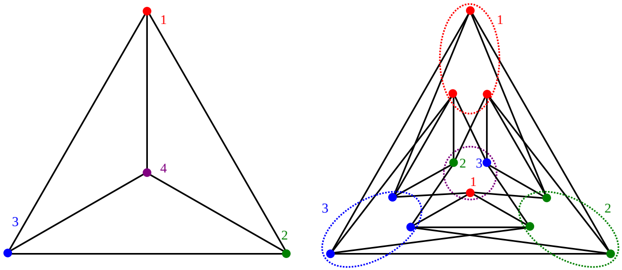


Figure: Change a color to the smallest possible integer

Let G be a graph with maximum degree Δ, chromatic number χ and edge chromatic number χ'. Let $\ell \geq 0$ be an integer and χ_ℓ be the chromatic number of the ℓ-link graph of G. Then

- If $\ell \geq 0$ is even, then $\chi_\ell \leq \min\{\chi, \lfloor(\frac{2}{3})^{\ell/2}(\chi - 3)\rfloor + 3\}$.

Let G be a graph with maximum degree Δ, chromatic number χ and edge chromatic number χ'. Let $\ell \geq 0$ be an integer and χ_ℓ be the chromatic number of the ℓ-link graph of G. Then

- If $\ell \geq 0$ is even, then $\chi_\ell \leq \min\{\chi, \lfloor(\frac{2}{3})^{\ell/2}(\chi - 3)\rfloor + 3\}$.
- If $\ell \geq 1$ is odd, then $\chi_\ell \leq \min\{\chi', \lfloor(\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3)\rfloor + 3\}$.

Let G be a graph with maximum degree Δ, chromatic number χ and edge chromatic number χ'. Let $\ell \geq 0$ be an integer and χ_ℓ be the chromatic number of the ℓ-link graph of G. Then

- If $\ell \geq 0$ is even, then $\chi_\ell \leq \min\{\chi, \lfloor (\frac{2}{3})^{\ell/2}(\chi - 3) \rfloor + 3\}$.
- If $\ell \geq 1$ is odd, then $\chi_\ell \leq \min\{\chi', \lfloor (\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3) \rfloor + 3\}$.
- If $\ell \geq 2$, then $\chi_\ell \leq \chi_{\ell-2}$.

Let G be a graph with maximum degree Δ, chromatic number χ and edge chromatic number χ'. Let $\ell \geq 0$ be an integer and χ_ℓ be the chromatic number of the ℓ-link graph of G. Then

- If $\ell \geq 0$ is even, then $\chi_\ell \leq \min\{\chi, \lfloor (\frac{2}{3})^{\ell/2}(\chi - 3) \rfloor + 3\}$.
- If $\ell \geq 1$ is odd, then $\chi_\ell \leq \min\{\chi', \lfloor (\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3) \rfloor + 3\}$.
- If $\ell \geq 2$, then $\chi_\ell \leq \chi_{\ell-2}$.
- Recall that $\chi' \leq \frac{3}{2}\Delta$ (Shannon, 1949).
- If $\ell \neq 1$, then $\chi_\ell \leq \Delta + 1$.

Let G be a graph with maximum degree Δ, chromatic number χ and edge chromatic number χ'. Let $\ell \geq 0$ be an integer and χ_ℓ be the chromatic number of the ℓ-link graph of G. Then

- If $\ell \geq 0$ is even, then $\chi_\ell \leq \min\{\chi, \lfloor (\frac{2}{3})^{\ell/2}(\chi - 3)\rfloor + 3\}$.
- If $\ell \geq 1$ is odd, then $\chi_\ell \leq \min\{\chi', \lfloor (\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3)\rfloor + 3\}$.
- If $\ell \geq 2$, then $\chi_\ell \leq \chi_{\ell-2}$.
- Recall that $\chi' \leq \frac{3}{2}\Delta$ (Shannon, 1949).
- If $\ell \neq 1$, then $\chi_\ell \leq \Delta + 1$.
- $\mathbb{L}_\ell(G)$ is tripartite for all $\ell > 5\ln(\Delta - 2) + 3$.

Let G be a graph with maximum degree Δ, chromatic number χ and edge chromatic number χ'. Let $\ell \geq 0$ be an integer and χ_ℓ be the chromatic number of the ℓ-link graph of G. Then

- If $\ell \geq 0$ is even, then $\chi_\ell \leq \min\{\chi, \lfloor(\frac{2}{3})^{\ell/2}(\chi - 3)\rfloor + 3\}$.
- If $\ell \geq 1$ is odd, then $\chi_\ell \leq \min\{\chi', \lfloor(\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3)\rfloor + 3\}$.
- If $\ell \geq 2$, then $\chi_\ell \leq \chi_{\ell-2}$.
- Recall that $\chi' \leq \frac{3}{2}\Delta$ (Shannon, 1949).
- If $\ell \neq 1$, then $\chi_\ell \leq \Delta + 1$.
- $\mathbb{L}_\ell(G)$ is tripartite for all $\ell > 5\ln(\Delta - 2) + 3$.
- $\chi_\ell \leq 6$ if $\ell > 5\ln(\Delta - 2) - 3.8$.

Edge Contraction

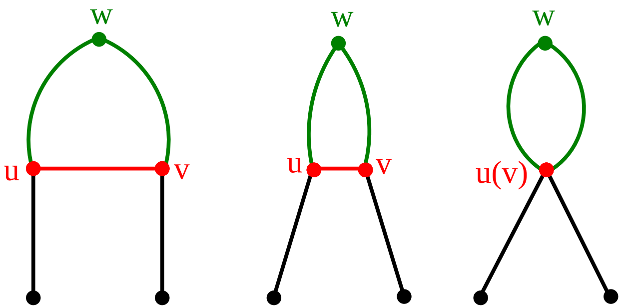


Figure: Contracting the edge *uv*

Graph minors

Let G be a finite graph.

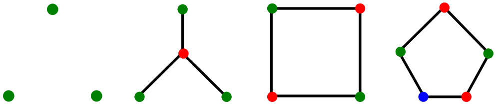
- A graph is said to be a minor of G if it can be obtained from a subgraph of G by contracting edges.

Graph minors

Let G be a finite graph.

- A graph is said to be a minor of G if it can be obtained from a subgraph of G by contracting edges.
- The Hadwiger number $\eta(G)$ is the maximal number t such that G has K_t as a minor.

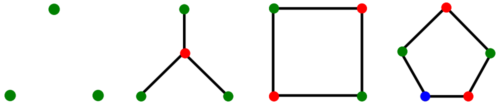
Hadwiger and chromatic numbers



Hadwiger number	1	2	3	3
Chromatic number	1	2	2	3

- In the above examples, we have $\eta(G) \geq \chi(G)$.

