On (Hoffman) graphs with smallest eigenvalue at least -3

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POSTECH

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Outline

1. Graphs and Eigenvalues
   - Definitions
   - Cameron-Goethals-Seidel-Shult
   - Hoffman and others
2. Hoffman Graphs
   - Definitions
3. (Hoffman) Graphs with given smallest eigenvalue
   - Smallest eigenvalue -2
4. Limit points
   - Limit points

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Defintion

Graph: $G = (V, E)$ where V vertex set, $E \subseteq \binom{V}{2}$ edge set.

- All graphs in this talk are simple.
- $x \sim y$ if $xy \in E$.
- $x \not\sim y$ if $xy \notin E$.
- $d(x, y)$: length of a shortest path connecting x and y.
- $D(G)$: diameter (maximum distance in G)

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- $\lambda_{\min}(G)$ denotes the smallest eigenvalue of G.

Line graphs

Let G be a graph.

- The **line graph** of G, denoted by $L(G)$ is the graph with vertex set $E(G)$ and $xy \sim uv$ if $\#(xy \cap uv) = 1$.
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- Not all graphs with smallest eigenvalue at least -2 are line graphs: For example the Petersen graph is a graph with smallest eigenvalue -2, but clearly it is not a line graph. **Why?**
- A graph G is a line graph if and only if there are edge-disjoint complete subgraphs $C_1, \dots, C_t$ (for some integer t) such that for each edge xy of G there is a unique i such $xy \in C_i$ and each vertex is in at most two C_i's.

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- Let for x a vertex c_x be the column of N associated with x.

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- Let for x a vertex c_x be the column of N associated with x.
- Now consider the inner product (c_x, c_y). This is 2 if $x = y$, 1 if $x \sim y$ and 0 otherwise.
- This means that the lattice generated by $\{c_x\}$ is a root lattice and this lattice is irreducible if G is connected.
- The irreducible root lattices are classified by Witt (1930's) and they are A_n, D_n $(n = 1, 2, 3, \ldots)$ and E_6, E_7, E_8. The lattices A_n, D_n can be embedded in $\mathbf{Z}^{n+1}$.

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- A graph is called a generalized line graph if there is a N with only integral coefficients. (I will give an other equivalent definition later)

Cameron-Goethals-Seidel-Shult

This gives:

Theorem(CGSS(1976))

Let G be a connected graph. If its smallest eigenvalue is at least -2, then G is a generalized line graph or the number of vertices of G is bounded by 36.

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Note: A generalized line graph is a combination of a line graph and some Cocktail Party graphs.

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- Let $-2 \geq \lambda > -1 - \sqrt{2}$. Then there exists a k_λ such that any connected graph with smallest eigenvalue at least λ and minimal valency at least k_λ is a generalized line graph.

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Hoffman 2

The second part of the last theorem can be reformulated as

Theorem (Hoffman (1977))

Let θ_k be the supremum of the smallest eigenvalue of graphs with smallest valency k and smallest eigenvalue < -2. Then $(\theta_k)_k$ forms a monotone decreasing sequence with limit $-1-\sqrt{2}$.

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Bussemaker et al. (198?) showed θ_1 is about -2.008.
Woo and Neumaier went below $-1-\sqrt{2}$.

Theorem (Woo and Neumaier (1995))

Let η_k be the supremum of the smallest eigenvalue of graphs with smallest valency k and smallest eigenvalue $< -1-\sqrt{2}$. Then $(\eta_k)_k$ forms a monotone decreasing sequence with limit $-2.48\ldots$

Regular graphs

For regular graphs, Yu showed

Theorem (Yu (2012))

Let $\hat{\theta}_k$ be the supremum of the smallest eigenvalue of k-regular graphs and smallest eigenvalue < -2. Then $(\hat{\theta}_k)_k$ forms a sequence with limit $-1 - \sqrt{2}$.

She also showed $\hat{\theta}_3 = 2.03...$

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She also showed $\hat{\theta}_3 = 2.03...$
In order to show the results of Woo-Neumaier, Yu, the best way is to use Hoffman graphs as introduced by Woo-Neumaier.

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Hoffman Graphs 1

Hoffman Graph

- A **Hoffman Graph** $\mathfrak{G} = (G = (V, E), \ell : V \to \{f, s\})$, such that any two vertices with label f are non-adjacent. In other words, it is a graph with a distinguished independent set $F = \{v \in V \mid \ell(v) = f\}$ of vertices.
- The vertices in the independent set F, we will call **fat** and the rest of the vertices we will call **slim**.

