Combinatorial lines with few intervals

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joint work with David Conlon and Christoph Spiegel

The Hales-Jewett Theorem

NOTATION

- Ground set $[m]^n$, usually $[3]^n$
- A *root* is a word w of length n in the symbols $[m] \cup \{*\}$ with at least one $*$.
- For $i \in [m]$, w_i is the word obtained from w by replacing each $*$ by i.
- A (*combinatorial*) *line* is a set of words $\{w_i : i \in [m]\}$.

EXAMPLE: a combinatorial line in $[3]^{12}$

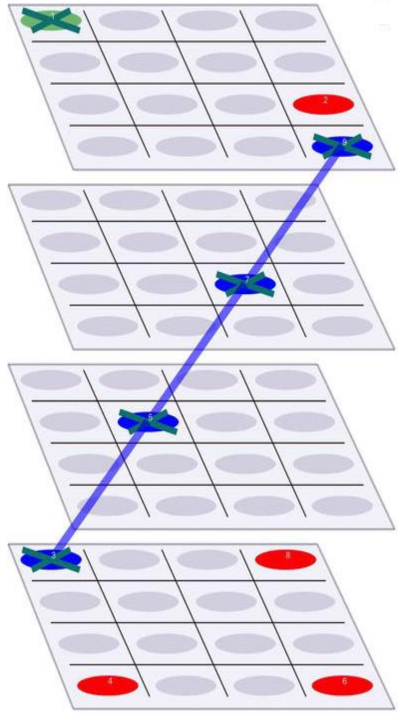
$$u \;=\; 22 * 32 * 12 *** 22$$

$$u_1 = 2\,2\,1\,3\,2\,1\,1\,2\,1\,1\,1\,2\,2$$
$$u_2 = 2\,2\,2\,3\,2\,2\,1\,2\,2\,2\,2\,2\,2$$
$$u_3 = 2\,2\,3\,3\,2\,3\,1\,2\,3\,3\,3\,2\,2$$

Wildcard set

Theorem (Hales, Jewett, ‘63). Given $m, r \in \mathbb{N}$, there is a natural number n such that any r-colouring of $[m]^n$ contains a monochromatic combinatorial line.

- $HJ(m, r)$ is the minimal n for which the conclusion holds



A line in $[4]^3$ with wildcard set $\{1,2,3\}$

A warm-up…

Claim. $HJ(2,r) = r$.

Proof. Let $c\colon [2]^r \to \mathbb{Z}_r$.

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Proof. Let $c\colon [2]^r \to \mathbb{Z}_r$. Consider the words

$$\begin{array}{l} 1111111\ldots1 \\ 11\ldots111\ldots2 \\ 11\ldots122\ldots2 \\ \quad\vdots \\ 1222222\ldots2 \\ 2222222\ldots2. \end{array}$$

Among $r+1$ words, two have the same colour

$\Rightarrow$ monochromatic line.

Shelah's proof of HJT

Consequences

- $HJ(m,r) \leq \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} 2HJ(m-1,r)$ times

- For $m = 3$, there exists an r-*interval* line (a line whose wildcard set consists of at most r intervals), e.g.
  $\underbrace{u = \ 33 * 2 * 21 * * * 33}_{\leq r \text{ intervals}}$

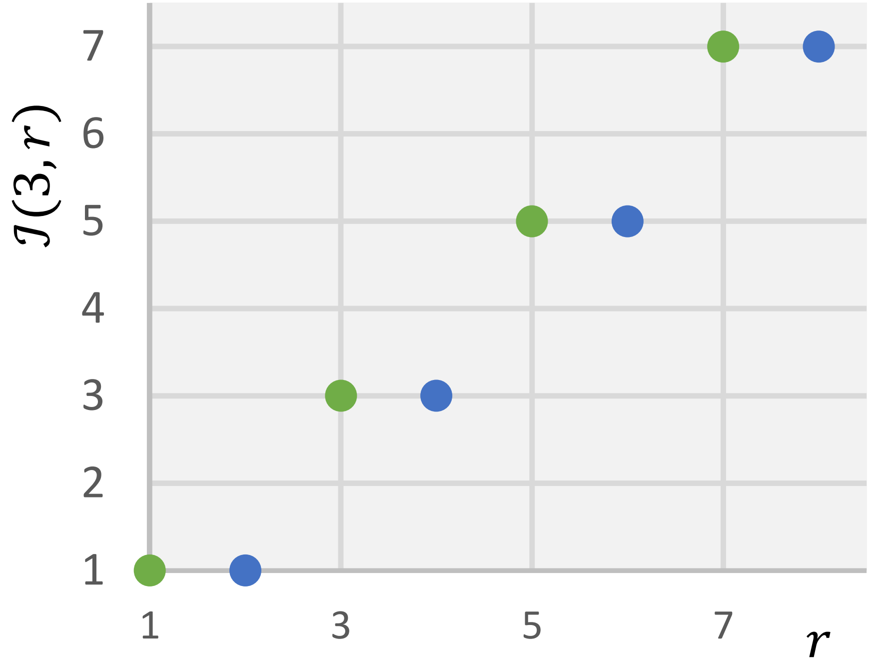
Is this the best possible?