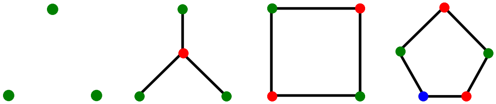
Hadwiger and chromatic numbers



Hadwiger number	1	2	3	3
Chromatic number	1	2	2	3

- In the above examples, we have $\eta(G) \geq \chi(G)$.
- Hadwiger's conjecture states that:

Hadwiger and chromatic numbers



Hadwiger number	1	2	3	3
Chromatic number	1	2	2	3

- In the above examples, we have $\eta(G) \geq \chi(G)$.
- Hadwiger's conjecture states that:
- For every finite graph G, $\eta(G) \geq \chi(G)$.

Graph minors of link graphs

Let G be a graph of $\delta(G) \geqslant 2$, and X be a connected subgraph of G containing at least one ℓ-link. Then

Graph minors of link graphs

Let G be a graph of $\delta(G) \geqslant 2$, and X be a connected subgraph of G containing at least one ℓ-link. Then

- For any two ℓ-links of X, one can be shunted to the other under the restriction that

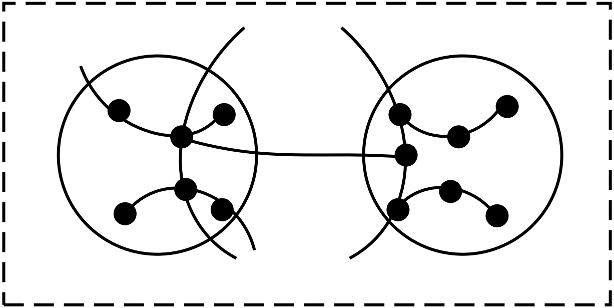
Graph minors of link graphs

Let G be a graph of $\delta(G) \geqslant 2$, and X be a connected subgraph of G containing at least one ℓ-link. Then

- For any two ℓ-links of X, one can be shunted to the other under the restriction that
- the middle vertices or edges of the images are inside X.

Graph minors of link graphs

Let G be a graph of $\delta(G) \geqslant 2$, and X be a connected subgraph of G containing at least one ℓ-link. Then

- For any two ℓ-links of X, one can be shunted to the other under the restriction that
- the middle vertices or edges of the images are inside X.
- 

Clique minors

If G contains an (ℓ, t)-system, then the ℓ-link graph of G contains a K_t-minor.

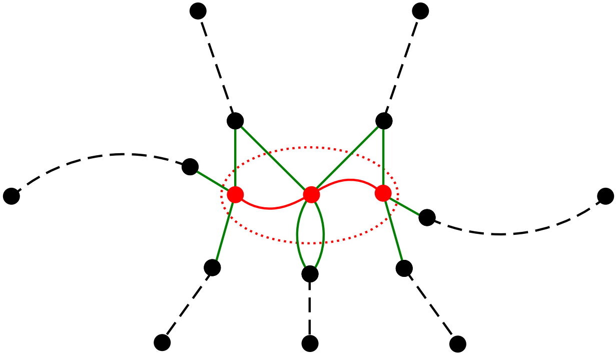


Figure: An $(\ell, 10)$-system implies a K_{10}-minor

Hadwiger's conjecture

- So far the best progress in Hadwiger's conjecture is achieved by Robertson, Seymour and Thomas in 1993.

Hadwiger's conjecture

- So far the best progress in Hadwiger's conjecture is achieved by Robertson, Seymour and Thomas in 1993.
- They proved the conjecture is true for all graphs G of $\chi(G) \leq 6$.

Hadwiger's conjecture

- So far the best progress in Hadwiger's conjecture is achieved by Robertson, Seymour and Thomas in 1993.
- They proved the conjecture is true for all graphs G of $\chi(G) \leq 6$.
- In 2004, Reed and Seymour proved that the conjecture is true for all line graphs,

Hadwiger's conjecture

- So far the best progress in Hadwiger's conjecture is achieved by Robertson, Seymour and Thomas in 1993.
- They proved the conjecture is true for all graphs G of $\chi(G) \leq 6$.
- In 2004, Reed and Seymour proved that the conjecture is true for all line graphs,
- which is equivalent to say, Hadwiger's conjecture is true for all 1-link graphs.

Our results

Hadwiger's conjecture for 0-link graphs is equivalent to the conjecture itself. We proved the conjecture for ℓ-link graphs of a graph G such that:

- $\ell > 5\ln(\Delta(G) - 2) - 3.8$.

Our results

Hadwiger's conjecture for 0-link graphs is equivalent to the conjecture itself. We proved the conjecture for ℓ-link graphs of a graph G such that:

- $\ell > 5\ln(\Delta(G) - 2) - 3.8$.
- $\delta(G) \geq 3$ and $\ell \geq 1$.

Our results

Hadwiger's conjecture for 0-link graphs is equivalent to the conjecture itself. We proved the conjecture for ℓ-link graphs of a graph G such that:

- $\ell > 5\ln(\Delta(G) - 2) - 3.8$.
- $\delta(G) \geq 3$ and $\ell \geq 1$.
- $\ell \geq 2$ is an even integer.

Our results

Hadwiger's conjecture for 0-link graphs is equivalent to the conjecture itself. We proved the conjecture for ℓ-link graphs of a graph G such that:

- $\ell > 5\ln(\Delta(G) - 2) - 3.8$.
- $\delta(G) \geq 3$ and $\ell \geq 1$.
- $\ell \geq 2$ is an even integer.
- G is biconnected and $\ell \geq 1$.

Our results

Hadwiger's conjecture for 0-link graphs is equivalent to the conjecture itself. We proved the conjecture for ℓ-link graphs of a graph G such that:

- $\ell > 5\ln(\Delta(G) - 2) - 3.8$.
- $\delta(G) \geq 3$ and $\ell \geq 1$.
- $\ell \geq 2$ is an even integer.
- G is biconnected and $\ell \geq 1$.
- Another interesting result we obtained was that, if G contains a cycle, then $\eta(\mathbb{L}_\ell(G)) \geq \eta(G)$ for all $\ell \geq 0$.

Minimal roots of cycles

Let $\mathbb{R}_\ell(H)$ be the set of ℓ-roots of H; that is, minimal graphs G such that $\mathbb{L}_\ell(G) \cong H$.

Minimal roots of cycles

Let $\mathbb{R}_\ell(H)$ be the set of ℓ-roots of H; that is, minimal graphs G such that $\mathbb{L}_\ell(G) \cong H$.

- By Whitney, $\mathbb{R}_1(K_3) = \{K_3, K_{1,3}\}$.