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- The vertices in the independent set F, we will call **fat** and the rest of the vertices we will call **slim**.
- The way to think about the fat vertices is that they behave like complete subgraphs. (We will show later a reason for this).

examples

Hoffman Graphs 2

Hoffman Graph 2

- A Hoffman graph $\mathfrak{H}$ is called **fat** if every slim vertex has at least one fat neighbour.
- The subgraph induced on $S := \{v \in V \mid \ell(v) = s\}$ is called the slim subgraph of $\mathfrak{H}$.
- The way to think about Hoffman graphs is that they are just (slim) graphs with some fat vertices attached.

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- The way to think about Hoffman graphs is that they are just (slim) graphs with some fat vertices attached.
- Hoffman graphs and especially fat Hoffman graphs give a good way to construct graphs with unbounded number of vertices such that the smallest eigenvalue is at least a fixed number.
- On the other hand graphs with a large minimum valency and fixed smallest eigenvalue are very close to fat Hoffman graphs. (I will try to make this more precise later)

Eigenvalues

Eigenvalues of Hoffman graphs

- Let $\mathfrak{H}$ be a Hoffman graph with fat vertex set F and slim vertex set S.
- The adjacency matrix A of $\mathfrak{H}$ can be written in the following form:

$$A := \left(\begin{array}{c|c} B & C \\ \hline C^T & 0 \end{array} \right),$$

where the block B corresponds to the adjacency matrix on the set S, and so on.

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- As CC^T is a positive semidefinite matrix $\lambda_{\min}(B) \geq \lambda_{\min}(\mathfrak{H})$.
- Note that $B(\mathfrak{H}) - \lambda_{\min}(\mathfrak{H})$ is a positive semidefinite matrix, and hence the Gram matrix of a set vectors $\{\phi_x \mid x \in F \cup S\}$, which is called the representation of $\mathfrak{H}$.

Replacing fat vertices by cliques

One reason for the definition of the smallest eigenvalue of a Hoffman graph is the following theorem of Hoffman and Ostrowski (1960's):

Theorem

Let $\mathfrak{H}$ be a Hoffman graph. Define the graph G_n as follows: Replace the fat vertices with complete graphs $C_f (f \in F)$ with n vertices and each vertex of C_f has the same neighbours in S as f. Then $\lim_{n \to \infty} \lambda_{\min}(G_n) = \lambda_{\min}(\mathfrak{H})$.

Line graph

How to construct a fat Hoffman graph with smallest eigenvalue -2 for a line graph? For each maximal clique add a fat vertex adjacent to each vertex of this clique.

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Direct sums

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Let $\mathfrak{H}' = (F' \cup S', E')$ and $\mathfrak{H}'' = (F'' \cup S'', E'')$ be two Hoffman graphs, such that

- $S' \cap S'' = \emptyset$;
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- $S' \cap S'' = \emptyset$;
- $s' \in S'$ and $s'' \in S''$ have at most one common fat neighbour in $F' \cap F''$.
- The Hoffman graph $\mathfrak{H}' \oplus \mathfrak{H}''$ has as vertex set $S \cup F$ where $S = S' \cup S''$ and $F = F' \cup F''$.
- The induced subgraphs on $S' \cup F'$ resp. $S'' \cup F''$ are $\mathcal{H}'$ resp. $\mathcal{H}''$.
- $s' \in S'$ and $s'' \in S''$ are adjacent if and only if they have exactly one common fat neighbour.

blackboard

Direct sums 2

- $\mathfrak{H}' \oplus \mathfrak{H}''$ is called the **direct sum** of $\mathfrak{H}'$ and $\mathfrak{H}''$.

Direct sums 2

- $\mathfrak{H}' \oplus \mathfrak{H}''$ is called the **direct sum** of $\mathfrak{H}'$ and $\mathfrak{H}''$.
- If $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_t$ are Hoffman graphs then we can define $\mathfrak{H}_1 \uplus \ldots \uplus \mathfrak{H}_t$ recursively by $((\ldots(\mathfrak{H}_1 \uplus \mathfrak{H}_2) \uplus \mathfrak{H}_3) \ldots \mathfrak{H}_t)$.

Theorem (Woo & Neumaier)

- Let $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ where $\mathfrak{H}'$ and $\mathfrak{H}''$ are Hoffman graphs.
- Then $\lambda_{\min}(\mathfrak{H}) = \min(\lambda_{\min}(\mathfrak{H}'), \lambda_{\min}(\mathfrak{H}''))$.

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$\mathcal{F}$-line graph

Let $\mathcal{F}$ be a family of Hoffman graphs. A graph is called **$\mathcal{F}$-line graph** if it is an induced subgraph of the slim subgraph of $\bigoplus_{i=1}^{t} \mathfrak{F}_i$ where $\mathfrak{F}_i \in \mathcal{F}$.