Lines with few intervals

Definition. $\mathcal{J}(m,r) = \min\{q$: for large n, any r-colouring of $[m]^n$ contains a monochromatic q-interval line$\}$.

Observations.

- $\mathcal{J}(2,r) = 1$ for all r.
- $\mathcal{J}(3,r) \leq r,$ generally $\mathcal{J}(m,r) \leq HJ(m-1,r)$



Theorem.

(i) $\mathcal{J}(3,r) = r$ for odd r (Conlon, K)

(ii) $\mathcal{J}(3,2) = 1$ (Leader, Räty)

(iii) $\mathcal{J}(3\ r) = r - 1$ for even r (K, Spiegel).

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$$[3]^n \xrightarrow{contract} [3]^{\leq n} \xrightarrow{count} \mathbb{Z}_r^3$$

$$u_1 = 221321121 1122 \longmapsto \overline{u_1} = 2132\ \ 1212 \longmapsto \varphi(\overline{u_1}) = (3,4,1)$$

$$u_2 = 2223221222222 \longmapsto \overline{u_2} = 2\ \ 32\ \ 12 \longmapsto \varphi(\overline{u_2}) = (1,3,1)$$

$$u_3 = 2233231233322 \longmapsto \overline{u_3} = 2\ \ 3231232 \longmapsto \varphi(\overline{u_3}) = (1,4,3)$$

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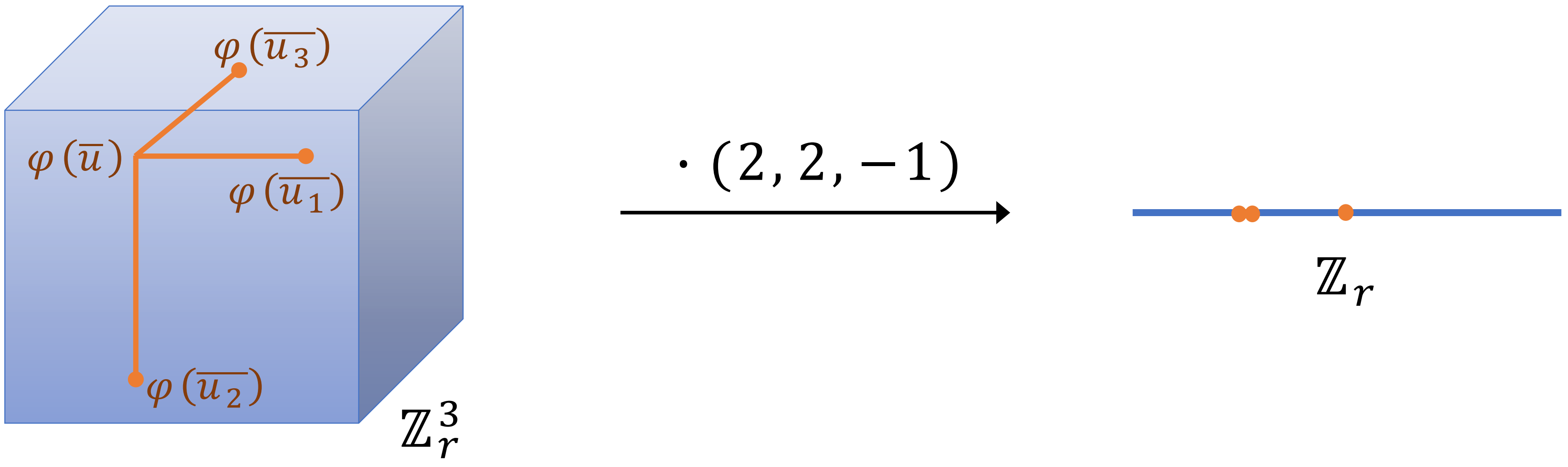
$$u = 22 * 32 * 12 *** 22 \mapsto \overline{u} = 2 * 32 * 12 * 2 \mapsto \varphi(\overline{u}) = (1,4,1)$$



(iii) The upper bound

Theorem (K, Spiegel). $\mathcal{J}(3, r) = r - 1$ for even r.

- Idea: reduce the problem to the 'linear' colourings

Proposition. For *any* r, if $\mathcal{J}(3\ r) > r - 1$, then there is a colouring T: $[3]^{\,n} \rightarrow \mathbb{Z}^r$ avoiding $(r-1)$-interval lines with

$$T(u) = T'\big(\varphi(\bar{u})\big) \text{ for all } u.$$

That is, T has the form

$$[3]^n \xrightarrow{\ \ \bar{}\ \ } [3]^{\leq n} \xrightarrow{\ \varphi\ } \mathbb{Z}_r^3 \xrightarrow{\ T'\ } \mathbb{Z}_r^3$$

- Odd r – there is a colouring
- Even r – contradiction.

Onwards and upwards

Improve the bounds $r \leq \mathcal{J}(m, r) \leq HJ(m-1, r)$.

New proofs of the Hales-Jewett theorem?

Thank you!