Minimal roots of cycles

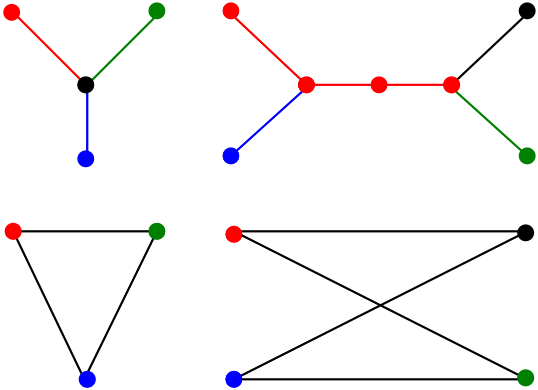
Let $\mathbb{R}_\ell(H)$ be the set of ℓ-roots of H; that is, minimal graphs G such that $\mathbb{L}_\ell(G) \cong H$.

- By Whitney, $\mathbb{R}_1(K_3) = \{K_3, K_{1,3}\}$.
- $|\mathbb{R}_\ell(O)| \in \{1, 2\}$.

Minimal roots of cycles

Let $\mathbb{R}_\ell(H)$ be the set of ℓ-roots of H; that is, minimal graphs G such that $\mathbb{L}_\ell(G) \cong H$.

- By Whitney, $\mathbb{R}_1(K_3) = \{K_3, K_{1,3}\}$.
- $|\mathbb{R}_\ell(O)| \in \{1, 2\}$.
- Cycles with two minimal ℓ-roots



Minimal $0, 1, 2$-roots of $2K_1$

- $\mathbb{R}_0(2K_1) = 2K_1$.

Minimal $0, 1, 2$-roots of $2K_1$

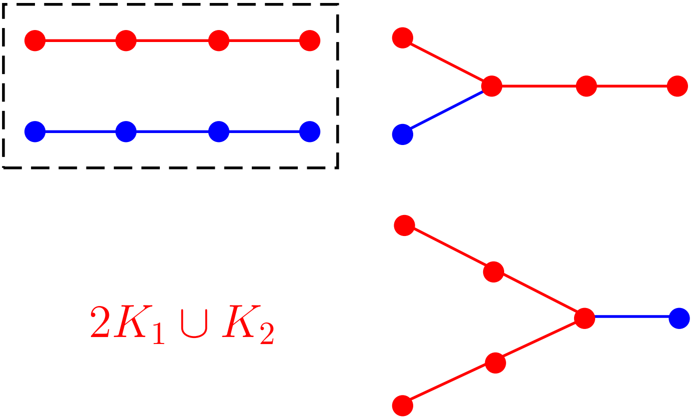
- $\mathbb{R}_0(2K_1) = 2K_1$.
- $\mathbb{R}_1(2K_1) = 2K_2$.

Minimal $0, 1, 2$-roots of $2K_1$

- $\mathbb{R}_0(2K_1) = 2K_1$.
- $\mathbb{R}_1(2K_1) = 2K_2$.
- $\mathbb{R}_2(2K_1) = 2P_2$.

Minimal 3-roots of $2K_1$

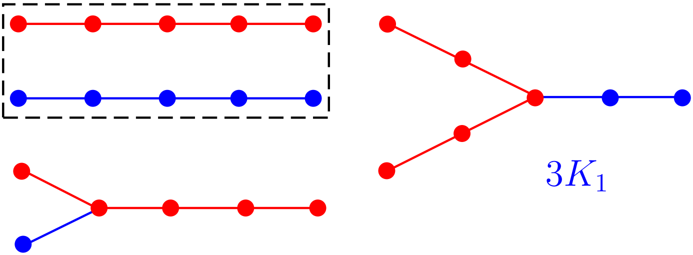
$|\mathbb{R}_3(2K_1)| = 2$



$2K_1 \cup K_2$

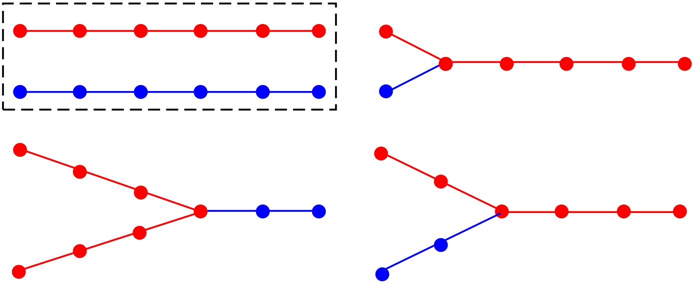
Minimal 4-roots of $2K_1$

$|\mathbb{R}_4(2K_1)| = 2$



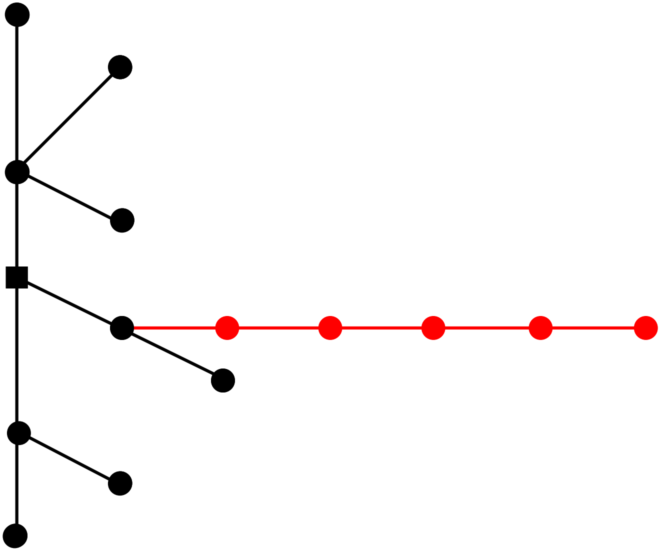
Minimal 5-roots of $2K_1$

$|\mathbb{R}_5(2K_1)| = 3$. In general, $|\mathbb{R}_\ell(2K_1)|$ is 1 if $\ell = 0$, and is $\lfloor \frac{\ell+1}{2} \rfloor$ if $\ell \geqslant 1$.



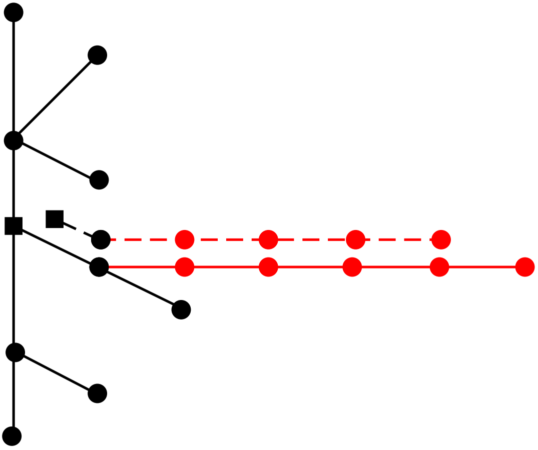
Minimal 5-roots of a tree

The 5-link graph of the whole tree TP is the black subtree T.



