Recent progress on graphs with smallest eigenvalue at least -3

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Outline

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Defintion

Graph: $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$.

- All graphs in this talk are undirected and simple.
- The adjacency matrix A of a graph Γ is the matrix whose rows and columns are indexed by its vertices such that $A_{xy} = 1$ if xy is an edge and 0 otherwise.

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- First I will introduce lattices. They form an important tool for us.

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- Let $U \subset R^n$ be a finite set.
- The lattice Λ generated by U is the set $\{\sum \alpha_u u \mid u \in U, \alpha_u \in Z \text{ for all } u\}$. The lattice Λ is called integral if $\langle u_1, u_2 \rangle$ is an integer for all $u_1, u_2 \in U$.

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- If Λ can be written as the orthogonal sum of two proper sublattices we say Λ is reducible, and otherwise it is called irreducible.
- We say an integral lattice is *s-integrable* if $\sqrt{s}\Lambda$ is isomorphic to a sublattice of the standard lattice.

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- D_n is the root lattice generated by $\{e_i - e_j, e_i + e_j \mid 1 \leq i, j \leq n\}$.

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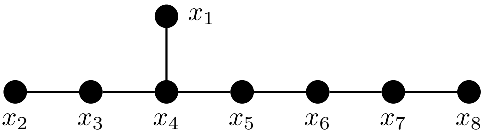
Theorem

(Witt (1941)) The only irreducible root lattices are A_n, D_n and E_6, E_7, E_8.

The root lattices E_6, E_7 and E_8 are 2-integrable, but can not be 1-integrated.



A basis of E_8

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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Results of Conway and Sloane

Definitions

Let s be a positive integer. We define $c(s)$ as the smallest positive integer t such that there exists an integral lattice Λ with $\dim(\Lambda) = t$, that can not be s-integrated. It is not a priori clear that this number exists.

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- $c(s) \to \infty \quad (s \to \infty)$

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- The main idea of Conway and Sloane is that for any integral lattice Λ, there exists a unimodular lattice Λ' that contains Λ as a sublattice and $\dim(\Lambda') - \dim(\Lambda)$ is at most three.
- For an unimodular lattice one needs the same number of positions to show its s-integrability.
- And then they classified the low dimensional unimodular lattices.

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- We say G is s-integrable if Λ is s-integrable.

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Smallest eigenvalue -2

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Note that if I can take N with entries only $0'$s and $1'$s, then G is a line graph. So a generalized line graph is a generalization of a line graph.

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We give now a sketch of proof for this result, as we will need this idea later in the talk.

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- Then Λ is an even lattice, generated by norm two vectors, so it is a root lattice and it is irreducible as G is connected.

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- The irreducible root lattices were classified by Witt, and are of type A_n, D_n or E_6, E_7, E_8.
- The first two lattices give us generalized line graphs, and for the last three lattices one can show that the number of vertices is at most 36.

Outline

Smallest eigenvalue $-1-\sqrt{2}$

Hoffman (1977) showed the following result:

Theorem

Let $2 < \lambda < 1+\sqrt{2}$. Then there is constant $K = K(\lambda)$ such that if Γ is a connected graph with minimal valency at least K and smallest eigenvalue $\lambda_{\min} \geq -\lambda$, then Γ is a generalised line graph. In particular, $\lambda_{\min} \geq -2$.

- This result means that there exists a real number $\tau(k) < -2$ such that any connected graph with minimal valency at least k has smallest eigenvalue either at least -2 or at most $\tau(k)$, and $\tau(k) \to -1-\sqrt{2}$ $(k \to \infty)$.

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- Woo and Neumaier (1995) generalized this result by Hoffman by going slightly below $-1-\sqrt{2}$.

Rephrasing the results of Cameron et al. and Hoffman

As the generalised line graphs are exactly the graphs with smallest eigenvalue at least -2 which are 1-integrable, we can rephrase the results of Cameron et al. and Hoffman as follows:

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Can we generalize these two results to graphs with smallest eigenvalue at least -3?

Can we generalize these two results to graphs with smallest eigenvalue at least -3? In this talk we will show a generalization of the result of Hoffman, and that to generalize the first result is probably difficult.

Outline

1. Introduction
   - Definitions
2. Integral lattices
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   - Results of Conway and Sloane
   - A lattice associated to a graph
3. Smallest eigenvalue -2
   - Smallest eigenvalue -2
   - A result of Hoffman
4. Smallest eigenvalue at least -3
   - Main result
5. Lattices with minimum norm 3
   - Lattices with norm 3

Main result

Our main result is:

Theorem

There exists a constant $K > 0$ such that any connected graph G with minimal valency at least K and $\lambda_{\min}$ at least -3 is 2-integrable.

Remarks

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- Using the regular two-graph on 276 vertices, K., Munemasa, Rehman and Yang showed that $K \geq 166$.
- We only know an implicit upper bound for K, but certainly our bound is far from the true value.
- We have a family of connected non 2-integrable graphs with unbounded number of vertices and smallest eigenvalue at least -3. So this means that the result of Cameron et al. is not so easy to be generalized.

Representations

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- We first need to discuss representations of graphs

Representations of graphs

- A representation of a graph G with norm m is a map $x \mapsto \bar{x}$ satisfying
  - $\langle \bar{x}, \bar{x} \rangle = m$
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- A representation of G is called **fat** if there exists a set U of orthonormal vectors such that
  - for all $x \in V(G)$ and all $u \in U$, the inner product $\langle \bar{x}, u \rangle \in \{0, 1\}$, and
  - for any $x \in V(G)$, there exists $u \in U$ such that $\langle \bar{x}, u \rangle = 1$.

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- Now we take the orthogonal projection on the orthogonal complement of subspace generated by the set U of the fat representation, to obtain the representation $x \mapsto \tilde{x}$.
- Note $\langle \tilde{x}, \tilde{x} \rangle \leq 2$ and $\langle \tilde{x}, \tilde{y} \rangle \in Z$.
- So now we are done by the classification of the root lattices and the fact that they are all 2-integrable.

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- He first showed that $\hat{G}$ with $\lambda_{\min} \geq -2$, such that its corresponding lattice is not 1-integrable, has at most 6 vertices.
- Then he used the classification of root lattices and that $\hat{G}$ has exactly one eigenvalue at most -2, by the star-complement method and interlacing.

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- By repeating the procedure we obtain the minimal graphs with a fat representation of norm > 3.
- Then it is fairly easy to finish the proof of our fat representation result.

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- Can we classify those lattices Λ'?

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- I think so, but to show this we need to develop a theory of signed Hoffman graphs.

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- Our main result also holds for signed graphs.
- There exists a positive integer s such that any connected signed graph with smallest eigenvalue at least -3 is s-integrable. (Maybe $s = 4$?)

The last conjecture would essentially give a classification of the lattices with minimum norm 3.

Question:

- Can we extend our result to smallest eigenvalue -4?
- Our method can not be extended to -4, because there are infinitely many minimal forbidden vertex-weighted signed graphs with smallest eigenvalue < -3.

Thank you for your attention.