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Examples

- A $\{\mathfrak{H}_1\}$-line graph is exactly the same as a line graph.
- A $\{\mathfrak{H}_1, \mathfrak{H}_2\}$-line graph is exactly the same as a generalized line graph. (You can take this as the definition)

CGSS revisited

We can reformulate the theorem of Cameron et al. as follows:

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Let G be a graph with smallest eigenvalue at least -2. Then either G is a $\{\mathfrak{H}_1, \mathfrak{H}_2\}$-line graph, or the number of vertices is bounded by 36.

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Woo and Neumaier (1995) showed

Theorem

There exists a constant C such that if G is a connected graph with smallest eigenvalue at least $-1-\sqrt{2}$ and minmal valency at least C, then G is a $\mathfrak{F}$-line graph, where $\mathfrak{F}$ is a family of nine fat Hoffman graphs.

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In how far can these theorems be generalized? It is unlikely that such a theorem holds for -3 but maybe it is true for all $\lambda > -3$.

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- Doob observed: Every value in the half-open interval $(-\infty, -\sqrt{2+\sqrt{5}}]$ is the limit point of a sequence of the smallest eigenvalues of connected graphs with increasing number of vertices. This is of course an immediate consequence of Shearers result.

Limit Points 2

- Let L be the closure of $\{\lambda_{\min}(G) \mid G \text{ connected graph }\}$, that is, include its limit points as well. Then by previous slide: $(-\infty, -\sqrt{2+\sqrt{5}}] \subset L$.

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- Let L be the closure of $\{\lambda_{\min}(G) \mid G \text{ connected graph }\}$, that is, include its limit points as well. Then by previous slide: $(-\infty, -\sqrt{2+\sqrt{5}}] \subset L$.
- Let $-1 > \lambda > -2$. Then Hoffman (1970's) showed $L \cap [\lambda, -1]$ contains a finite number of limit points and each of these limit points is the smallest eigenvalue of a connected Hoffman graph with exactly one fat vertex. Moreover the largest limit point is $(-1-\sqrt{5})/2$.

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- It is not clear what happens in the interval $(-\sqrt{2+\sqrt{5}}, -2)$.

Limit Points of Hoffman Graphs

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- If G a graph, then the Hoffman graph $\mathfrak{H}(G)$ by attaching a unique fat vertex to each vertex of G has $\lambda_{\min}(\mathfrak{H}) = -1 + \lambda_{\min}(G)$.
- Using Doobs result and the above item, we find each value in the half-open interval $(-\infty, -1 - \sqrt{2+\sqrt{5}}]$ is the limit point of a sequence of the smallest eigenvalues of connected fat Hoffman graphs with increasing number of slim vertices, that is, $(-\infty, -1 - \sqrt{2+\sqrt{5}}] \subset L_H$.

Limit Points of Hoffman Graphs 2

- With the same method as Hoffman one can show that for $-2 > \lambda > -3$, the set $L_H \cap [\lambda, -2]$ contains a finite number of limit points and each of these limit points is the smallest eigenvalue of a connected fat Hoffman graph with exactly one slim vertex adjacent to exactly two fat vertices. Moreover, the largest limit point is $(-3 - \sqrt{5})/2$.

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- An open question is: $L_H = \{\lambda - 1 \mid \lambda \in L\}$? By the construction in second item of last slide, L_H contains $\{\lambda - 1 \mid \lambda \in L\}$.

Regular graphs and limit points

Theorem (Yu and K.)

- Let G be a connected graph.
- Then there exists a sequence of k_n-regular graphs $(G_n)_n$ with $k_n \to \infty \quad (n \to \infty)$, which are $\mathfrak{H}(G)$-line graphs. In particular $\lambda_{\min}(G_n) \to \lambda_{\min}(\mathfrak{H}(G)) \quad (n \to \infty)$.

A consequence:

Theorem (Yu)

Let $\hat{\theta}_k$ be the supremum of the smallest eigenvalue of k-regular graphs and smallest eigenvalue < -2. Then $(\hat{\theta}_k)_k$ forms a sequence with limit $-1 - \sqrt{2}$.

Eigenvalue -3

With A. Munemasa and T. Taniguchi, we are determining the fat Hoffman graphs with smallest eigenvalue -3. For this classification we again need to classification of the root lattices.

Some questions

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- What can you say about the exceptional (not coming from fat Hoffman graph with smallest eigenvalue -3) graphs with smallest eigenvalue at least -3? (The Hoffman-Singleton graph is an example)

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- What can you say about the exceptional (not coming from fat Hoffman graph with smallest eigenvalue -3) graphs with smallest eigenvalue at least -3? (The Hoffman-Singleton graph is an example)
- Can we classify the exceptional SRG with smallest eigenvalue -3?

Thank you for your attention.