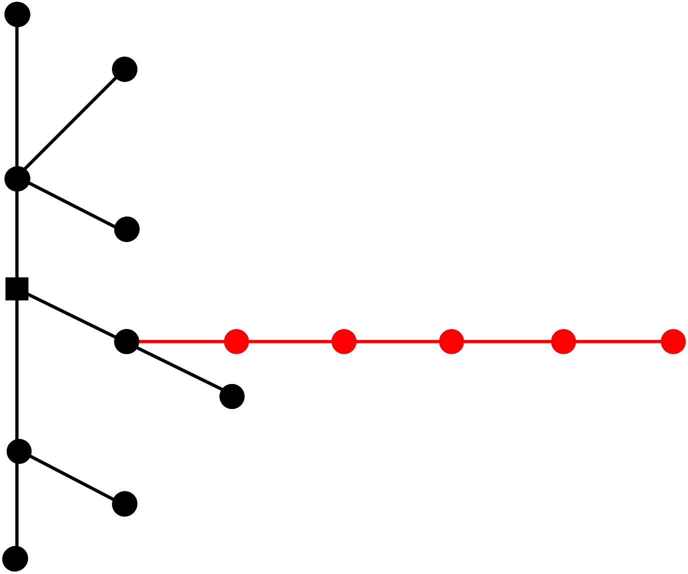
Minimal 5-roots of a tree

$\mathbb{L}_5(TP) \cong T$. Every 5-link has a unique black end; Every black node is the end of a unique 5-link. This gives a bijection between the 5-links of the whole tree TP and the nodes of the black subtree T.



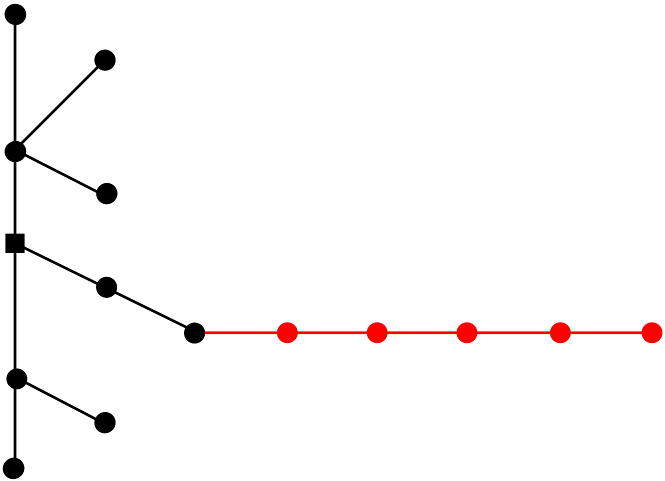
TP1

$\mathbb{L}_5(TP1) \cong T.$



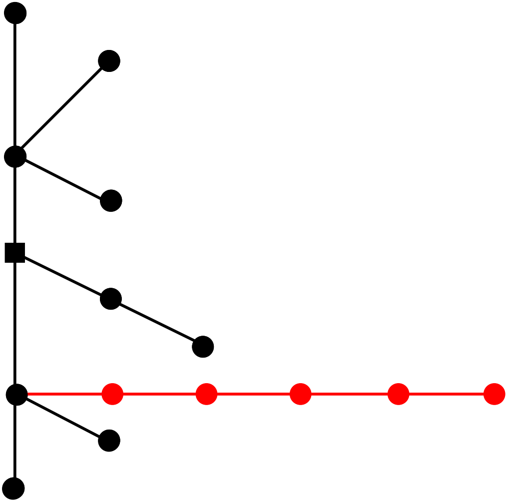
TP2

$\mathbb{L}_5(TP2) \cong T.$



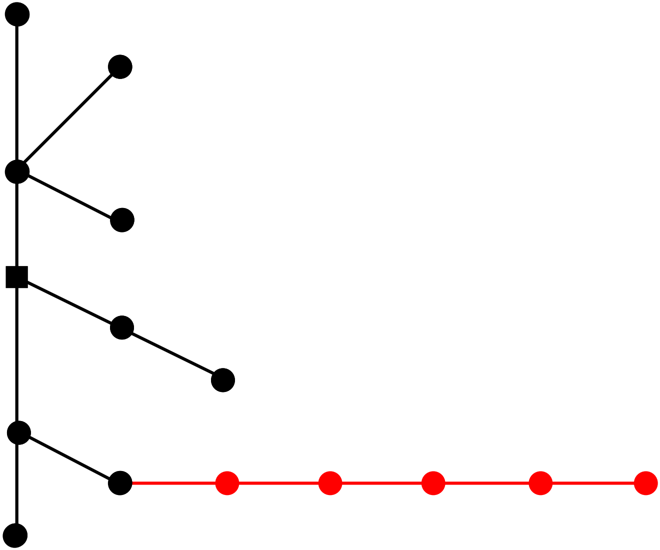
TP3

$\mathbb{L}_5(TP3) \cong T$.



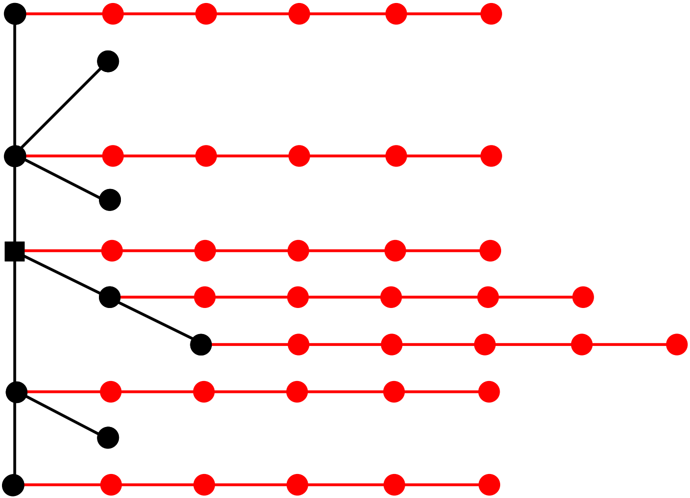
TP4

$\mathbb{L}_5(T\,P4) \cong T.$



Minimal 5-roots of a tree

$\mathbb{L}_5(T\,P) \cong T.$



Minimal ℓ-roots of a tree

For fixed $\ell \geqslant 4$ and any given number k, there exists a tree T with $|\mathbb{R}_\ell(T)| > k$.

Bounding minimal roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. Then the maximum degree, order, size, and total number of minimal ℓ-roots of H are finite and bounded by functions of H and ℓ.

Bounding minimal roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. Then the maximum degree, order, size, and total number of minimal ℓ-roots of H are finite and bounded by functions of H and ℓ.

- If the order and size are bounded, then other parameters above are trivially bounded.

Bounding minimal roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. Then the maximum degree, order, size, and total number of minimal ℓ-roots of H are finite and bounded by functions of H and ℓ.

- If the order and size are bounded, then other parameters above are trivially bounded.
- However, we improve the upper bounds by further investigating the structure of ℓ-link graphs.

Bounding minimal roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. Then the maximum degree, order, size, and total number of minimal ℓ-roots of H are finite and bounded by functions of H and ℓ.

- If the order and size are bounded, then other parameters above are trivially bounded.
- However, we improve the upper bounds by further investigating the structure of ℓ-link graphs.
- This is important in proving that the recognition problem belongs to $\mathcal{NP}$.

Minimal path roots

We say G is an ℓ-path root of H if $\mathbb{P}_\ell(G) \cong H$. Let $\mathbb{Q}_\ell(H)$ be the set of minimal ℓ-path roots of H.

Minimal path roots

We say G is an ℓ-path root of H if $\mathbb{P}_\ell(G) \cong H$. Let $\mathbb{Q}_\ell(H)$ be the set of minimal ℓ-path roots of H.

- Xueliang Li (1996) proved that H has at most one simple 2-path root of minimum degree at least 3.

Minimal path roots

We say G is an ℓ-path root of H if $\mathbb{P}_\ell(G) \cong H$. Let $\mathbb{Q}_\ell(H)$ be the set of minimal ℓ-path roots of H.

- Xueliang Li (1996) proved that H has at most one simple 2-path root of minimum degree at least 3.
- Prisner (2000) showed that $\mathbb{Q}_\ell(H)$ contains at most one simple graph of minimum degree greater than ℓ.

Minimal path roots

We say G is an ℓ-path root of H if $\mathbb{P}_\ell(G) \cong H$. Let $\mathbb{Q}_\ell(H)$ be the set of minimal ℓ-path roots of H.

- Xueliang Li (1996) proved that H has at most one simple 2-path root of minimum degree at least 3.
- Prisner (2000) showed that $\mathbb{Q}_\ell(H)$ contains at most one simple graph of minimum degree greater than ℓ.
- Li and Yan Liu (2008) proved that, if H is connected and nonnull, then $\mathbb{Q}_2(H)$ contains at most two simple graphs.

Minimal path roots

We say G is an ℓ-path root of H if $\mathbb{P}_\ell(G) \cong H$. Let $\mathbb{Q}_\ell(H)$ be the set of minimal ℓ-path roots of H.

- Xueliang Li (1996) proved that H has at most one simple 2-path root of minimum degree at least 3.
- Prisner (2000) showed that $\mathbb{Q}_\ell(H)$ contains at most one simple graph of minimum degree greater than ℓ.
- Li and Yan Liu (2008) proved that, if H is connected and nonnull, then $\mathbb{Q}_2(H)$ contains at most two simple graphs.
- The finite graphs having exactly two simple minimal 2-path roots have been characterised by Aldred, Ellingham, Hemminger and Jipsen (1997).

Bounding minimal path roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. We prove the following:

Bounding minimal path roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. We prove the following:

- The order, size, and total number of minimal ℓ-path roots of H are finite and bounded by functions of H and ℓ.

Infiniteness of ℓ-roots

The maximum degree, order, size, and total number of ℓ-roots of a finite graph might be infinite.

Infiniteness of ℓ-roots

The maximum degree, order, size, and total number of ℓ-roots of a finite graph might be infinite.

- The ℓ-roots of K_0 are trees of diameter at most $\ell - 1$.

Infiniteness of ℓ-roots

The maximum degree, order, size, and total number of ℓ-roots of a finite graph might be infinite.

- The ℓ-roots of K_0 are trees of diameter at most $\ell - 1$.

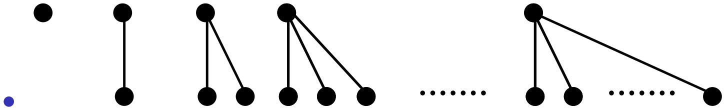


Figure: All stars are 3-roots of K_0

4-roots of K_1

The maximum degree, order, size, and total number of ℓ-roots of a finite graph might be infinite.

4-roots of K_1

The maximum degree, order, size, and total number of ℓ-roots of a finite graph might be infinite.

- An ℓ-root of K_1 is a forest containing exactly one ℓ-path as a subgraph.

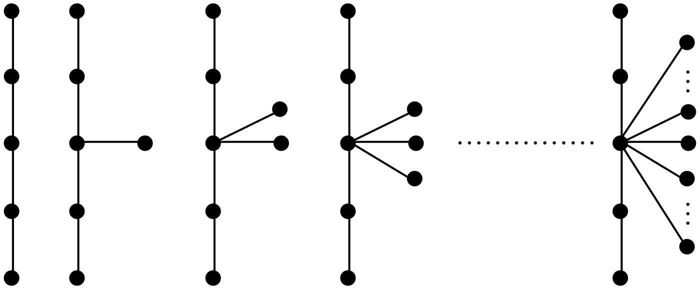


Figure: Connected 4-roots of K_1

Constructing all ℓ-roots

Let G be the minimal graph of an ℓ-equivalence class. Then a graph belongs to this class if and only if it can be obtained from G as follows:

Constructing all ℓ-roots

Let G be the minimal graph of an ℓ-equivalence class. Then a graph belongs to this class if and only if it can be obtained from G as follows:

- For each acyclic component T of G of diameter within $[\ell, 2\ell - 4]$, and every vertex u of eccentricity s in T such that $\lceil \ell/2 \rceil \leqslant s \leqslant \ell - 2$,

Constructing all ℓ-roots

Let G be the minimal graph of an ℓ-equivalence class. Then a graph belongs to this class if and only if it can be obtained from G as follows:

- For each acyclic component T of G of diameter within $[\ell, 2\ell - 4]$, and every vertex u of eccentricity s in T such that $\lceil \ell/2 \rceil \leqslant s \leqslant \ell - 2$,
- paste to u the root of a rooted tree of height at most $\ell - s - 1$.

Constructing all ℓ-roots

Let G be the minimal graph of an ℓ-equivalence class. Then a graph belongs to this class if and only if it can be obtained from G as follows:

- For each acyclic component T of G of diameter within $[\ell, 2\ell - 4]$, and every vertex u of eccentricity s in T such that $\lceil \ell/2 \rceil \leqslant s \leqslant \ell - 2$,
- paste to u the root of a rooted tree of height at most $\ell - s - 1$.
- Add to G zero or more acyclic components of diameter at most $\ell - 1$.

Constructing 4-roots of K_1

Every ℓ-root of a finite graph H can be constructed by a certain combination of a minimal ℓ-root of H and trees of bounded diameter.

Constructing 4-roots of K_1

Every ℓ-root of a finite graph H can be constructed by a certain combination of a minimal ℓ-root of H and trees of bounded diameter.

- An ℓ-root of K_1 is a forest containing exactly one ℓ-path as a subgraph.

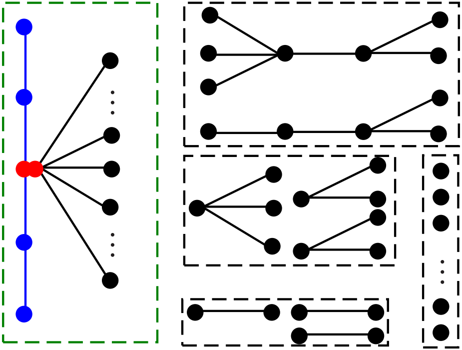
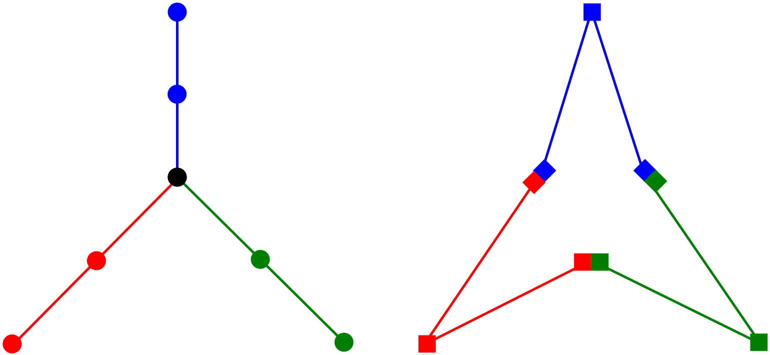
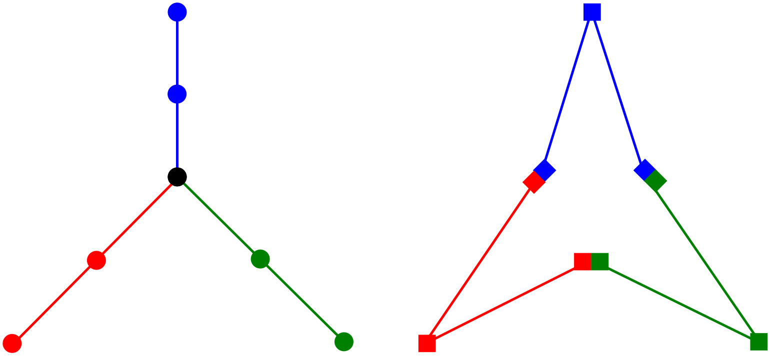


Figure: Constructing 4-roots of K_1 from a 4-path

ℓ-roots of a 3ℓ-cycles

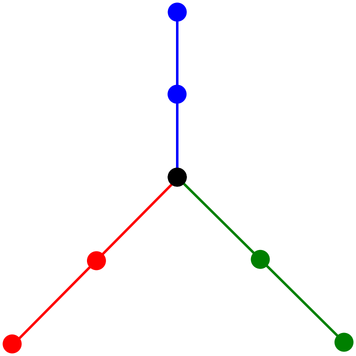


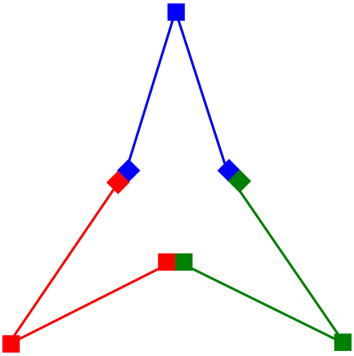
ℓ-roots of a 3ℓ-cycles



- A 6-cycle has two minimal 2-roots.

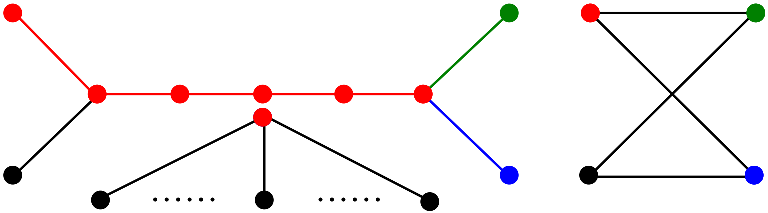
ℓ-roots of a 3ℓ-cycles



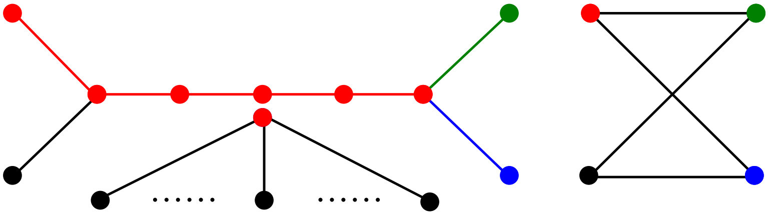


- A 6-cycle has two minimal 2-roots.
- all 2-roots can be obtained by adding to one of them disjoint vertices or edges.

ℓ-roots of a $4s$-cycles

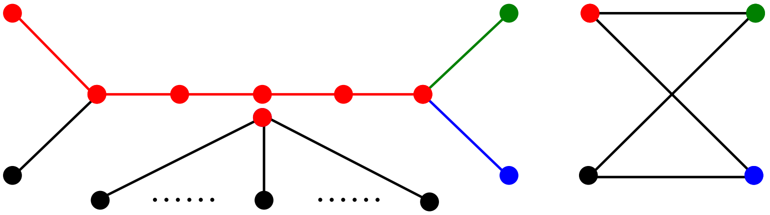


ℓ-roots of a $4s$-cycles



- A 4-cycle has two minimal 5-roots.

ℓ-roots of a $4s$-cycles



- A 4-cycle has two minimal 5-roots.
- There are infinitely many trees of which the 5-link graph is a 4-cycle.

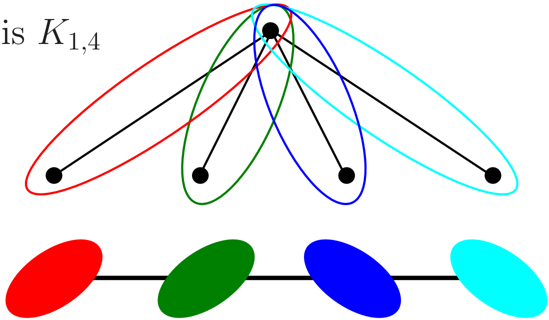
Tree-decompositions

Let G be a graph, T be a tree, and $\mathcal{V} := \{V_w | w \in V(T)\}$ be a set cover of $V(G)$ indexed by the nodes of T. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if . . .

G is $K_{1,4}$

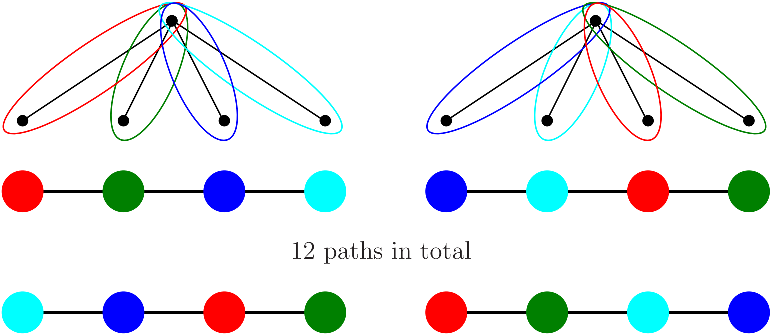
$\mathcal{V}$

T is a 4-path



Different indexes

Even if G, $\mathcal{V}$ and T are given, $\mathcal{V}$ can be indexed by $V(T)$ in different ways.



Different T

Even if G, $\mathcal{V}$ are given, there may be different T.

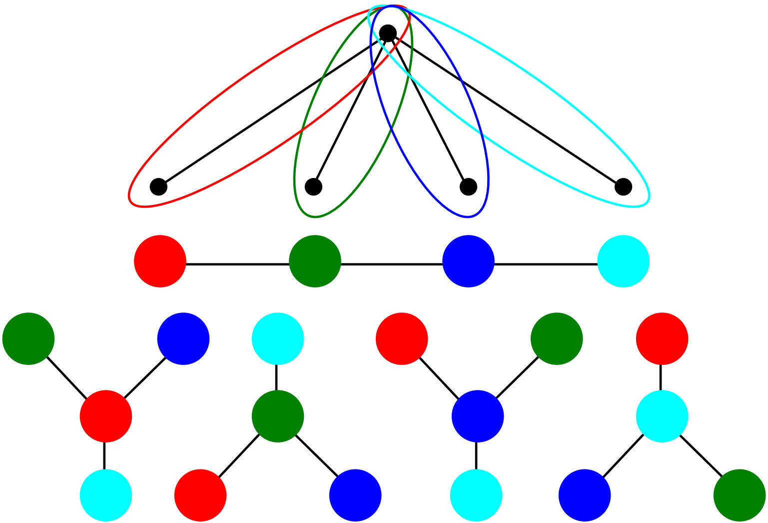


Figure: star- VS path-decomposition: diam(T)

Tree-diameter of ℓ-roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. Then

Tree-diameter of ℓ-roots

Let $\ell \geqslant 0$ be an integer, and H be a finite graph. Then

- the tree-width and tree-diameter of the ℓ-roots and ℓ-path roots of H are finite and bounded by functions of H and ℓ.

Quasi-ordering

A quasi-ordering is a binary relation $\leqslant$ that is:

Quasi-ordering

A quasi-ordering is a binary relation $\leqslant$ that is:

- reflexive: $x \leqslant x$; and

Quasi-ordering

A quasi-ordering is a binary relation $\leqslant$ that is:

- reflexive: $x \leqslant x$; and
- transitive: if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

Well-quasi-ordering

Let $\leqslant$ be a quasi-ordering on X. X is said to be well-quasi-ordered if

Well-quasi-ordering

Let $\leqslant$ be a quasi-ordering on X. X is said to be well-quasi-ordered if

- for any infinite sequence $x_1, x_2, \dots,$ of X

Well-quasi-ordering

Let $\leqslant$ be a quasi-ordering on X. X is said to be well-quasi-ordered if

- for any infinite sequence $x_1, x_2, \ldots,$ of X
- there are indices $i < j$ such that $x_i \leqslant x_j$.

Well-quasi-ordering

Let $\leqslant$ be a quasi-ordering on X. X is said to be well-quasi-ordered if

- for any infinite sequence $x_1, x_2, \ldots,$ of X
- there are indices $i < j$ such that $x_i \leqslant x_j$.
- The indices for well-quasi-ordering are $1 < 2 < 3 < \ldots$

Better-quasi-ordering: indices

1	2	4	5
2	4	5	7
4	5	7	8	9

| 1 |
| 2 |
| 4 |

Better-quasi-ordering: indices

$$
\begin{array}{llll}
|\,1\,|\,2\,|\,4\,|\,5\,| & & & |\,1\,| \\
\quad |\,2\,|\,4\,|\,5\,|\,7\,| & & & \quad |\,2\,| \\
\qquad |\,4\,|\,5\,|\,7\,|\,8\,|\,9\,| & & & \qquad |\,4\,|
\end{array}
$$

- It includes the indices for well-quasi-ordering.

Better-quasi-ordering: indices

$$
\begin{array}{l}
|\,1\,|\,2\,|\,4\,|\,5\,| \\
\quad|\,2\,|\,4\,|\,5\,|\,7\,| \\
\qquad|\,4\,|\,5\,|\,7\,|\,8\,|\,9\,|
\end{array}
\qquad\qquad
\begin{array}{l}
|\,1\,| \\
\quad|\,2\,| \\
\qquad|\,4\,|
\end{array}
$$

- It includes the indices for well-quasi-ordering.
- $1 \triangleleft 2 \triangleleft 4 \triangleleft \dots$.

Better-quasi-ordering: indices

$$
\begin{array}{l}
|1|2|4|5| \\
\quad |2|4|5|7| \\
\quad\quad |4|5|7|8|9|
\end{array}
\qquad\qquad
\begin{array}{l}
|1| \\
\quad |2| \\
\quad\quad |4|
\end{array}
$$

- It includes the indices for well-quasi-ordering.
- $1 \lhd 2 \lhd 4 \lhd \ldots$.
- $(1, 2, 4, 5) \lhd (2, 4, 5, 7) \lhd (4, 5, 7, 8, 9)$.

Better-quasi-ordering: blocks

A *block* $\mathcal{B}$ is a set of finite increasing sequences $B_1, B_2, \ldots$ such that:

Better-quasi-ordering: blocks

A *block* $\mathcal{B}$ is a set of finite increasing sequences $B_1, B_2, \ldots$ such that:

- every infinite increasing sequence with elements in $B_1 \cup B_2 \cup \ldots$

Better-quasi-ordering: blocks

A *block* $\mathcal{B}$ is a set of finite increasing sequences $B_1, B_2, \ldots$ such that:

- every infinite increasing sequence with elements in $B_1 \cup B_2 \cup \ldots$
- has an initial finite subsequence that belongs to $\mathcal{B}$.

Better-quasi-ordering: blocks

A *block* $\mathcal{B}$ is a set of finite increasing sequences $B_1, B_2, \ldots$ such that:

- every infinite increasing sequence with elements in $B_1 \cup B_2 \cup \ldots$
- has an initial finite subsequence that belongs to $\mathcal{B}$.
- $\{1, 2, \ldots\}$ is a block.

Better-quasi-ordering: blocks

A *block* $\mathcal{B}$ is a set of finite increasing sequences $B_1, B_2, \ldots$ such that:

- every infinite increasing sequence with elements in $B_1 \cup B_2 \cup \ldots$
- has an initial finite subsequence that belongs to $\mathcal{B}$.
- $\{1, 2, \ldots\}$ is a block.
- For example, for each $n \geqslant 1$, the set of increasing sequence of $\mathbb{N}$ with n elements is a block.

Better-quasi-ordering

Let $\mathcal{Q}$ be a set quasi-ordered by $\leqslant$.

Better-quasi-ordering

Let $\mathcal{Q}$ be a set quasi-ordered by $\leqslant$.

- A $\mathcal{Q}$-pattern is a sequence of elements from $\mathcal{Q}$ that is indexed by a block $\mathcal{B}$.

Better-quasi-ordering

Let $\mathcal{Q}$ be a set quasi-ordered by $\leqslant$.

- A $\mathcal{Q}$-pattern is a sequence of elements from $\mathcal{Q}$ that is indexed by a block $\mathcal{B}$.
- A $\mathcal{Q}$-pattern is good if there are $B \triangleleft C$ in $\mathcal{B}$, such that

Better-quasi-ordering

Let $\mathcal{Q}$ be a set quasi-ordered by $\leqslant$.

- A $\mathcal{Q}$-pattern is a sequence of elements from $\mathcal{Q}$ that is indexed by a block $\mathcal{B}$.
- A $\mathcal{Q}$-pattern is good if there are $B \lhd C$ in $\mathcal{B}$, such that
- $q_B \leqslant q_C$. Otherwise, it is bad.

Better-quasi-ordering

Let $\mathcal{Q}$ be a set quasi-ordered by $\leqslant$.

- A $\mathcal{Q}$-pattern is a sequence of elements from $\mathcal{Q}$ that is indexed by a block $\mathcal{B}$.
- A $\mathcal{Q}$-pattern is good if there are $B \triangleleft C$ in $\mathcal{B}$, such that
- $q_B \leqslant q_C$. Otherwise, it is bad.
- $\mathcal{Q}$ is better-quasi-ordered if there is NO bad $\mathcal{Q}$-pattern.

Better-quasi-ordering: ℓ-roots

The ℓ-roots of a finite graph are better-quasi-ordered by the induced subgraph relation.

Better-quasi-ordering: ℓ-roots

The ℓ-roots of a finite graph are better-quasi-ordered by the induced subgraph relation.

- The ℓ-path roots of a finite graph are better-quasi-ordered by the subgraph relation.

Better-quasi-ordering: ℓ-roots

The ℓ-roots of a finite graph are better-quasi-ordered by the induced subgraph relation.

- The ℓ-path roots of a finite graph are better-quasi-ordered by the subgraph relation.
- The ℓ-path roots of bounded multiplicity of a finite graph are better-quasi-ordered by the induced subgraph relation.

Better-quasi-ordering: ℓ-roots

The ℓ-roots of a finite graph are better-quasi-ordered by the induced subgraph relation.

- The ℓ-path roots of a finite graph are better-quasi-ordered by the subgraph relation.
- The ℓ-path roots of bounded multiplicity of a finite graph are better-quasi-ordered by the induced subgraph relation.
- Trees of bounded diameter are better-quasi-ordered by the induced subgraph relation.

ℓ-path-free graphs

Let H be a finite planar graph. Robertson and Seymour (1986, 1990) proved that finite H-minor-free graphs are well-quasi-ordered by the minor relation.

ℓ-path-free graphs

Let H be a finite planar graph. Robertson and Seymour (1986, 1990) proved that finite H-minor-free graphs are well-quasi-ordered by the minor relation.

- Thomas (1989) generalised this result by showing that H-minor-free graphs are better-quasi-ordered by the minor relation.

ℓ-path-free graphs

Let H be a finite planar graph. Robertson and Seymour (1986, 1990) proved that finite H-minor-free graphs are well-quasi-ordered by the minor relation.

- Thomas (1989) generalised this result by showing that H-minor-free graphs are better-quasi-ordered by the minor relation.
- Ding (1992) proved that finite simple ℓ-path-free graphs are well-quasi-ordered by the induced subgraph relation.

ℓ-path-free graphs

Let H be a finite planar graph. Robertson and Seymour (1986, 1990) proved that finite H-minor-free graphs are well-quasi-ordered by the minor relation.

- Thomas (1989) generalised this result by showing that H-minor-free graphs are better-quasi-ordered by the minor relation.
- Ding (1992) proved that finite simple ℓ-path-free graphs are well-quasi-ordered by the induced subgraph relation.
- Another proof, based on *tree-depth* was given by Nešetřil and Ossona de Mendez (2012).

ℓ-path-free graphs

Let H be a finite planar graph. Robertson and Seymour (1986, 1990) proved that finite H-minor-free graphs are well-quasi-ordered by the minor relation.

- Thomas (1989) generalised this result by showing that H-minor-free graphs are better-quasi-ordered by the minor relation.
- Ding (1992) proved that finite simple ℓ-path-free graphs are well-quasi-ordered by the induced subgraph relation.
- Another proof, based on *tree-depth* was given by Nešetřil and Ossona de Mendez (2012).
- The H-minor-free graphs are better-quasi-ordered by the subgraph relation if and only if H is a disjoint union of paths.

ℓ-path-free graphs

Let H be a finite planar graph. Robertson and Seymour (1986, 1990) proved that finite H-minor-free graphs are well-quasi-ordered by the minor relation.

- Thomas (1989) generalised this result by showing that H-minor-free graphs are better-quasi-ordered by the minor relation.
- Ding (1992) proved that finite simple ℓ-path-free graphs are well-quasi-ordered by the induced subgraph relation.
- Another proof, based on *tree-depth* was given by Nešetřil and Ossona de Mendez (2012).
- The H-minor-free graphs are better-quasi-ordered by the subgraph relation if and only if H is a disjoint union of paths.
- The H-minor-free graphs of bounded multiplicity are better-quasi-ordered by the induced subgraph relation if and only if H is a disjoint union of paths.

Thanks

Thank You!

Thanks

Thank You for listening!